Short Communication

A Lagrange-based generalised formulation for the equations of motion of simple walking models

Michael McGrath[1], David Howard[2] , Richard Baker[1]

[1] School of Health Sciences, University of Salford

[2] School of Computing, Science and Engineering

Introduction

There are numerous examples of researchers using relatively simple dynamic models to investigate the way in which human beings walk (Baker et al., 2004; Buczek et al., 2006; Kuo, 2007; McGrath et al., 2015b; Millard et al., 2011). Some have further expanded to models of 'moderate' complexity (Martin and Schmiedeler, 2014; McGrath et al., 2015a; Pandy and Berme, 1988a, b). Often these latter models consist of a number of rigid links connected by frictionless hinge joints, forming a chain. These represent the segments and joints of a person's limbs. In order for these models to provide forward dynamic simulations of a person's movement, their equations of motion (EOM) must be derived.

General formulae for the EOM of *n*-link chains have been previously developed for use in gait modelling, using a Newtonian approach (Pandy and Berme, 1988a). A great advantage of these general formulae is the time saved in developing the EOM for models with a large number of degrees-of-freedom (DOFs), where a manual approach is very time consuming. This paper describes a similar approach but using Lagrangian mechanics to develop the formulae instead, which are independent of the chosen coordinate frame. Also, because they use energy calculations, rather than forces, prior knowledge of the ground reaction force (GRF) is not required.

Once these equations are developed, walking simulations can be performed using the same methods as the complex models, such as using optimisation to estimate internal kinetics and joint activations (Anderson and Pandy, 2003). This study gives an example of such a simulation.

Method

Open-loop chains

The Lagrange equation to derive EOM for an open-loop chain is given (Onyshko and Winter, 1980).

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Equation 1

Where L is the Lagrangian function – the difference between the kinetic and potential energy – and q_i is a generalised coordinate for the i^{th} link of the chain.

Equation 1 shows the Lagrange equation equal to zero. This is valid when there are no external forces or moments acting on the system. For the derivations outlined here, moments will be acting at the joints between links so the Lagrange equation is adapted.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Equation 2

Where Q_i are the generalised forces derived from a consideration of virtual work (δw):

$$\delta w = \sum_i Q_i \delta q_i$$

Equation 3

Two choices for q_i are joint angle (φ_i) or link angle (θ_i) to the vertical.

$$\delta w = \sum_i -M_i \delta \varphi_i = \sum_i M_i (\theta_{i-1} - \theta_i) = \sum_i (M_{i+1} - M_i) \theta_i$$

Equation 4

Where M_i is the moment acting at the distal joint of the i^{th} link of the chain. This means Q_i is equal to $-M_i$ if joint angles are used or $M_{i+1} - M_i$ if the link angles to the vertical are used. Although selecting the joint angles would decouple the generalised force terms, it makes the functions for the energy calculations more complex. Consequently, link angles to the vertical are preferable and are used throughout this paper.

The following derivation is for an open-loop chain consisting of *n* rigid links, where the ground acts as a workless constraint at one end of the chain and the other end is free. Each link has the characteristics shown in Figure 1. The angular position of the i^{th} link is defined as the link's angle to the vertical. Anticlockwise is positive for angles and moments. The total length of the link is l_i. It has a mass, m_i, acting at a single point, with a moment of inertia, I_i. The position of the centre-of-mass (CM) of the link is defined by two values, d_i and e_i, where d_i is parallel to the length of the link and e_i is perpendicular to it. The direction of progression is in the positive x direction and upwards is the positive y direction. The acceleration due to gravity is written as g.

Assumptions are made for these generalised formulae to be valid. There is no branching and each link is connected to adjacent links by frictionless hinge joints. The model is 2D, in the sagittal plane, and the hinge joints are the only DOFs. For each link, there are two controlled muscle moments acting on the proximal and distal ends, respectively.

