# Scattering of Electromagnetic Waves by Cantor Screens: Rayleigh-Sommerfeld Integrals on Complex Domains


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The scattering of light from fractal screens has been a topic of sustained interest in optics for many decades. A common thread weaving together much of the theoretical literature is the scalar approximation, wherein the polarization state of the electromagnetic field is not a primary concern. Experiments and their supporting analyses have also been confined largely to Fraunhofer [1,2] or, more recently, Fresnel [3] regimes. Here, these various simplifications are rolled back in order to address the scattering problem in a more fundamental way.

We consider the classic middle-third Cantor set [4] as a model for a perfectly-conducting fractal screen with negligible thickness [5]. The initial level, denoted by the index $n = 0$, is taken to describe a bounded gap of arbitrary width $2a$ along the $x$ axis (identical to the standard diffraction problem for an infinitely long slit). The first level, $n = 1$, fills-in the unbounded middle third of the gap to produce two gaps with reduced width $2a/3$. At subsequent levels $n = 2, 3, 4, \ldots$, this initiator-generator algorithm creates a set of $2^n$ bounded gaps, each of width $2a/3^n$, so that the fully-developed fractal (as $n \rightarrow \infty$) has a Hausdorff dimension $D = \log(2)/\log(3) \approx 0.631$ and a Lebesque measure of zero [4]. The limit of the Cantor screen may thus be regarded as having zero area. Yet, remarkably, rigorous application of functional analysis has recently proved that similar screens (for instance, in acoustics contexts) are still capable of supporting a transmitted wave with non-zero amplitude [6].

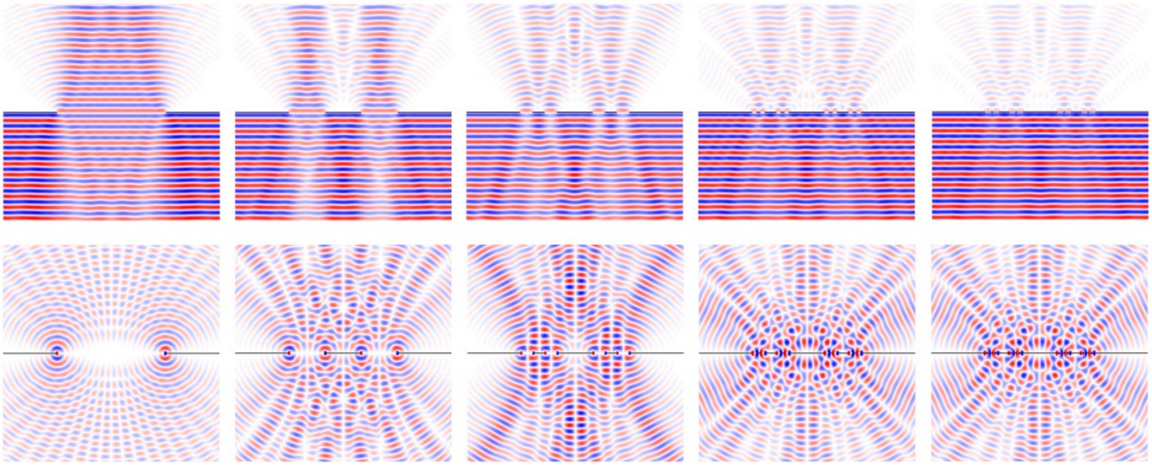

**Fig. 1** Top row: Snapshot in time of the total electric field $E_z(x,y,t)$ for the initial ($n = 0$) and first four pre-fractal levels ($n = 1 - 4$) of a Cantor screen (black line). Illumination is by a normally-incident plane wave from below. The field behind the screen is obtained directly from RS integrals, while that in front of the screen is reconstructed from symmetries in Maxwell's equations [5]. Bottom row: Corresponding longitudinal magnetic field component $B_y(x,y,t)$.

In this presentation, we construct the vector scattering problem for a Cantor screen by solving the 2D Helmholtz equation in conjunction with Sommerfeld's radiation condition and Kirchhoff's approximation of the boundary conditions [4]. Formal solutions for the diffracted electric field then turn out to be Rayleigh-Sommerfeld (RS) integrals whose kernels (involving the free Green's function in 2D) are selected depending upon what information is specified for the field in the gaps of the screen (see Fig. 1, top row) [7]. The (out-of-page) translational invariance of the screen means that these RS solutions are, in essence, also solutions to the underlying Maxwell equations. We have derived similar integral expressions for the remaining electromagnetic field components (see Fig. 1, bottom row), and proved that the divergence-free nature of **E** and **B** is fully respected in the entire space. A moderate form of Babinet's Principle also appears to provide access to the complementary scattering problem. We conclude by highlighting the advantages and disadvantages of an RS approach on pre-fractal domains, and considering alternative solution methods such as boundary elements.

## References


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