OSCILLATORY OSEENLETS

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Oscillatory Oseenlets

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Contents

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Abstract

Consider uniform flow past an oscillating body. Assume that the resulting far-field flow consists of both steady and time periodic components. The time periodic component can be decomposed into a Fourier expansion series of time harmonic terms. The form of the steady terms given by the steady oseenlets are well-known. However, the time-harmonic terms given by the oscillatory oseenlets are not. In particular, the Green's functions associated with these terms are presented.

In this thesis, the oscillatory oseenlet solution is presented for the velocity and pressure, and the forces generated by them are calculated. A physical interpretation is given so that the consequences for moving oscillating bodies can be determined.

As the frequency of the oscillations tend to zero, it is shown that the steady oseenlet solution is recovered. Also, as the Reynolds number of the flow tends to zero, it is shown that the oscillatory stokeslet solution is recovered. In this latter case, the oscillatory oseenlets solution is an outer matching to the inner oscillatory stokeslet solution. An application of this new representation is discussed for future work.

Chapter 1

Introduction

The problem of uniform flow past an oscillating body is a general one, examples being the flapping flight of birds and insects, the swimming of mammals, fish and micro-organisms, the oscillating of Micro-ElectroMechanical Systems (MEMS), and acoustic devices under water. What differentiates this diverse set is the Reynolds numbers for stream velocity and velocity of oscillation, and the dimensionalised frequency of oscillation. What is common in most of this diverse set is the aim to achieve a steady forward propulsion, this lends itself to the Oseen linearisation in the far-field.

The current literature on time dependent Oseen and associated Stokes flows subdivides into transient analysis and oscillatory analysis, with the majority of work on transient rather than oscillatory analysis.

Price [1] use transient oseenlets in order to model ship motions. Also, Chan and Chwang in [2], and Lu and Chwang in [3] describe the unsteady (transient) stokeslet and oseenlet and give applications related to acceleration and free surface waves. Childress [4] uses transient oseenlets to model the effect of flapping of a swimming mollusc. A numerical solution of the transient oseenlet analysis is employed. However, for a steady oscillatory motion of the swimming mollusc, an oscillatory oseenlet would be beneficial.

Riley's work [5] [6] and Amin's work [7] do employ an oscillating rather than transient analysis to model the flow generated by fixed oscillating bodies. Here the focus is on matching the inner Stokes- type flow to an outer flow. However, there is no uniform stream for these problems and the outer flow is not an Oseen flow and very different from it. Clarke et. al [8] consider the problem of a MEMS device vibrating in a fluid at rest. The device is treated as a slender body and the Stokes approximation is used. The oscillatory stokeslet given by Pozrikidis [9] is used. A further development on Clarke's work would be to consider the effect of a uniform rather than stationary flow field, which would for example replicate blood flow. Within such a development, the oscillatory stokeslet is an inner near-field description to be matched to an outer far-field oscillatory oseenlet. In order to enable this, there is a requirement for the oscillatory oseenlet solution. Iima [10] considers a butterfly flapping and whether it can sustain hovering motion. He formulates a far-field periodic Oseen representation for a small steady uniform flow motion and then lets that motion tend to zero. This representation is not expressed in terms of oseenlets, and instead uses an approach based upon that of Imai [11]. Yet the representation by singular (stokeslet, oseenlet) solutions has many advantages, one being that a body can be represented in a straightforward way by a distributed superposition of them [12], and another being the additional insight into the physical understanding of the flow such a model provides.

The omission of the oscillatory oseenlet representation within the literature is noticeable, and restricting the armoury of techniques to be used on these important problems. In the present work, we therefore give the oscillatory oseenlet solution and indicate how it can be applied to good effect on these problems.

In this thesis, we shall give the time-harmonic oscillatory oseenlet representation and the force it generates. Furthermore, we shall show that it reduces to the steady oseenlet and oscillatory stokeslet solutions in appropriate limiting cases. It is noted that a steady streaming velocity perturbation is also expected in practice, but this shall not be detailed as this steady Oseen solution is well known,see [13]. Also, it is noted that the flow may not be time-harmonic but time-periodic, for example in the formulation given by Lighthill in [14]. However, the time-periodic solution can be expressed as fourier series of time-harmonic terms, see for example Iima [10], for those problems which require a time-periodic rather than time-harmonic solution.

1.1 Basic concepts

We give here definitions of some concepts that are used in this thesis.

1.1.1 Viscous flow

Fluids are divided into viscous and inviscid fluids depending on their resistance to stress. Fluids which resist a stress are called viscous fluid and the viscosity measures the fluid resistance to a shear force or to flow. Hence, water has low viscosity relative to honey which has a high viscosity. Fluids which have no resistance to stress are known as inviscid fluids.

Fluid with high viscosity is called slow viscous flow, and the viscous effects are dominant the flow over the inertial effects. A dimensionless number which parameterise the flow, is used to measure the relative importance of the inertial effects to the viscous effects, that is Reynolds number

Reynolds number

The Reynolds number is a dimensionless number which determines the relative importance of inertial and viscous effects, defined as:

$$Reynolds\ number = \frac{fluid\ density \times speed \times length}{viscosity}. \tag{1.1}$$

The Reynolds number can be written as a ratio of the convective acceleration (convective acceleration unit volume has dimension $\frac{\rho U^2}{L}$) to the viscous forces (viscous force unit volume has dimension $\frac{\mu U}{L^2}$), where ρ is the fluid density, U is a velocity scale which could be the body velocity, L is a length scale, which could be a body length and μ is the fluid viscosity:

$$Re = \frac{\rho U^2}{L} / \frac{\mu U}{L^2} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}. \tag{1.2}$$

The Reynolds number may be small ($Re \ll 1$) in the sense of slow velocity U, high viscosity ν, small size length L, or for the fluid density ρ is much lower than fluid viscosity μ even for the case where the viscosity is very low. The cases of small Reynolds number flow are called slow viscous flows, in which the inertial forces associated with acceleration of fluid particles are small compared to the viscous forces arising from shearing motions of the fluid, see [15], [16], and [17].

1.1.2 Incompressible and Newtonian fluid

In this thesis we deal with a Newtonian, incompressible fluid. In particular, an incompressible fluid with constant density is considered. That means, it is fluid in which the volume of any material region is unchanged with time, see [18].

The implication of incompressibility Consider a fixed closed surface S in the fluid, with outward unit normal $\mathbf{n}$. At some points on the surface S, the fluid is entering the region V which is bounded by S, and at some other points on the surface S, the fluid is leaving. The velocity along the normal $\mathbf{n}$ is $\mathbf{u.n}$, where $\mathbf{u}$ is the fluid velocity and taking a small surface element dS of the surface S, then the volume of the fluid leaving through dS is $\mathbf{u.n}\ dS$. Thus, the net volume rate at which fluid is leaving V is $\int\int_S \mathbf{u.n}\ dS$.