Firstly, the coordinates of the CMs of each segment are considered:

$$x_i = \sum_{h=1}^{i-1}\left(-l_h sin\theta_h\right) - d_i sin\theta_i + e_i cos\theta_i$$

$$y_i = \sum_{h=1}^{i-1}\left(l_h cos\theta_h\right) + d_i cos\theta_i + e_i sin\theta_i$$

Equations 5, 6

The linear velocities of these CMs are defined by the first derivatives.

$$\dot{x}_i = \sum_{h=1}^{i-1}\left(-l_h cos\theta_h\dot{\theta}_h\right) - d_i cos\theta_i\dot{\theta}_i - e_i sin\theta_i\dot{\theta}_i$$

$$\dot{y}_i = \sum_{h=1}^{i-1}\left(-l_h sin\theta_h\dot{\theta}_h\right) - d_i sin\theta_i\dot{\theta}_i + e_i cos\theta_i\dot{\theta}_i$$

Equations 7, 8

The resultant velocities are calculated for each CM.

$$v_i^2 = \dot{x}_i^2 + \dot{y}_i^2$$

Equation 9

The kinetic energy, T, and the potential energy, V, of the system are calculated.

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2 + \frac{1}{2}I_i\dot{\theta}_i^2\right)$$

Equation 10

$$V = mgh = \sum_{i=1}^{n}\left(m_i\left(\sum_{h=1}^{i-1}(l_h g cos\theta_h) + d_i g cos\theta_i + e_i g \sin\theta_i\right)\right)$$

Equation 11

The Lagrangian function is calculated by subtracting the potential energy from the kinetic.

$$L = T - V$$

Equation 12

Partial differentials of L with respect to $\dot{\theta}_i$ and θ_i are taken in order to evaluate the terms in the Lagrangian equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = \sum_i (M_{i+1} - M_i)\theta_i$$

Equation 13

From the calculation of these terms, the EOM can be written in matrix form.

$$B.\ddot{\theta} = C \qquad \text{where, } \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \dots & b_{n,n} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \vdots \\ \ddot{\theta}_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Equation 14

For a given row, p, and a given column, q:

$$b_{p,q} = \begin{cases} \left(m_p d_p^2 + m_p e_p^2 + \left(\sum_{j=p}^{n} m_{j+1} \right) l_p^2 + I_p \right) & if\ p = q \\ \left(\left(m_p d_p + \left(\sum_{j=p}^{n} m_{j+1} \right) l_p \right) l_q \cos\left(\theta_q - \theta_p\right) \right) + \left(m_p e_p l_q \sin\left(\theta_p - \theta_q\right) \right) & if\ p > q \\ \left(\left(m_q d_q + \left(\sum_{j=q}^{n} m_{j+1} \right) l_q \right) l_p \cos\left(\theta_p - \theta_q\right) \right) + \left(m_q e_q l_p \sin\left(\theta_q - \theta_p\right) \right) & if\ q > p \end{cases}$$

Equation 15

$$c_p = \sum_{h=1}^{\{n|p \neq h\}} \left(\dot{\theta}_h^{\,2} \left(\left(\left(\begin{cases} \left(m_p d_p + \sum_{j=p}^{n} \left(m_{j+1}\right) l_p \right) l_h \sin\left(\theta_h - \theta_p\right) & if\ h < p \\ -\left(m_h d_h + \sum_{j=h}^{n} \left(m_{j+1}\right) l_h \right) l_p \sin\left(\theta_p - \theta_h\right) & otherwise \end{cases} \right) \right. \right. \right.$$

$$\left. \left. \left. + \left(\begin{cases} \left(m_p e_p l_h\right) \cos\left(\theta_p - \theta_h\right) & if\ h < p \\ -\left(m_h e_h l_p\right) \cos\left(\theta_h - \theta_p\right) & otherwise \end{cases} \right) \right) \right) \right)$$

$$+ \left(m_p d_p + \left(\sum_{j=p}^{n} m_{j+1} \right) l_p \right) g sin\theta_p - m_p e_p g \cos\theta_p + M_{p+1} - M_p$$

Equation 16

The sigma notation $\sum_{h=1}^{\{n|p \neq h\}}$ means h covers all of the values from 1 to n, but is never the same as p.

