# Electromagnetic scattering problems on perfectly-conducting complex domains: from Rayleigh-Sommerfeld integrals toward fractal screens

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**Abstract**

The diffraction of light by an aperture in an otherwise perfectly conducting plane screen of infinite extent is a phenomenon of fundamental interest in electromagnetics. Here, we consider classes of problems where the aperture domain is *complex* (possessing self-similar structure across a range of spatial scales) and modelled on finite iterations of the fractal shapes devised by Cantor and Sierpinski.

Rayleigh-Sommerfeld (RS) integrals are deployed to predict electric fields in the space behind the screen. This approach captures more fully the details of wave scattering, eliminating many of the approximations inherent with simpler analyses in Fraunhofer and Fresnel regimes. The solutions are essentially exact for Cantor-set apertures, at least within Kirchhoff's treatment of the boundary conditions. Diffraction patterns from Cantor dust and Sierpinski triangle apertures are computed by transforming integrations over the domain into circulations around the constituent subdomain boundaries.

**Keywords:** Fractal screens, Cantor set, Cantor dust, Sierpinski triangle

## 1 Introduction

We consider an infinite screen $\Gamma_\infty$ that is a perfect conductor of zero thickness and which occupies an entire axis (in 2D) or an entire plane (in 3D). If $\Gamma$ denotes a bounded aperture in $\Gamma_\infty$, then the *Dirichlet* and *Neumann* RS integrals for the electric field $E$ behind the screen are

$$E^D(\mathbf{x}) = -2\int_\Gamma \mathrm{d}\Gamma'\, E(\mathbf{x}')\frac{\partial}{\partial n'}G_0(\mathbf{x}|\mathbf{x}'), \quad (1a)$$

$$E^N(\mathbf{x}) = 2\int_\Gamma \mathrm{d}\Gamma'\, G_0(\mathbf{x}|\mathbf{x}')\frac{\partial}{\partial n'}E(\mathbf{x}'), \quad (1b)$$

respectively, where $G_0$ is the free space Green's function of the corresponding Helmholtz equation [1]. These formulations of the diffraction problem inherently respect the Sommerfeld radiation condition [2]. Since either $E(\mathbf{x}')$ or its (outward) normal derivative $(\partial/\partial n')E(\mathbf{x}')$ are anticipated to vanish on the screen, one needs to specify their values on $\Gamma$. Following Kirchhoff's prescription, we set these quantities to match those of the incident plane wave; Eqs. (1a) and (1b) are then internally self-consistent [1].

## 2 Cantor set

Consider removing a closed interval of width $2a$ from the centre of an infinite screen $\Gamma_\infty$ that is defined along a straight line. This initiator stage, labelled by index $n = 0$, creates a gap of empty space $[-a, a]$ which represents a bounded aperture $\Gamma$ (see Fig. 1). At the first pre-fractal level ($n = 1$), the open middle third of that gap is filled-in to produce two closed sub-intervals of empty space, $[-a, -a/3]$ and $[a/3, a]$. The iterative process of filling-in the open middle thirds may continue indefinitely, with the limit $n \to \infty$ defining a Cantor set whose capacity dimension is $\log 2/\log 3 \approx 0.63$ [3]. We then take the complex domain $\Gamma$ as the union of $2^n$ closed aperture sub-intervals, each of width $2a/3^n$.

When the electric vector of the incident wave is linearly polarized and perpendicular to the propagation plane, Eqs. (1a) and (1b) prescribe

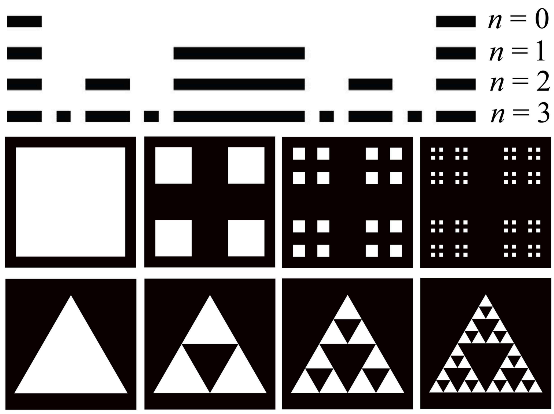


Figure 1: Examples of complex domains—the initiator and first three pre-fractal levels of the Cantor set (top), Cantor dust (middle), and Sierpinski triangle (bottom). White: bounded aperture $\Gamma$. Black: unbounded screen $\Gamma_\infty \backslash \Gamma$.