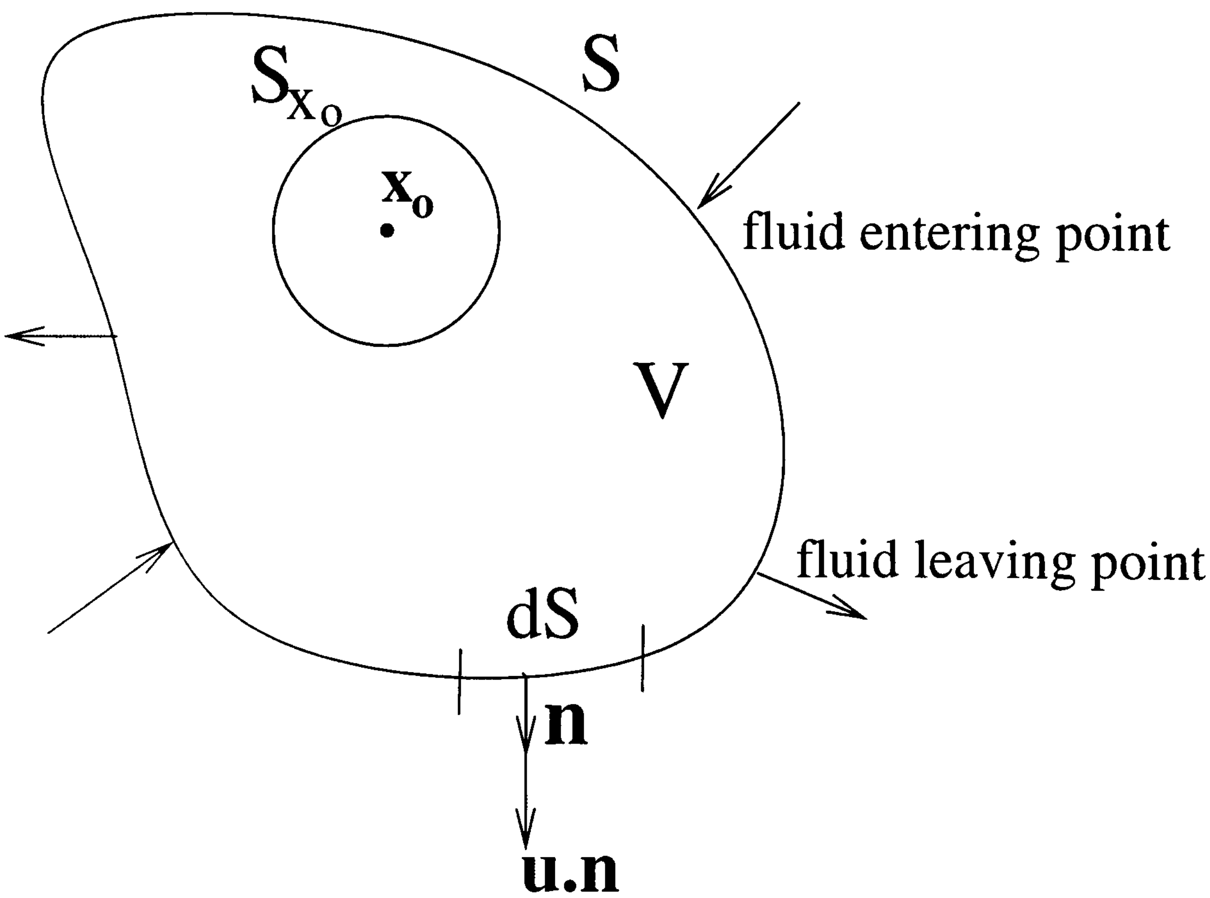

Figure 1.1: region V

Because of the incompressibility $\int\int_S \mathbf{u.n}\ dS = 0$ which is called no outflow condition.

using the divergence theorem gives

$$\int\int_S \mathbf{u.n}\, dS = \int\int\int_V \nabla.\mathbf{u}\, dV = 0 \tag{1.3}$$

for any region V within the fluid, where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ denotes the gradient operator [19]. It can be shown that $\nabla.\, \mathbf{u} = 0$ must be true for all fluid points, as follows. Suppose that $\nabla.\, \mathbf{u} \neq 0$ then either $\nabla.\, \mathbf{u} > 0$ or $\nabla.\, \mathbf{u} < 0$ at some points. If $\nabla.\, \mathbf{u} > 0$ for some point x_o in the fluid and assuming that $\nabla.\, \mathbf{u}$ is continuous, then $\nabla.\, \mathbf{u} > 0$ in some small sphere S_{x_o}around the point x_o. Taking $V = S_{x_o}$ leads to

$$0 < \int\int\int_{V(=S_{x_o})} \nabla.\mathbf{u}\, dV \neq 0, \tag{1.4}$$

and if $\nabla.\mathbf{u} < 0$ for some point x_o in the fluid then in a similar way to above, we have

$$0 > \int\int\int_{V(=S_{x_o})} \nabla.\mathbf{u}\, dV \neq 0. \tag{1.5}$$

There is a contradiction between both cases and (1.3). Hence, $\nabla.\, \mathbf{u} = 0$ for any point within the incompressible fluid, see [20].

The fluid is called Newtonian if the shear stress and the velocity gradient are related linearly, and the constant of proportionality is known as the viscosity. The Newtonian fluid agrees with Newton's law of viscosity, which is described by the equation

$$\underbrace{\sigma_{ij}}_{\text{shear stress exerted by fluid}} = \overbrace{\mu}^{viscosity} \underbrace{\frac{\partial u_i}{\partial x_j}}_{\text{velocity gradient}}, \tag{1.6}$$

where: σ_{ij} denotes the shear stress; u_i is the velocity component in the i direction of a Cartesian coordinate system x_i and $i, j = 1,\ 2,\ 3.$

1.1.3 Velocity potential of incompressible fluid

In certain cases, the fluid velocity may be expressed in terms of a single valued function ϕ, and such a function is called the velocity potential. (If a velocity potential exists, then it can be chosen such that the density of the fluid is either a function of the pressure only or a constant, [21], Art. 17). The velocity is described by

$$\mathbf{u} = \nabla\phi.$$

For incompressible fluid the velocity potential exists and is harmonic, resulting from applying the divergence which gives

$$\nabla^2\phi = \nabla \cdot \mathbf{u} = 0,$$

where ∇ is gradient operator, $\nabla^2 = \nabla \cdot \nabla$ is the Laplace operator and the dot denotes the inner product. The existence of the velocity potential for incompressible fluid is satisfied by the fundamental theorem of vector calculus, which states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (zero curl) vector field and a solenoidal (zero divergence) vector field; This implies that any such vector field F can be considered to be generated by a pair of potentials: a scalar potential ϕ and a vector potential A.

1.1.4 Steady and unsteady flow

Flows can be classified as steady flow and unsteady flow.

Steady flow: when all conditions of a flow remain unchanged over time, the flow is said to be steady. The conditions may vary from one point to another within the flow but remain unchanged at the same point.

Unsteady flow: when the flow conditions change with time at any point, the flow is said to be unsteady. Unsteady flow may be classified itself into transient flow (time-non periodic) and oscillatory flow (time-periodic).

1.1.5 Uniform flow

Flow is said to be uniform if the velocity remains unchanged at every point within the fluid, in other words, it is a flow during which the instantaneous velocity is always constant. Hence acceleration is zero, and a constant velocity implies that the direction of the flow is along a straight line and average velocity and instantaneous velocity have the same magnitude. An example of uniform flow is the flow with constant velocity U in the x_i-direction of coordinate system $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$;

$$u_i = \pm U\hat{x}_i, u_j = 0, j \neq i,$$

where the unit vector $\hat{x}_i$ is perpendicular to the plane $x_j x_k, j, k \neq i$. For $i = 1$, the uniform flow $u_i = U\hat{x}$, is in positive direction of x.

1.2 Thesis Overview

In the second chapter, the derivation of the equations of motion has been considered both in dimensional and dimensionless form to give the background to the thesis. In chapter three, we introduce the steady Stokes flow, and give the Green integral representation of the steady Stokes flow and the construction of the steady stokeslets is given in terms of potentials using the approach used by Lamb [21] for the Oseen flow. Also, the Green's Integral Representation of the Steady Stokes Velocity is given and the integral representation of the force. Finally, we compute the force generated by the steady stokeslet. The new result in chapter three is constructing the steady stokeslets using a different approach,involving the Oseen potentials, to the approach that has been used in literature to obtain the stokeslets.

In chapter four, we consider the oscillatory Stokes flow. The Oscillatory stokeslets are first given using the singularity method by Pozrikidis [9], we obtain the oscillatory stokeslet in terms of potentials, using a similar approach which we used in chapter three for the steady stokeslets. The potentials representation will enable us later to show that the oscillatory stokeslets can be recovered from the oscillatory oseenlet at a particular limit. The Green's surface integral representation of the flow is given and we establish the behaviour of the flow in the far field and at high frequencies. The representation of the flow velocity in terms of the oscillatory Stokes solutions which requires to know the behaviour of the stokeslets close to the point force, is given. Also, we present the force integral representation and the force generated by the oscillatory stokeslets.This is the first time in literature to represent the stokeslets in terms of potentials and results which are given in this chapter are identical to the existing results using the singular method.

In chapter five, the steady Oseen flow is considered and well known results are given in some details. These include obtaining the steady Oseenlet, flow Green's surface integral representation, and the force generated by a steady oseenlet.

In chapter six, we consider a uniform flow past an oscillatory body in an unbounded fluid region. The Green's integral representation for oscillatory Oseen flow is given. Lamb and Goldstein used potential decomposition of fluid velocity to obtain the steady oseenlet, we use a similar decomposition for Green functions rather than fluid velocity to obtain the oscillatory oseenlet. At particular limits we demonstrate that the oscillatory oseenlets reduce to known cases. The asymptotic series of the oseenlet around zero are presented, which are then used to obtain the Green's integral representation of the Oseen velocity.

We show that the oscillatory oseenlets reduce to the steady oseenlets when the frequency tends to zero. The problem of uniform flow past a steady body in an unbounded region and Oseen's approximation are given as well as the Green's integral representation of Oseen flow. Following the Lamb and Goldstein decomposition, the steady oseenlets and the asymptotic series of the oseenlet around zero are presented, which are then used to obtain the Green's integral representation of Oseen velocity. Finally, the force is given as a far field integral in more detail.