This method does, however, rely on an estimation of joint moments. Later in this study, an optimisation algorithm is described, which uses measured kinematics and estimates these moments. This means that Matrix B can then be inverted and used to produce the vector $\ddot{\theta}$, which gives the angular acceleration for each link of the chain.

Closed-loop chains

Equation 14 is only applicable for open-loop chains, i.e. single support walking models. In order to create double support models, closed-loop chains are required. An advantage of

Lagrange mechanics is that constraints can be applied relatively simply using *'Lagrange multipliers'*.

In order to apply a constraint, the *jth* constraint function (f_j) is defined such that:

$$f_j = 0$$

Equation 17

The governing Lagrange equation is modified to include the Lagrange multipliers:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} - \sum_j \left(\lambda_j \frac{\partial f_j}{\partial q_i}\right) = Q_i$$

Equation 18

Where λ_j is the Lagrange multiplier for the j^{th} constraint. For a number of constraint equations, *r*, the same number of new unknown variables need to be solved. This is done by incorporating the constraint equations into the matrix formulation of the EOM, thus solving for $\ddot{q}_i$ and λ_j simultaneously. If the constraint equations are purely positional (only contain q_i terms), they need to be differentiated twice so that they contain $\ddot{q}_i$ terms. This new equation then needs to be separated into two functions; one that contains only the $\ddot{q}_i$ terms, g_j, and one that contains the rest of the terms h_j (Equation 19). These terms can now be incorporated into the matrix formulation (Equation 20).

$$\frac{d^2 f_j}{dt^2}\left(\ddot{q}_i, \dot{q}_i, q_i, t\right) = g_j\left(\ddot{q}_i, t\right) + h_j\left(\dot{q}_i, q_i, t\right) = 0$$

Equation 19

$$\begin{bmatrix} b_{i,i} & -\frac{\partial f_j}{\partial q_i} \\ \frac{g_j\left(\ddot{q}_i, t\right)}{\ddot{q}_i} & 0 \end{bmatrix} \begin{bmatrix} \ddot{q}_i \\ \lambda_j \end{bmatrix} = \begin{bmatrix} c_i \\ -h_j\left(\dot{q}_i, q_i, t\right) \end{bmatrix}$$

Equation 20

It's important to note that the $\ddot{q}_i$ terms are no longer all independent. For a chain with *n* DOFs and *r* constraint equations, only *n-r* are independent. If the initial conditions satisfy the constraints, then computing $\ddot{q}_i$ and integrating to solve for all DOFs should produce solutions which are consistent with the constraint equations. These can be validated using the constraint equations (Ülker, 2010). If $\ddot{q}_i$ is known for the first *n-r* links in the chain, the constraint equations can be used to compute $\ddot{q}_i$ for the final *r* links. A worked example is given in the appendix.

Ground reaction force calculations

Inverse dynamics can be used to calculate the total GRF acting on a walking model. For open-loop chains, this is the GRF where the chain is in contact with the ground (the single supporting foot). For closed-loop chains, a method is required to determine how the total GRF is distributed between the two ground contact points, which is an indeterminate problem. The following derivation is for the vertical and horizontal components of the total GRF.

By considering the vertical direction first, Newton’s second law of motion is used:

$$GRF_y - mg = \sum_{i=1}^{n} m_i \ddot{y}_i$$

Equation 21

Differentiating Equation 8:

$$\ddot{y}_i = \sum_{h=1}^{i-1} l_h\left(-\ddot{\theta}_h \sin\theta_h - \dot{\theta}_h^{\,2} \cos\theta_h\right) + d_i\left(-\ddot{\theta}_i \sin\theta_i - \dot{\theta}_i^{\,2} \cos\theta_i\right) + e_i\left(\ddot{\theta}_i \cos\theta_i - \dot{\theta}_i^{\,2} \sin\theta_i\right)$$