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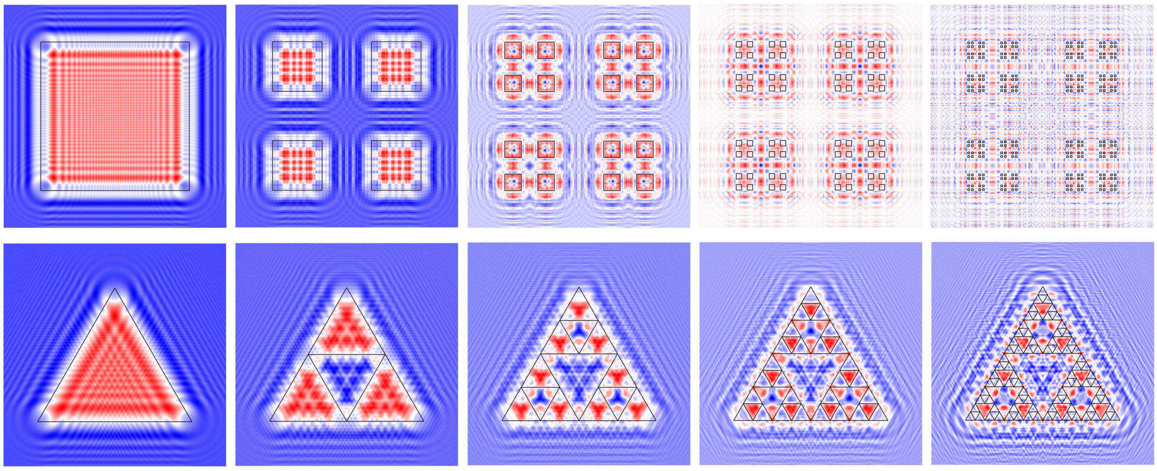

Figure 2: Numerical calculations of the Dirichlet RS integral [that is, $\Re\mathrm{e}(E^D)$] for the initiator and first four pre-fractal levels of the (top) Cantor dust and (bottom) Sierpinski triangle. Black lines correspond to the geometrical projections of the aperture domain boundaries onto the observation plane.

the vector $\mathbf{E} = (0, 0, E^{D,N})$. In this case, $E^{D,N}$ and its partial derivatives must be zero on the unlit surface of the screen. One may then calculate the magnetic field components from $\mathbf{B} = -(i/ck)\nabla\times\mathbf{E}$, prove that $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{B} = 0$ (as required by Maxwell's equations), and work out the energy flow from the Poynting vector. It is also possible to reconstruct the electromagnetic field in front of the screen by restoring the incident and reflected waves, and to devise a moderate form of Babinet's principle by considering a complementary problem [2].

## 3 Cantor dust & Sierpinski triangle

We now consider apertures based on the Cantor dust and Sierpinski triangle (see Fig. 1) [3]. In the limit, these shapes have capacity dimensions of $\log 4/\log 3 \approx 1.26$ and $\log 3/\log 2 \approx 1.58$, respectively. In these cases, the relationship between the scalar fields of Eqs. (1a) and (1b) and the full vector solution for the electromagnetic wave is not so obvious. Polarization effects are thus neglected here, but we expect $E^{D,N}$ to capture the dominant contribution in $\mathbf{E}$.

Evaluating the RS integrals for a given 2D domain is nontrivial, but progress can be facilitated by applying the divergence theorem and transforming integrations over area $\Gamma$ into circulations around the boundaries $\partial\Gamma$ of all the constituent subdomains. Such a technique renders the calculations more manageable (see Fig. 2) but they can still remain computationally expensive as the pre-fractal level increases.

## 4 Concluding remarks

The RS diffraction formulae are best suited to high-frequency regimes and have many advantages over their far field (Fraunhofer) and paraxial (Fresnel) counterparts. One must be particularly mindful of limitations [1, 2], however, within the complex-domains arena. All three apertures have a Lebesgue measure of zero, interpreted physically as *vanishing area* for $n \to \infty$. Equations (1a) and (1b) will inevitably return a wave with zero amplitude as $\Gamma$ shrinks to a set of points (though the validity of Kirchhoff's approximation will have been compromised well before then). Formulating the scattering problem more rigorously, it has recently been proved that classes of zero-measure screens *can* sometimes support a transmitted wave [4].

## References

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