For the first time in the literature, the time-harmonic oscillatory Oseenlets for velocity and pressure are represented in chapter six. The oscillatory oseenlets are constructed in terms of potentials and the reduction to the steady oseenlets and to the oscillatory stokeslets in appropriate limits are given. We give the Green's integral representation of the oscillatory Oseen equation and we demonstrate that the oscillatory oseenlets can be written in Pozrikidis's form of the oscillatory stokeslets. The integral representation of the oscillatory Oseen velocity and the expansion of the oseenlets around zero are represented. The force generated by the oscillatory oseenlets is given in terms of the velocity, pressure and the frequency. In the last chapter, applications discussion of our results and future work are presented.

Chapter 2

Equations of Motion

Fluids display such properties as not resisting deformation, or resisting it only slightly, and the ability to flow which can be described as the ability to take on the shape of a container. Ideal fluids (inviscid and incompressible) can only be subjected to normal, compressive stress which is called pressure and real fluids are capable of being subjected to shear stress.

In fluids, shear stress is a function of the rate of strain, and depending on the form of this relation between shear stress and the rate of strain and its derivatives, fluids can be characterised as non-Newtonian fluids (where stress is proportional to rate of strain, its higher powers and derivatives) and Newtonian fluids (where stress is directly proportional to rate of strain).

In this thesis, we will deal only with Newtonian Fluid which is named after Sir Isaac Newton, and the constant of proportionality is known as the viscosity. The behaviour of fluids can be described by the Navier-Stokes equations which are a set of partial differential equations based on:

- Continuity (conservation of mass)
- Conservation of linear momentum (Newton's second law of motion)

The study of fluids in motion (fluid flow) is fluid dynamics which is a sub-discipline of fluid mechanics. It has several sub-disciplines itself, including aerodynamics (the study of gases in motion) and hydrodynamics (the study of liquids in motion).

2.1 Continuity Equation

The continuity equation is a differential equation that describes conservation of mass. In fluid dynamics, it is a mathematical statement that the rate at which mass enters a system is equal to the rate at which mass leaves the system.

The continuity equation is governed by the physical laws of the moving fluid in a fixed control volume V, taking into account the flow through the surface S enclosing V and the forces which act on the fluid.

Ensuring that mass is conserved in V gives the rate of increase of mass in V equal to the rate at which the fluid is flowing into V through the surface S.
This is can be written as

$$\frac{\partial}{\partial t}(\int\int\int_V \rho dV) = -\int\int_S \rho u_j n_j ds$$

where the minus sign is necessary because $\mathbf{n}$ is the outward unit normal.
So

$$\int\int\int_V \frac{\partial \rho}{\partial t} dV + \int\int_S \rho u_j n_j ds = 0.$$

Applying the divergence theorem $\int\int_S \mathbf{u}.\mathbf{n}\, ds = \int\int\int_V \nabla.\mathbf{u}\, dV$, where $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ for a continuously differentiable vector field $\mathbf{u}$, gives

$$\int\int\int_V \{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\} dV = 0.$$

Since V is an arbitrary volume, then

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0.$$

This is the continuity equation, which is the local form of the conservation of mass law.

In the case of an incompressible fluid, for which the density ρ is a constant, the continuity equation simplifies to

$$\frac{\partial u_j}{\partial x_j} = 0 \; or \; \nabla.\mathbf{u} = 0, \tag{2.1}$$

which means that the divergence of the velocity field is zero everywhere, see [16] or [22].

2.2 Stress

When external force is applied on a body, internal forces are produced within the body as reaction. The measurement of these forces is called stress. Types of external forces are:

- Body forces, such as gravity and electronic magnetic forces, which are force per unit of mass acting on all volume elements ΔV. We let body forces be denoted by $\mathbf{b}$.
- Surface forces, such as pressure, act across an internal or external surface element ΔS in a material body. Let surface forces be denoted by $\mathbf{f}$.

- Point forces, when a force is applied on very small area which can be consider as a point, then the force is called a point force.

In general, the stress is not uniformly distributed across a section of the material body. Therefore, it is necessary to define the stress at a specific point **P** in the body, which is assumed to be a continuum.

2.2.1 Cauchy's stress principle

Consider a continuum body subjected to a surface force **f** and body force **b**. Let V be an arbitrary volume enclosed by the surface S and **n** be the outward normal at point $\mathbf{P} \in \Delta S$. The resultant forces are given by Δf_i exerted across ΔS upon the material within V by the material of outside V.

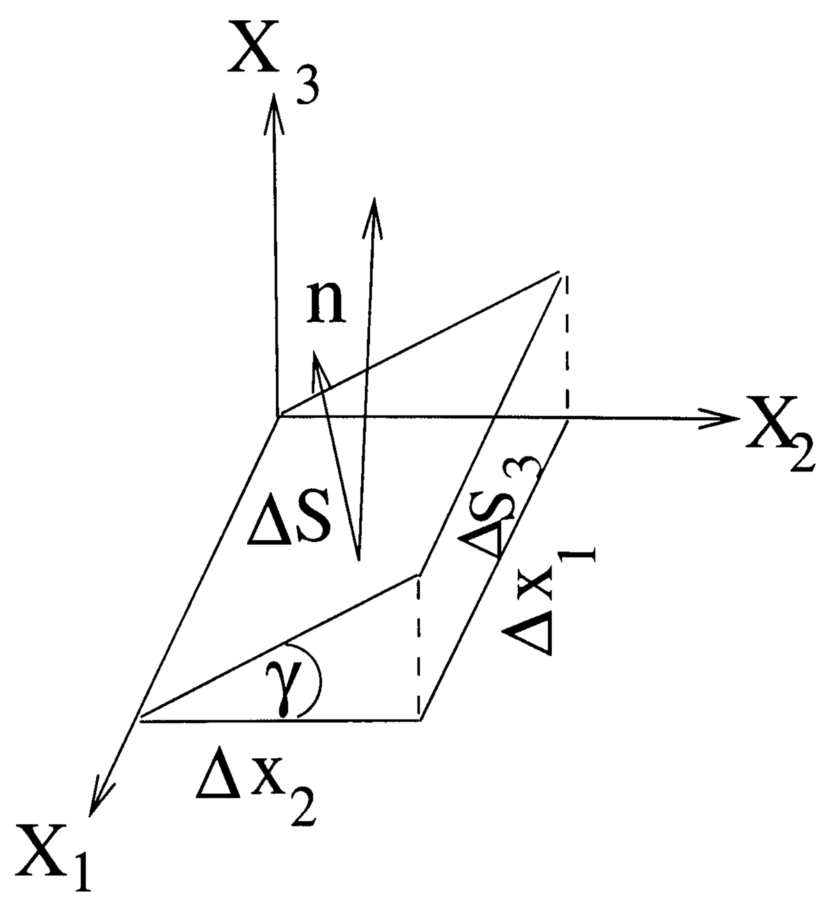

Figure 2.1: ΔS_3 is the projection of ΔS on the x_1x_2.plane

The distribution of force on the area ΔS is not always uniform, as there may be a moment ΔM at **P** due to the force $\Delta\mathbf{f}$.

Cauchy's stress principle states that as ΔS tends to zero, in the limit $\frac{\Delta \mathbf{f}}{\Delta S}$ becomes $\frac{df_i}{dS}$ and ΔM vanishes. The resulting vector $\frac{df_i}{dS}$ is defined as the stress vector $t^{(\mathbf{n})}$ at the point **P**,

$$t^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} \quad or \; t_i^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta f_i}{\Delta S}. \tag{2.2}$$