Equation 22

Similarly, for the horizontal direction:

$$GRF_x = ma = \sum_{i=1}^{n} m_i \ddot{x}_i$$

Equation 23

Differentiating Equation 7:

$$\ddot{x}_i = \sum_{h=1}^{i-1} l_h \left(-\ddot{\theta}_h \cos\theta_h + \dot{\theta}_h{}^2 \sin\theta_h\right) + d_i \left(-\ddot{\theta}_i \cos\theta_i + \dot{\theta}_i{}^2 \sin\theta_i\right) + e_i \left(-\ddot{\theta}_i \sin\theta_i - \dot{\theta}_i{}^2 \cos\theta_i\right)$$

Equation 24

During double support, although the total GRF can be calculated, there is an infinite number of ways this can be distributed between the two feet. Ren et al. (Ren et al., 2007), solved this problem by making a smooth transition assumption. The Lagrange multipliers method used here offers an alternative approach because the multipliers can be used to calculate the force required to maintain a given constraint. In the case of this study, the forces required to hold the trailing foot fixed to the ground can be used to calculate the GRF under that foot. By using inverse dynamics, in the same way as before, to calculate the total GRF, a simple subtraction can be used to obtain the GRF under the leading foot.

Since the constraint forces are acting upon the trailing foot and it is stationary, it can be assumed that the GRF components beneath it are equal to these constraint forces. The forces the constraints produce can be expressed:

$$F_{q_i} = \lambda \frac{\partial f}{\partial q_i}$$

Equation 25

In order to calculate the constraint forces in the x and y directions, the following equations are used:

$$F_x = \lambda_{f_1} \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial \theta_i} \frac{\partial \theta_i}{\partial x} \right) = \lambda_{f_1} \sum_{i=1}^{n} \left(-l_i \cos \theta_i \, . \frac{1}{-l_i \cos \theta_i} \right) = \lambda_{f_1}$$

Equation 26

$$F_y = \lambda_{f_2} \sum_{i=1}^{n} \left(\frac{\partial f_2}{\partial \theta_i} \frac{\partial \theta_i}{\partial y} \right) = \lambda_{f_2} \sum_{i=1}^{n} \left(-l_i \sin \theta_i \, . \frac{1}{-l_i \sin \theta_i} \right) = \lambda_{f_2}$$

Equation 27

These values relate to the GRF components at the trailing foot. Subtracting these from their respective total GRF components give the GRF components beneath the leading foot.

Example simulation

Gait laboratory data was collected for a single, healthy, female participant (28 years old, 65kg, 162cm). Ethical approval for the study was granted by the Institutional Ethics Panel (ref HSCR13/18). A Vicon 3D motion capture system (Oxford Metrics plc., Oxford, UK) and Kistler force plates (Kistler Group, Winterthur, Switzerland) were used to capture kinematic and kinetic data, respectively.

The derived generalised formulae were used to generate a seven degree-of-freedom model (previously described by McGrath et al. (2015a)). For the simulation model, the participants anthropometric data were used and segment masses were estimated using Winter's formulae (1979, 1991).

The simulation was split into two: a single support (open chain) and a double support (closed chain). For both double and single support simulations, a global optimisation was performed using the MATLAB function *'GlobalSearch'* (Ugray et al., 2007). The input parameters were the initial kinematic state (segment angular positions and velocities) and the joint moments over the whole simulation. The initial kinematic state was known from the gait lab measurements but since the temporal profiles of the joint moments were unknown, the initial estimate was taken from Winter's data (1979, 1991). The cost function was the root mean square difference of the predicted kinematics, to those measured in the gait lab. Consequently, the optimiser was designed to 'track' the motion.

The results of this simulation are illustrated in Figure 2.

Discussion

A general formulation for the EOM of an open-link chain has been derived and presented here, with the application of modelling bipedal walking. Using Lagrangian mechanics to derive these formulae has been shown to be independent of coordinate frames and requires less prior kinetic knowledge than alternative approaches, such as Newton-Euler mechanics. In terms of walking, this means that the GRF does not need to be known or estimated in order to perform forward dynamics calculations.

