Discrete nonlinear Schrödinger equations for periodic optical systems: pattern formation in $\chi^{(3)}$ coupled waveguide arrays

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Abstract

Discrete nonlinear Schrödinger equations have been used for many years to model the propagation of light in optical architectures whose refractive index profile is modulated periodically in the transverse direction. Typically, one considers a modal decomposition of the electric field where the complex amplitudes satisfy a coupled system that accommodates nearest neighbour linear interactions and a local intensity dependent term whose origin lies in the $\chi^{(3)}$ contribution to the medium's dielectric response.

In this presentation, two classic continuum configurations are discretized in ways that have received little attention in the literature: the *ring cavity* and *counterpropagating waves*. Both of these systems are defined by distinct types of boundary condition. Moreover, they are susceptible to spatial instabilities that are ultimately responsible for generating spontaneous patterns from arbitrarily small background disturbances. Good agreement between analytical predictions and simulations will be demonstrated.

Keywords: Ring cavity, counterpropagation, Turing instability

1 Introduction

In optics, discrete nonlinear Schrödinger (dNLS) equations are often used to describe the way in which electromagnetic waves propagate through structures whose dielectric properties vary periodically [1]. The archetypal waveguide array, for instance, comprises a refractive index distribution engineered in the form of a square wave with period D. Light confined to each channel in the array is coupled to that in its nearest neighbours due to evanescent fields.

While dNLS equations have been studied extensively for over thirty years, here we address two geometries that have received little attention. First, the *ring cavity* involves confining a periodic array between a set of mirrors and feeding the output back into the input. Such systems are known in dNLS contexts, but here we relax the mean field considerations that have tended to underpin previous works. Second, the *counterpropagating waves* scenario appears to be entirely new to the dNLS realm and does not lend itself well to a mean-field theory. Both proposed dNLS models turn out to support spontaneous pattern formation through the universal mechanism discovered by Turing [2].

2 Ring cavity

In dimensionless form, the slowly-varying envelope confined within waveguide channel $n = 0, \pm1, \pm2, ...$ is denoted by E_n. It follows that

$$\begin{aligned} i\mathrm{d}_z E_n + cL(E_{n+1} - 2E_n + E_{n-1}) \\ + \chi L|E_n|^2 E_n = 0, \end{aligned} \tag{1a}$$

where $z \in [0, 1]$ denotes the (scaled) longitudinal position along an array of length L, c is a coupling constant, and χ parametrizes the intensity dependent polarization of the medium. To capture the essence of the cavity, we enforce the traditional boundary condition [3]

$$E_n(0) = tE_{\mathrm{in}} + r\exp(i\delta)E_n(1). \tag{1b}$$

Here, E_{in} is the complex amplitude of the plane wave pump field, the transmission t and reflection r coefficients of the coupling mirror satisfy $t^2 + r^2 = 1$, and δ allows for interferometric mistuning between the pump and intracavity waves.

3 Counterpropagating waves

The continuum counterpropagating waves configuration [4] may be reformulated within a discrete framework by following much the same approach as with the cavity. The array is illuminated from both ends by plane waves travelling in exactly opposite directions down the longitudinal axis z. Channel n then supports a forward and a backward wave whose slowly-varying envelopes are denoted by $F_n \equiv F_n(z,t)$ and $B_n \equiv B_n(z,t)$, respectively. In dimensionless form, we find that F_n and B_n must satisfy

$$\begin{aligned} i\left(\partial_z + \partial_t\right)F_n + cL\left(F_{n+1} - 2F_n + F_{n-1}\right) \\ + \chi L\left(|F_n|^2 + G|B_n|^2\right)F_n = 0, \end{aligned} \tag{2a}$$

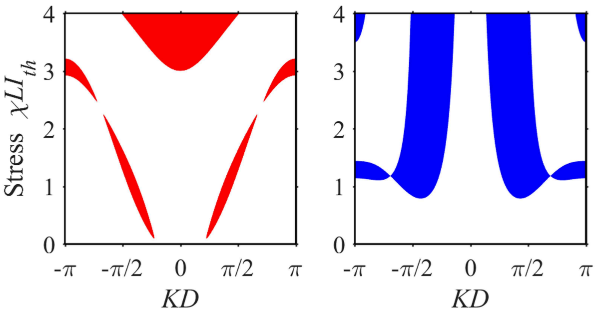

Figure 1: Typical threshold instability spectra obtained for (left) the ring cavity and (right) counterpropagating waves.

$$i\left(-\partial_z + \partial_t\right) B_n + cL\left(B_{n+1} - 2B_n + B_{n-1}\right) + \chi L\left(|B_n|^2 + G|F_n|^2\right) B_n = 0, \qquad \text{(2b)}$$

where t is now the time coordinate and $1 \leq G \leq 2$ is the grating factor [4]. Equations (2a) and (2b) belong to the "$1+2$" class of problem [note that Eq. (1a) is the simpler "$1+1$" class]. They are supplemented by the plane wave pumping boundary conditions $F_n(0,t) = F_0$ and $B_n(1,t) = B_0$ for all n, where F_0 and B_0 are constants.

4 Spontaneous patterns

The uniform states of Eqs. (1a) and (2a)–(2b) can be identified, and the discrete stability analysis subsequently proceeds in a way that takes its inspiration from the corresponding continuum model [3, 4]. These states are perturbed, and one seeks solutions to the linearized problem that prescribe Fourier modes with transverse spatial frequency K and which must respect the appropriate boundary conditions. It turns out that counterpropagation is the more difficult case to analyze, involving the exponentiation of a non-diagonal 4×4 matrix.

The threshold instability spectrum has been derived for both systems, which predicts the minimum wave intensity required to drive the growth of a perturbation mode at any given K. These spectra comprise discrete sets of islands or bands whose structure is periodic in 2π along the KD axis (see Fig. 1). The most unstable frequency, denoted by K_0, is defined to be that K with the lowest threshold and which hence possesses the highest growth rate.

In regimes with $KD \ll \mathcal{O}(1)$, the threshold conditions revert to those of their continuum counterparts [3,4]. This type of asymptotic behaviour is required on physical grounds since perturbation wavelengths $2\pi/K$ much greater than D do not 'see' the periodic structure.

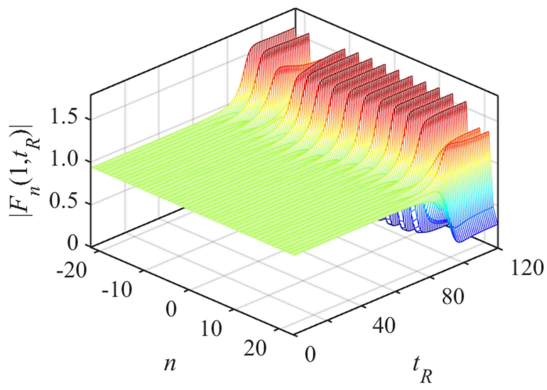

Figure 2: Emergence in time of a static pattern in the output forward wave of the counterpropagation problem. The discrete index t_R denotes the number of passes through the medium.

Simulations have been performed for both systems, wherein the uniform state is initialized with a small level of (filtered) noise added to mimic a random background fluctuation. When the intensity of the uniform state *just* exceeds threshold, a simple pattern is seen to grow spontaneously after a sufficient number of transits through the medium. Such patterns appear to be static in nature and they have a dominant spatial scalelength given by $2\pi/K_0$ (see Fig. 2). The numerics have thus provided some encouraging evidence to confirm the threshold predictions of K_0 made by linear stability analysis.

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