Considering the projection onto the plane normal to n_3, gives

$$\Delta S = \frac{\Delta x_1 \Delta x_2}{\cos \gamma}, \tag{2.3}$$

where γ is the angle between the plane ΔS and the plane $\Delta x_1 \Delta x_2$, see figure (2.1). As **n** is perpendicular to ΔS, then

$$\cos \gamma = \frac{\mathbf{n}.\mathbf{e}_3}{|\mathbf{n}||\mathbf{e}_3|} = \mathbf{n}.\mathbf{e}_3 = n_3, \tag{2.4}$$

where **n** is the outward pointing normal to the surface and $\mathbf{e}_i = \widehat{x}_i$ is the unit vector in the ith coordinate direction. So

$$\Delta S = \frac{\Delta x_1 \Delta x_2}{n_3}, \tag{2.5}$$

gives $\Delta x_1 \Delta x_2 = \Delta S n_3 = \Delta S_3$, where ΔS_3 is the projection of ΔS on the plane perpendicular to x_3 axis. A similar argument holds for ΔS_1 and ΔS_2, giving

$$\Delta S_j = \Delta S n_j.$$

Taking the limit when $\Delta S \longrightarrow 0$ yields

$$\lim_{\Delta S \to 0} \frac{\Delta S_j}{\Delta S} = \frac{dS_j}{dS} = n_j.$$

From (2.2) we have

$$t_i^{(n)} = \frac{df_i}{dS} = \frac{df_i}{dS_j}\frac{dS_j}{dS} = \frac{df_i}{dS_j} n_j \tag{2.6}$$

where the repeated suffix implies a summation over j. This gives the stress vector at the point **P** across a surface S. The stress depends on both the location in the body and also the plane across which it is acting. From Newton's third law (law of action and reaction), the stress vector acting on opposite sides of the same surface are equal in magnitude and opposite in direction,

$$t^{(\mathbf{n})} = -t^{(-\mathbf{n})}.$$

2.2.2 Stress Tensor

The stress at a point **P** in the body is defined by all the stress vectors $t^{(\mathbf{n})}$ and normal vectors **n** associated with all planes that pass through that point. Fortunately, according to Cauchy's fundamental theorem, we need to know the stress on three mutually perpendicular planes, then the stress vector on any other plane passing through that point can be found through coordinate transformation.

Since the unit vector $\hat{x}_i$ is perpendicular to the plane $x_j x_k, j, k \neq i$, we can write

$$t^{(\mathbf{n})} = t_1^{(\mathbf{n})}\hat{x}_1 + t_2^{(\mathbf{n})}\hat{x}_2 + t_3^{(\mathbf{n})}\hat{x}_3 = t_i^{(\mathbf{n})}\hat{x}_i.$$

For the three coordinate planes, the stress vector can be written by taking $\mathbf{n} = \hat{\mathbf{x}}$,

$$
\begin{aligned}
t^{(\hat{x}_1)} &= t_1^{(\hat{x}_1)}\hat{x}_1 + t_2^{(\hat{x}_1)}\hat{x}_2 + t_3^{(\hat{x}_1)}\hat{x}_3 \\
t^{(\hat{x}_2)} &= t_1^{(\hat{x}_2)}\hat{x}_1 + t_2^{(\hat{x}_2)}\hat{x}_2 + t_3^{(\hat{x}_2)}\hat{x}_3 \\
t^{(\hat{x}_3)} &= t_1^{(\hat{x}_3)}\hat{x}_1 + t_2^{(\hat{x}_3)}\hat{x}_2 + t_3^{(\hat{x}_3)}\hat{x}_3. \qquad (2.7)
\end{aligned}
$$

In index notation

$$t^{(\hat{x}_j)} = t_1^{(\hat{x}_j)}\hat{x}_1 + t_2^{(\hat{x}_j)}\hat{x}_2 + t_3^{(\hat{x}_j)}\hat{x}_3 = t_i^{(\hat{x}_j)}\hat{e}_i.$$

In (2.6) replacing $\mathbf{n}$ by $\hat{\mathbf{x}}$, gives

$$t_i^{(\hat{x}_j)} = \frac{df_i}{dS_j}.$$

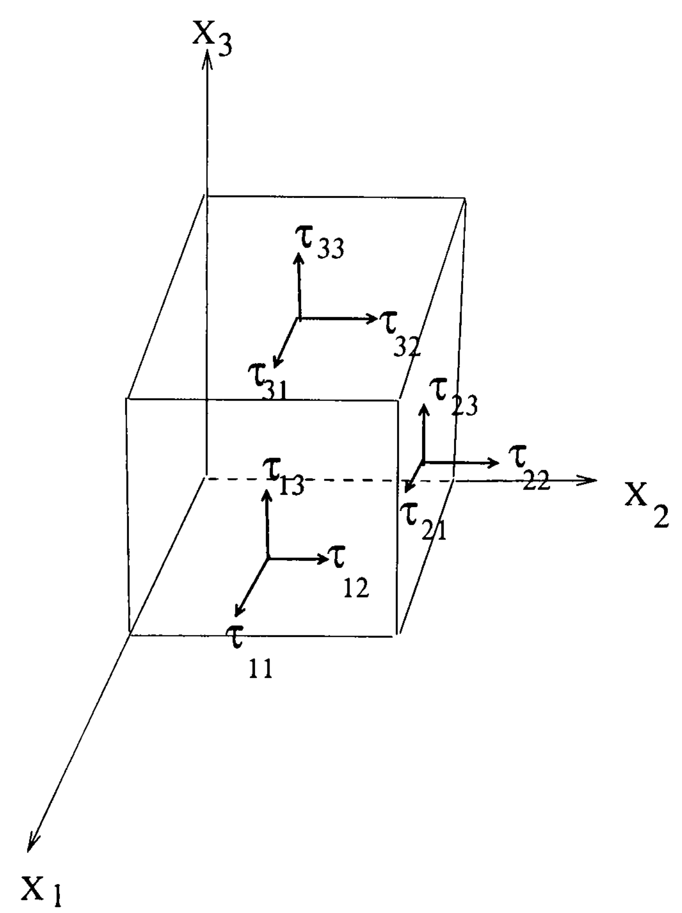

Figure 2.2: Components of stress in three dimensions

Denote τ_{ji} by $t_i^{(\hat{x}_j)}$, the nine stress vector components. Therefore

$$
\begin{aligned}
t^{(\mathbf{n})} &= \left(t_1^{(\mathbf{n})},\ t_2^{(\mathbf{n})},\ t_3^{(\mathbf{n})} \right) = \left(\frac{df_1}{dS},\ \frac{df_2}{dS},\ \frac{df_3}{dS} \right) \\
&= \left(\frac{df_1}{dS_j} n_j,\ \frac{df_2}{dS_j} n_j,\ \frac{df_3}{dS_j} n_j \right) \\
&= \left(\frac{df_1}{dS_1} n_1 + \frac{df_1}{dS_2} n_2 + \frac{df_1}{dS_3} n_3,\ \frac{df_2}{dS_1} n_1 + \frac{df_2}{dS_2} n_2 + \frac{df_2}{dS_3} n_3,\ \frac{df_3}{dS_1} n_1 + \frac{df_3}{dS_2} n_2 + \frac{df_3}{dS_3} n_3 \right) \\
&= \left(\tau_{11} n_1 + \tau_{21} n_2 + \tau_{31} n_3,\ \ \tau_{12} n_1 + \tau_{22} n_2 + \tau_{32} n_3,\ \ \tau_{13} n_1 + \tau_{23} n_2 + \tau_{33} n_3 \right) \\
&= \left(\begin{pmatrix} n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \right)^T = \tau_{ij} n_j. \qquad (2.8)
\end{aligned}
$$

This gives the relation between the stress vector $t^{(\mathbf{n})}$ and the stress tensor τ_{ij}.

2.2.3 Normal and Shear stresses

Normal stresses are the stress vector components which are perpendicular to the planes $(\tau_{11}, \tau_{22}, \tau_{33})$, while the shear stresses are the stress vector components which are tangents to the plane $(\tau_{12}, \tau_{13}, \tau_{21}, \tau_{23}, \tau_{31}, \tau_{32})$.

2.2.4 The shear stress and the strain rate tensor

The shear stress σ_{ij} is related to the velocity gradient $\frac{\partial u_i}{\partial x_j}$, which can be decomposed into symmetric part which is the strain rate tensor e_{ij} and antisymmetric part that is the vorticity tensor Ω_{ij},

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) \tag{2.9}$$

where $e_{ij} \equiv \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, and $\Omega_{ij} \equiv \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i})$. The antisymmetric tensor Ω_{ij} represents fluid rotation without deformation , and cannot by itself generate stress. While the strain tensor e_{ij} generates the stresses alone [19] For Newtonian fluid, the relation between the stress σ_{ij} and the velocity gradient $\frac{\partial u_i}{\partial x_j}$, is linear. The most general linear relation is

$$\sigma_{ij} = A_{ijmp}\frac{\partial u_m}{\partial x_p} = A_{ijmp}e_{mp} \tag{2.10}$$

where

$$A_{ijmp} = \lambda\delta_{ij}\delta_{mp} + 2\mu\delta_{im}\delta_{jp}$$

which is an isotropic tensor (that is defined as the tensor whose components do not change under a rotation of the coordinate system, see [19]). Taking into account that there is no stress generated by the vorticity, λ and μ are scalar constants, and A_{ijmp} must be symmetric in i and j because σ_{ij} is also symmetric.