However, joint moments do need to be estimated. This can be executed using an optimisation procedure, a similar method to how Anderson and Pandy (2003) estimated muscle activations in a more complex model with a higher number of degrees-of-freedom. The advantage of the model described here is that a solution can be achieved within a matter of hours, rather than days, which is particularly important when a forward dynamics simulation is used within an iterative optimisation procedure. Additionally, with simpler models, it can be easier to identify cause-and-effect relationships, to gain a better understanding of the relationships between form and function in gait biomechanics. With more complex models, this process becomes much more challenging because the internal model calculations are less amenable to inspection.

Another advantage of Lagrangian mechanics is that Lagrange multipliers can be incorporated into the calculations to apply constraints. This enables the modelling of a closed-loop chain, which, in terms of walking, equates to the double support phase. Additionally, it has been shown that these multipliers can be used to estimate the distribution of the GRF when both feet are contacting the floor; something that was previously an indeterminate problem.

Word count: 1990

Acknowledgements

There were no external contributors or sponsors to the work detailed throughout this manuscript.

Conflict of interest statement

There are no conflicts of interest related to this work.

References

Anderson, F.C., Pandy, M.G., 2003. Individual muscle contributions to support in normal walking. Gait & Posture 17, 159-169.

Baker, R., Kirkwood, C., Pandy, M., 2004. Minimising the vertical excursion of the center of mass is not the primary aim of walking, Eighth International Symposium on the 3D Analysis of Human Movement, Tampa, Florida, pp. 101-104.

Buczek, F.L., Cooney, K.M., Walker, M.R., Rainbow, M.J., Concha, M.C., Sanders, J.O., 2006. Performance of an inverted pendulum model directly applied to normal human gait. Clinical Biomechanics 21, 288-296.

Kuo, A.D., 2007. The six determinants of gait and the inverted pendulum analogy: A dynamic walking perspective. Human Movement Science 26, 617-656.

Martin, A.E., Schmiedeler, J.P., 2014. Predicting human walking gaits with a simple planar model. Journal of biomechanics 47, 1416-1421.

McGrath, M., Howard, D., Baker, R., 2015a. A Forward Dynamic Modelling Investigation of Cause-and-Effect Relationships in Single Support Phase of Human Walking. Computational and Mathematical Methods in Medicine 2015, 9.

McGrath, M., Howard, D., Baker, R., 2015b. The strengths and weaknesses of inverted pendulum models of human walking. Gait & posture 41, 389-394.

Millard, M., Kubica, E., McPhee, J., 2011. Forward dynamic human gait simulation using a SLIP target model. Procedia IUTAM 2, 142-157.

Onyshko, S., Winter, D.A., 1980. A mathematical model for the dynamics of human locomotion. Journal of Biomechanics 13, 361-368.

Pandy, M.G., Berme, N., 1988a. A numerical method for simulating the dynamics of human walking. Journal of Biomechanics 21, 1043-1051.

Pandy, M.G., Berme, N., 1988b. Synthesis of human walking: A planar model for single support. Journal of Biomechanics 21, 1053-1060.

Ren, L., Jones, R.K., Howard, D., 2007. Predictive modelling of human walking over a complete gait cycle. Journal of Biomechanics 40, 1567-1574.

Ugray, Z., Lasdon, L., Plummer, J., Glover, F., Kelly, J., Mart\, R., \#237, 2007. Scatter Search and Local NLP Solvers: A Multistart Framework for Global Optimization. INFORMS J. on Computing 19, 328-340.

Ülker, H., 2010. Dynamic Analysis of Flexible Mechanisms by Finite Element Method. İzmir Institute of Technology.

Winter, D.A., 1979. Biomechanics of Human Movement. John Wiley & Sons, Inc.

Winter, D.A., 1991. The biomechanics and motor control of human gait: Normal, Elderly and Pathological, 2nd ed. Waterloo Biomechanics, Waterloo.