2.2.5 The Constitutive Relation for Newtonian Fluid

At rest, fluid has no tangential stress acting on its surface, only the normal component of stress on the surface appears, which is internal stress due to the pressure p. The stress tensor is isotropic, and any isotropic second order tensor is proportional to the Kronecker delta, therefore

$$\tau_{ij} = -p\delta_{ij}.$$

For fluid in motion, due to viscosity additional components of stress appear and the shear stress develops. Now we can decompose the stress τ_{ij} into two parts,

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{2.11}$$

Substituting (2.10) into (2.11) and using the symmetric property of e_{ij}, gives

$$\begin{aligned}
\tau_{ij} &= -p\delta_{ij} + A_{ijmp}e_{mp} = -p\delta_{ij} + (\lambda\delta_{ij}\delta_{mp} + 2\mu\delta_{im}\delta_{jp})e_{mp} \\
&= -p\delta_{ij} + \lambda\delta_{ij}\delta_{mp}e_{mp} + 2\mu\delta_{im}\delta_{jp}e_{mp} \\
&= -p\delta_{ij} + \lambda\delta_{ij}\delta_{mp}e_{pm} + 2\mu\delta_{im}\delta_{jp}e_{pm} \\
&= -p\delta_{ij} + \lambda\delta_{ij}e_{mm} + 2\mu\delta_{im}e_{jm} \\
&= -p\delta_{ij} + \lambda\delta_{ij}e_{mm} + 2\mu e_{ij}.
\end{aligned} \tag{2.12}$$

Since $e_{mm} = \nabla \cdot \mathbf{u} = 0$ for an incompressible fluid, then

$$\tau_{ij} = -p\delta_{ij} + 2\mu e_{ij} = -p\delta_{ij} + 2\mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}). \tag{2.13}$$

This relation is called the **constitutive relation.**

2.3 The Navier-Stokes Equation

The Navier-Stokes equation describes the motion of a Newtonian fluid. It is a non-linear differential equation which does not explicitly establish a relation among the variables of interest (velocity and pressure). Rather they establish a relation between the rates of change of the variables. The non-linearity is due to the convective acceleration, which is an acceleration associated with the change in velocity over position.

Usually, the Navier-Stokes equation is too complicated to be solved in a closed form. However, in some cases, such as Stokes and Oseen flows, the equation can be simplified to a linear equation.

2.3.1 Derivation

Newton's second law implies that the rate of increase of momentum in a control volume V must equal the sum of the rate at which momentum is flowing in through the boundary S and the total forces acting on the contents of V. We can write this in mathematical form as

$$\int\int\int_V \frac{\partial}{\partial t}(\rho u_i^\dagger)dV = -\int\int_S (\rho u_j^\dagger n_j)u_i^\dagger dS + \int\int\int_V \rho F_i dV + \int\int_S \tau_{ij} n_j dS$$

where: ρ is density; $u_i^\dagger$ is Navier-Stokes velocity; $\mathbf{F}$ is body force per unit mass acting on the fluid and τ_{ij} is the stress tensor.

Applying the divergence theorem gives

$$\int\int\int_V [\frac{\partial}{\partial t}(\rho u_i^\dagger) + \frac{\partial}{\partial x_j}(\rho u_i^\dagger u_j^\dagger) - \frac{\partial \tau_{ij}}{\partial x_j} - \rho F_i]dV = 0.$$

Since V is an arbitrary volume, we can write

$$\frac{\partial}{\partial t}(\rho u_i^\dagger) + \frac{\partial}{\partial x_j}(\rho u_i^\dagger u_j^\dagger) - \frac{\partial \tau_{ij}}{\partial x_j} - \rho F_i = 0.$$

Under the incompressible assumptions, the density ρ is a constant, and it follows that the second term will simplify to

$$\frac{\partial}{\partial x_j}(\rho u_j^\dagger u_i^\dagger) = \rho u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j}, \tag{2.14}$$

because $\frac{\partial u_j^\dagger}{\partial x_j} = 0$ from the continuity equation. So we get the following

$$\rho \frac{\partial u_i^\dagger}{\partial t} + \rho u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j} = \frac{\partial \tau_{ij}}{\partial x_j} + \rho F_i. \tag{2.15}$$

From (2.13), $\tau_{ij} = -p^\dagger \delta_{ij} + \mu(\frac{\partial u_i^\dagger}{\partial x_j} + \frac{\partial u_j^\dagger}{\partial x_i})$, which yields

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p^\dagger}{\partial x_i} + \mu(\frac{\partial^2 u_i^\dagger}{\partial x_j \partial x_j} + \frac{\partial^2 u_j^\dagger}{\partial x_i \partial x_j}), \tag{2.16}$$

where $p^\dagger$ is the fluid pressure. Using the continuity equation (2.1), gives

$$\frac{\partial \tau_{ij}}{\partial x_j} = -\frac{\partial p^\dagger}{\partial x_i} + \mu \frac{\partial^2 u_i^\dagger}{\partial x_j \partial x_j}. \tag{2.17}$$

By substituting (2.17) into (2.15), we get

$$\rho(\frac{\partial u_i^\dagger}{\partial t} + u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j}) = -\frac{\partial p^\dagger}{\partial x_i} + \mu \frac{\partial^2 u_i^\dagger}{\partial x_j \partial x_j} + \rho F_i. \tag{2.18}$$

The force **F** can be absorbed in the pressure for a conservative force field $F_i = \frac{\partial \phi}{\partial x_i}$, in which case the last term in (2.18) will be absent. The resulting equation (2.18) is the Navier-Stokes equation, which can be written in terms of the material derivative $\frac{D}{Dt} =$

$\frac{\partial}{\partial t} + u_j^\dagger \frac{\partial}{\partial x_j}$ as

$$\rho \frac{Du_i^\dagger}{Dt} = -\frac{\partial p^\dagger}{\partial x_i} + \mu \frac{\partial^2 u_i^\dagger}{\partial x_j \partial x_j} \tag{2.19}$$

with continuity equation $\frac{\partial u_i^\dagger}{\partial x_i} = 0$.

In operator form this is

$$\rho \frac{D\mathbf{u}^\dagger}{Dt} = -\nabla p^\dagger + \mu \nabla^2 \mathbf{u}^\dagger, \quad \nabla.\mathbf{u}^\dagger = 0. \tag{2.20}$$

It is worth observing the meaning of each term in the Navier-Stokes equation:

$$\underbrace{\rho\Big(\overbrace{\frac{\partial \mathbf{u}^\dagger}{\partial t}}^{\text{unsteady acceleration}} + \overbrace{\mathbf{u}^\dagger.\nabla\mathbf{u}^\dagger}^{\text{convective acceleration}} \Big)}_{\text{inertial force}} = \overbrace{\underbrace{-\nabla p^\dagger}_{\text{pressure and other body forces}} + \underbrace{\mu\nabla^2\mathbf{u}^\dagger}_{\text{viscous term}}}^{\text{Divergence of stress}}. \tag{2.21}$$

Only the convective term is non-linear and it is an acceleration caused by a change in velocity $\mathbf{u}^\dagger$ over position. The pressure term includes any other conservatives, such as gravity.

2.3.2 Stokes and Oseen Approximation

In the Navier-Stokes equation (2.21), we have three terms: the inertial term of two components (which are unsteady acceleration and convective acceleration), the pressure term and the viscous term. In the low viscous flow limit, we estimate the inertial term to be small, more specifically the convective acceleration is assumed small. It is negligible only if the remaining terms are not small by comparison.

Neglecting the convective acceleration term is called Stokes's approximation, while Oseen's approximation represents the convective acceleration by a linear term which is the combination of a uniform flow velocity and a velocity due to a body which is passing by the uniform flow.

2.3.3 Why Oseen's approximation is needed

For uniform flow past a solid body, the pressure and the viscous terms near the body are certainly not small, but far from the body both terms are expected to decay toward zero, see figure (2.3). So the Stokes' approximation may fail in the far field, where a better approximation may be to neglect the viscous term and let the pressure and inertial terms balance each other. Thus, Oseen replaces the inertial term $\frac{Du_i^\dagger}{Dt} = \frac{\partial u_i^\dagger}{\partial t} + u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j}$ in the Navier-Stokes equation by $\frac{\partial u_i}{\partial t} + U \frac{\partial u_i}{\partial x_1}$ where U is uniform velocity. In the far field region, the fluid velocity tends to the uniform flow velocity U. Therefore the Oseen's approximation is valid far from the body, but the condition $\mathbf{u}^\dagger.\mathbf{n} = 0$ must be satisfied on the body surface. This approximation fails near the body where the inertial term is small compared to pressure and viscous terms, but it becomes more appropriate far from the body, in the far field, see [21] and [17].

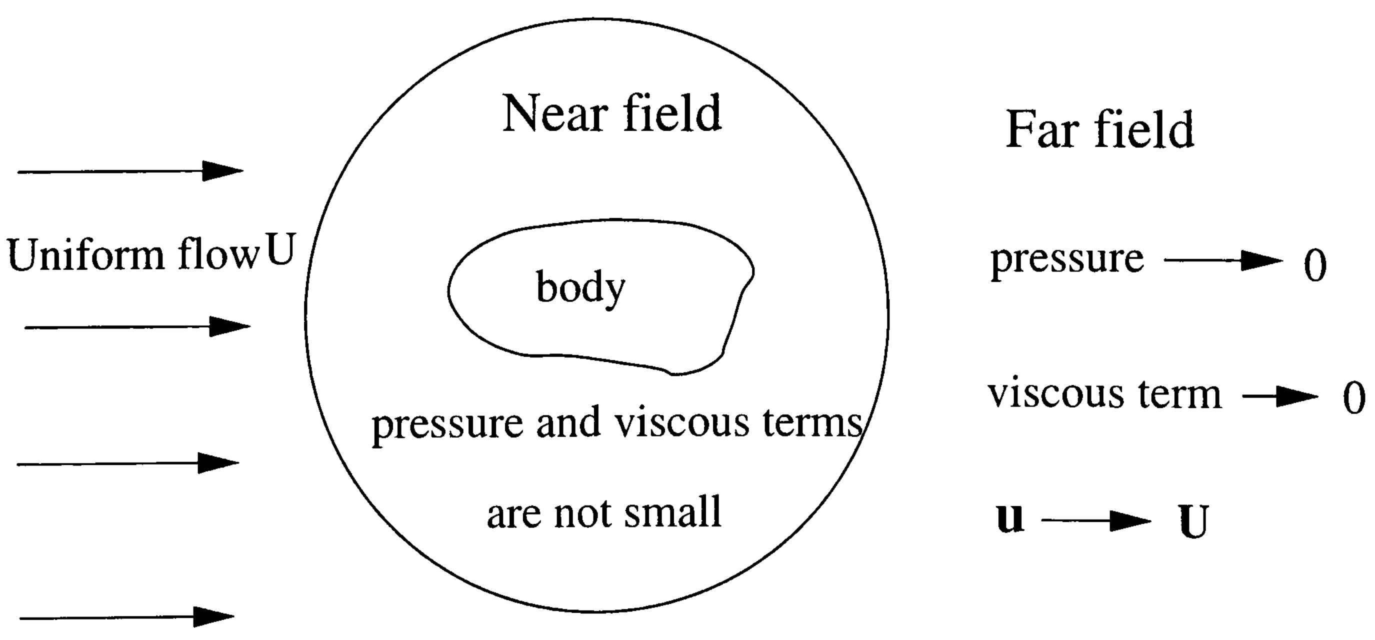

Figure 2.3: Near and Far Fields

2.4 The Stokes Equation

The Stokes equation describes slow viscous or low Reynolds number flows, in which the convective acceleration term on the left-hand side of the Navier-Stokes equation (2.19), is small compared to the rest of the terms and may be neglected. Thus the non-linear term $\rho\mathbf{u}.\nabla\mathbf{u}$ is negligible and then the inertial term can be approximated by the unsteady acceleration $\rho\frac{\partial\mathbf{u}}{\partial t}$.

2.4.1 Derivation of the Stokes Equation

Starting with the Navier-Stokes equation

$$\rho(\frac{\partial u_i^\dagger}{\partial t} + u_j^\dagger \frac{\partial u_i^\dagger}{\partial x_j}) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i^\dagger}{\partial x_j \partial x_j}, \tag{2.22}$$

the non-linear term can be neglected, which approximates (2.22) to

$$\rho\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}. \tag{2.23}$$

The resulting equation is the unsteady Stokes equation. For steady flow the term $\frac{\partial u_i}{\partial t}$ will be zero, so the steady Stokes equation will be

$$0 = -\nabla p + \mu\nabla^2\mathbf{u}, \tag{2.24}$$

where $\nabla^2 = \frac{\partial^2}{\partial x_1 \partial x_1} + \frac{\partial^2}{\partial x_2 \partial x_2} + \frac{\partial^2}{\partial x_3 \partial x_3} = \frac{\partial^2}{\partial x_j \partial x_j}$ is the Laplacian operator. Taking the divergence gives

$$\nabla^2 p = 0. \tag{2.25}$$

2.5 The Oseen Equation

The Oseen equation describes the flow of a viscous and incompressible fluid in a uniform flow field, as formulated by Carl Wilhelm Oseen in 1910 [23], [17]. In 1911 Horace Lamb was able to use the Oseen equation to derive improved expressions for the viscous flow around a sphere, improving on Stokes flow, and deriving a solution for the viscous flow around a circular cylinder, see [24], [17].

Consider a uniform flow with velocity U past a body. Far from the body the flow may be decomposed into the incident flow and a disturbance flow with velocity $\mathbf{u}$ due to the body. A similar decomposition can be introduced for the pressure.

2.5.1 Derivation of the Oseen equation

We have seen that the incompressible, Newtonian fluid flow is governed by the Navier-Stokes equations and continuity equation

$$\rho\frac{D\mathbf{u}^\dagger}{Dt} = -\nabla p^\dagger + \mu\nabla^2\mathbf{u}^\dagger, \quad \nabla.\mathbf{u}^\dagger = 0. \tag{2.26}$$

Let U be a uniform flow which is parallel to the x_1-axis. A body with an arbitrary shape is fixed in the stream. Oseen decomposes both the velocity and pressure as

$$u_i^\dagger = U\delta_{i1} + u_i + O(\varepsilon^2), \quad p^\dagger = p_0 + p + O(\varepsilon^2). \tag{2.27}$$

The notation 'O 'means 'of order of ', δ_{ij} is the Kronecker delta, and $\varepsilon \ll 1$, $\varepsilon = O(|\frac{\mathbf{u}}{U}|)$. The perturbation velocity and pressure are $\mathbf{u}$ and p, respectively, which depend on the position and time in the unsteady case and on position only in the steady case.

By Oseen's approximation the velocity $\mathbf{u}$ is approximately equal to the uniform flow velocity U thus the Oseen approximation depends on the condition that $|\frac{\mathbf{u}}{U}| \ll 1$.

Applying Oseen's approximation to the non-linear term $\rho u_j^\dagger \frac{\partial}{\partial x_j}$ of the Navier-Stokes equations, gives

$$\rho u_j^\dagger \frac{\partial}{\partial x_j} = \rho U(\frac{\partial}{\partial x_1} + \frac{u_j}{U}\frac{\partial}{\partial x_j}).$$

After neglecting quadratic terms in $\mathbf{u}$, the resulting equations are the Oseen equations, which are

$$\rho(\frac{\partial u_i}{\partial t} + U\frac{\partial u_i}{\partial x_1}) = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad \nabla.\mathbf{u} = 0. \tag{2.28}$$

Taking the divergence of the above equation yields

$$\rho(\frac{\partial}{\partial t}(\nabla.\mathbf{u}) + U\frac{\partial}{\partial x_1}(\nabla.\mathbf{u})) = -\nabla^2 p + \mu\nabla^2(\nabla.\mathbf{u}).$$

Therefore

$$\nabla^2 p = 0. \tag{2.29}$$

Away from the body, the velocity $\mathbf{u}$ tends to zero. That means the fluid flow perturbation to the uniform flow U in the far field is small. Now taking the Oseen equation to the far field and applying the condition that $\mathbf{u} \to 0$, yields $\nabla p \to 0$, thus we may choose $p \to 0$ in the far field.

2.6 Force Integral Equation

As shown in section (2.2), the force on a body can be divided into body forces and surface forces. The surface forces can further be resolved into normal and tangential components.

Let us consider dA as an area element of a surface A and dF as the force on the element dA, then the stress on the element dA is

$$\tau_A \equiv \frac{dF}{dA},$$

which implies

$$dF = \tau_A dA \Rightarrow F = \int\int_A \tau_A \, dA.$$

The total force acting on the surface is

$$\begin{aligned} F_i &= \int\int_A \tau_{ij} n_j \, dA. \\ &= \int\int_A (-p\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}))n_j \, dA \end{aligned}$$

For any volume V that enclosed the surface A and for an incompressible fluid $\frac{\partial^2 u_j}{\partial x_i \partial x_j} = 0$, then from the divergence theorem, see [16], the last integral is zero

$$\mu \int\int_A \frac{\partial u_j}{\partial x_i} n_j \, dA = \mu \int\int\int_V \frac{\partial}{\partial x_j}(\frac{\partial u_j}{\partial x_i}) \, dV = 0 \tag{2.30}$$

This gives

$$F_i = \int\int_A (-p\delta_{ij} + \mu\frac{\partial u_i}{\partial x_j}) \, n_j \, dA \tag{2.31}$$

which gives the integral equation of the force per unit area acting on the body.

2.7 Navier-Stokes Equations in Dimensionless form

In this section we derive the dimensionless Navier-Stokes equation, which gives better understanding of physical meaning and implications. Partial or full removal of units from an equation is known as dimensionalisation, which is used to write an equation in terms of dimensionless variables. This transformation gives the following advantages;

1. Gives minimum set of parameters on which the field depends
2. Simplifying the notation
3. Easier assessment to the relative impact of the terms
4. In dynamic similarity problems, dimensionless form allows the system to be adapted to a similar one.

2.7.1 How to obtain the dimensionless form

The dimensionless equation is obtained by following the next procedure;

1. Writing the differential equation, initial and boundary conditions which describes the problem.
2. Identify a parameter for each dependent or independent variable that needs to be parameterised. Parameters are dimensional constants with the same dimensions as the variable they parameterise and they depend on the nature of the problem. These parameters should be kept to small set. For example, if a problem involving position x, velocity u and time t, these three variables involve only two dimensions, length L and velocity U, so two parameters should be chosen. Time parameter is either defined as the ratio of the velocity and the distance ($t = \frac{L}{U}$) or identified with the frequency of oscillation for oscillatory flow, see [25].
3. Substitute the parameters in the equation to give dimensionless form of the equation.

We use dimensional analysis throughout this thesis and we only use the dimensionless form to show the physical implications and in the applications discussion. In the introduction, we define the Reynolds number which is dimensionless number, here we introduce another dimensionless number, Womersley number. Reynolds number and Womersley number are necessary to solve an incompressible fluid flow problem. Since the magnitude of the dimensionless variables and their derivatives is of order unity, the importance of the terms is determined by the magnitude of their multiplication to the Reynolds number and/or Womersley number [25].

2.7.2 Womersley Number

The Womersley number, named after John R Womersley (1907-1958), is dimensionless number which is defined as the ratio of the unsteady (transient or oscillatory) acceleration to the viscous term in Navier-Stokes equations. This number arises in solving the Navier-Stokes equations for oscillatory flow and it expressing the ratio between the flow frequency and the viscous forces. We denote it by R_ω, and it can be written as

$$R_\omega = L\sqrt{\frac{\omega\rho}{\mu}}. \tag{2.32}$$

The Womersley number is also called transient (or oscillatory) Reynolds number. In case that the viscous forces are dominant in the flow, the Womersley number is low and in the case the flow is dominated by oscillatory inertial forces, the Womersley number is large.

2.7.3 Derivation of the dimensionless Equations

We consider a flow of an compressible viscous fluid past a body of a finite size, recalling the Navier-Stokes equations

$$\rho\frac{D\mathbf{u}^{\dagger}}{Dt} = -\nabla p^{\dagger} + \mu\nabla^2\mathbf{u}^{\dagger}, \quad \nabla.\mathbf{u}^{\dagger} = 0. \tag{2.33}$$

We use a parameter velocity U and a parameter length L, and define the dimensionless variables by

$$\tilde{x} = \frac{x}{L}, \tilde{u} = \frac{u^{\dagger}}{U}, \tilde{t} = \omega t,$$

where ω is the period of oscillation, and tildes denote dimensionless quantities. This implies $\nabla = \frac{1}{L}\tilde{\nabla}$, $\nabla^2 = \frac{1}{L^2}\tilde{\nabla}^2$ where $\tilde{\nabla}$ is the gradient with respect to the dimensionless position vector $\tilde{x}$ and $\frac{\partial}{\partial t} = \omega\frac{\partial}{\partial \tilde{t}}$. By using this scale the terms of the Navier-Stokes equation will have the following orders

$$\rho\frac{\partial \tilde{u}_i}{\partial \tilde{t}} = O(\rho U\omega), \rho\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = O(\frac{\rho U^2}{L}), \mu\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l} = O(\frac{\mu U}{L^2}).$$

The choice of a correct scale for the pressure depends on the flow under consideration, if we consider Stokes flow, then for steady case both terms on the left-hand side tend to zero and the pressure term will balance with the viscous term. Hence, the pressure term $\frac{\partial \tilde{p}}{\partial \tilde{x}_i} = O(\frac{\mu U}{L^2})$ so that $\tilde{p} = O(\frac{\mu U}{L})$. Therefore, $\tilde{p} = \frac{L}{\mu U}p$. Substituting the dimensionless variables into the Navier-Stokes equation becomes

$$\rho\omega U\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \frac{\rho U^2}{L}\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\mu U}{L^2}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\mu U}{L^2}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}. \tag{2.34}$$

Focusing on the nonlinear term we multiply the equation by the factor $\frac{L}{\rho U^2}$

$$\frac{\omega L}{U}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\mu}{\rho U L}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\mu}{\rho U L}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}, \tag{2.35}$$

The factor $\frac{L\omega}{U}$ can be written as

$$\frac{L\omega}{U} = (\frac{\rho\omega L^2}{\mu})(\frac{\mu}{\rho U L}) = \frac{R_\omega}{Re}$$

where $R_\omega^2 = \frac{\rho\omega L^2}{\mu}$ is the dimensionless Womersley number, and Re is the Reynolds number (1.2).

So

$$\frac{R_\omega^2}{Re}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{1}{Re}\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}. \tag{2.36}$$

The dimensionless Navier-Stokes equations are

$$R_\omega^2\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + Re\tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}. \tag{2.37}$$

One can see that for very small values the dimensionless numbers, $Re \ll 1$ and $R_\omega \ll 1$, terms on the left-hand side of equation (3.7) may be neglected and then the Stokes equation governs the flow

$$-\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l} = 0. \tag{2.38}$$

From this equation we can see that the Stokes flow has only dependence on time through time-dependent boundary conditions, and the flow can be found without the knowledge of the flow at any other time. Also the Stokes equation is linear which allows superposition of solutions.

When flow has small Reynold number $Re \ll 1$ and R_ω approximately equals to unity

($R_\omega \approx 1$), the second term on the left-hand side, that is the inertial convective term, is small compared to the other terms and may be neglected, then the unsteady Stokes equation reveal;

$$\frac{\partial \tilde{u}_i}{\partial \tilde{t}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}. \tag{2.39}$$

As the acceleration term is present in (2.39), then the flow depends on the history of the motion. Also, the unsteady Stokes equation is linear, which gives ability to use variety of solution methods, such as Laplace and Fourier transform, and superposition of solutions.

Stokes flow has variety of applications in biology hydrodynamics, engineering and physics, such as flow due to movement of micro-organism (when *Re* is small due to the very small size), flow due to the motion of an air bubble in honey (*Re* is small due to honey high viscosity), and flow past a red cell blood (which has small diameter).

For Oseen's flow, we need to introduce another dimensionless variable for pressure, scaling the terms using a velocity parameter U, length L, and frequency ω and define dimensionless variables as

$$\tilde{x} = \frac{x}{L}, \tilde{u} = \frac{u^\dagger}{U}, \tilde{t} = \omega t, \tilde{p} = \frac{p^\dagger}{\rho U^2}$$

where tildes denote dimensionless quantities. This implies $\nabla = \frac{1}{L}\tilde{\nabla}$, $\nabla^2 = \frac{1}{L}\tilde{\nabla}^2$ where $\tilde{\nabla}$ is the gradient with respect to the dimensionless position vector $\tilde{x}$ and $\frac{\partial}{\partial t} = \omega \frac{\partial}{\partial \tilde{t}}$. Substituting the dimensionless variables into the equations leads to

$$\rho \omega U \frac{\partial \tilde{u}_i}{\partial \tilde{t}} + (\frac{\rho U^2}{L})\tilde{u}_j \frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -(\frac{\rho U^2}{L})\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + (\frac{\mu U}{L^2})\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}$$

To focus on the nonlinear term we multiply the equation by factor $\frac{L}{\rho U^2}$ (from the convective acceleration term),

$$(\frac{\omega L}{U})\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + (\frac{\mu}{\rho U L})\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}$$

since $Re = \frac{\rho U L}{\mu}$, then we can write

$$\frac{\omega L}{U}\frac{\partial \tilde{u}_i}{\partial \tilde{t}} + \tilde{u}_j\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}_l \partial \tilde{x}_l}.$$

Now, since the factor $\frac{\omega L}{U}$ can be written in terms of both the Reynolds and Womersley numbers

$$R_\omega^2 = L^2\frac{\rho\omega}{\mu} = \frac{\omega L}{U}(\frac{U\rho L}{\mu}) = \frac{\omega L}{U}\,Re$$

Then the dimensionless form of the Navier-Stokes equations is

$$\frac{R_\omega^2}{Re}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}}{\partial \tilde{x}_i} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}_l \partial \tilde{x}_l} \tag{2.40}$$

and

$$\tilde{\nabla}.\tilde{\mathbf{u}} = 0.$$

In the case of R_ω tends to zero the steady Oseen equation is recovered

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_i} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}_l \partial \tilde{x}_l}. \tag{2.41}$$

The equation (2.40) can be apply when the Reynolds number is small or high. For $Re \ll 1$, the equation represent the far field . When $Re \approx 1$ both Stokes and Oseen equations give good representation of the near field with equal accuracy. The unsteady Oseen equation will recove when $R_\omega \approx 1$. Applications will be discussed in chapter 7.

Chapter 3

Steady Stokes Flow

3.1 Introduction

Steady (or time-independent) flow is when the change of fluid velocity with time is zero, the equation of motion for steady flow is obtained by omitting the time- dependent terms of the Navier-Stokes equations. Stokes steady flow has been in the literature for many years, it is named after George Gabriel Stokes also it is called creeping flow(for which $Re \rightarrow 0$). In this chapter, we introduce the steady Stokes flow in section one, and in section two we give the Green integral representation of the steady Stokes flow. In section three, the construction of the steady stokeslets is given in terms of potentials using similar approach to the approach used by Lamb [21] for the Oseen flow. The Green's Integral Representation of the Steady Stokes Velocity is given and also the integral representation of the force. Finally, we compute the force generated by the steady stokeslet. Throughout this thesis, the superscripts s denotes the Stokes solutions.

3.2 Statement of the problem

Consider uniform steady incompressible fluid flow in an unbounded domain past a stationary body. At low Reynolds number flow, the Stokes approximation is valid in the near field and linearises the Navier-Stokes equations. Recalling (2.19) the Navier-Stokes equation for an incompressible fluid is

$$\rho \frac{Du_j^\dagger}{Dt} = -\frac{\partial p^\dagger}{\partial x_j} + \mu \frac{\partial^2 u_i^\dagger}{\partial x_l \partial x_l}, \tag{3.1}$$

where $i, l = 1, 2, 3$. Dropping the inertial force $\rho \frac{D\mathbf{u}^\dagger}{Dt}$ which encloses the time-dependent term $\rho \frac{\partial \mathbf{u}^\dagger}{\partial t}$ and the convective acceleration term $\rho \mathbf{u}^\dagger . \nabla \mathbf{u}^\dagger$, the viscous force is balanced by the pressure and the body force which can be absorbed into the pressure. This gives the steady Stokes equations

$$-\frac{\partial p^s(\mathbf{x})}{\partial x_j} + \mu \frac{\partial^2 u_j^s(\mathbf{x})}{\partial x_l \partial x_l} = 0 \tag{3.2}$$

with the continuity equation

$$\nabla \cdot \mathbf{u}^s = 0. \tag{3.3}$$

Taking the divergence of (3.2) and using the continuity equation, we find that the associated pressure satisfies the Laplace equation

$$\nabla^2 p^s = 0. \tag{3.4}$$

The equations

$$
\begin{aligned}
-\nabla p^s + \mu \nabla^2 \mathbf{u}^s &= 0, \\
\nabla \cdot \mathbf{u}^s &= 0, \\
\nabla^2 p^s &= 0,
\end{aligned} \tag{3.5}
$$

represent the Steady Stokes flow. One can note that the Stokes' approximation reduces the degree of the Navier-Stokes equations (2.19) from two to one. Recalling the dimensionless Navier-Stokes equations

$$
\frac{R_\omega^2}{Re}\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}}{\partial \tilde{x}_i} = -\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}}{\partial \tilde{x}_l \partial \tilde{x}_l} \tag{3.6}
$$

And

$$
\tilde{\nabla}.\tilde{\mathbf{u}} = 0.
$$

We can see that when both Reynolds and Womersley numbers are small, $Re \ll 1$ and $R_\omega \ll 1$, the left-hand side terms are small compare to the right hand-side terms, and may be neglected. In this case, the pressure term balance with the viscous forces term which means that the Stokes equation recovers

$$
-\frac{\partial \tilde{p}}{\partial \tilde{x}_i} + \frac{\partial^2 \tilde{u}}{\partial \tilde{x}_l \partial \tilde{x}_l} = 0 \tag{3.7}
$$

3.3 Green's Surface Integral Representation

We introduce the Green's functions $(\mathbf{u}^{s(m)}, p^{s(m)})$, $m = 1,\ 2,\ 3$, for the velocity and pressure field, which with the general velocity $\mathbf{u}^s$ and pressure p^s, satisfy the surface integral representation of steady Stokes equation which we shall construct. Consider distinct Cartesian coordinates y_j and $z_j = x_j - y_j$, the coordinate $\mathbf{y}$ parameterise a point on or within a fixed closed surface and the coordinate $\mathbf{x}$ shall refer to a general fluid point. The four solutions $\mathbf{u}^s, p^s, \mathbf{u}^{s(m)}$ and $p^{s(m)}$ then satisfy the equations

$$-\frac{\partial p^s(\mathbf{y})}{\partial y_j} + \mu \frac{\partial^2 u_j^s(\mathbf{y})}{\partial y_l \partial y_l} = 0 \tag{3.8}$$

and the Green equations are

$$-\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} + \mu \frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial z_l \partial z_l} = 0. \tag{3.9}$$

Since $\mathbf{z} = \mathbf{x} - \mathbf{y}$, then the adjoint equation in $\mathbf{y}$ is satisfied as $\frac{\partial}{\partial z_j} = -\frac{\partial}{\partial y_j}$, which is

$$\frac{\partial p^{s(m)}(\mathbf{z})}{\partial y_j} + \mu \frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial y_l \partial y_l} = 0. \tag{3.10}$$

Next, following Oseen [13], we dot product (3.8) with $u_j^{s(m)}(\mathbf{z})$ and take it from the dot product of (3.10) with $u_j^s(\mathbf{y})$, to obtain the following equation

$$\frac{\partial p^{s(m)}(\mathbf{z})}{\partial y_j} u_j^s(\mathbf{y}) + \frac{\partial p^s(\mathbf{y})}{\partial y_j} u_j^{s(m)}(\mathbf{z}) + \mu \frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial y_l \partial y_l} u_j^s(\mathbf{y}) - \mu \frac{\partial^2 u_j^s(\mathbf{y})}{\partial y_l \partial y_l} u_j^{s(m)}(\mathbf{z}) = 0. \tag{3.11}$$

Applying the continuity equation $\nabla \cdot \mathbf{u}^s = 0$ gives

$$\frac{\partial}{\partial y_j}\left(p^{s(m)}(\mathbf{z}) u_j^s(\mathbf{y}) + p^s(\mathbf{y}) u_j^{s(m)}(\mathbf{z})\right) + \mu \frac{\partial}{\partial y_l}\left(u_j^s(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} - u_j^{s(m)}(\mathbf{z}) \frac{\partial u_j^s(\mathbf{y})}{\partial y_l}\right) = 0. \tag{3.12}$$

Consider a volume V of fluid bounded by the surface S enclosing the body. The divergence theorem applied to (3.12) enables us to write

$$\int\int_S [u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j - \mu u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}n_l$$
$$+\mu u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}n_l]ds = 0, \tag{3.13}$$

the equation (3.13) represents the Green's surface integral of the steady Stokes equations.

3.4 Steady Stokeslets

The Green's functions (the fundamental singular solutions) for a point force are called stokeslets, the steady stokeslets due to a point force in unbounded fluid are well known and have been in the literature such as [25] and [26]. Consider solutions of the steady Stokes equations (3.2) and the continuity equation (3.3), $\mathbf{u}^{s(m)} = (u_j^{s(m)}),\ j, m = 1,\ 2,\ 3$, where m corresponds to a stokeslet pointing in x_m direction and j corresponds to the components of velocity in the x_j direction. In this section, we obtain the steady stokeslets in terms of the potentials ϕ and χ similar to the approach used by Lamb [21] for Oseen flow, but which is also applicable for the steady Stokes flow. Then, we employ the stokeslets to represent the steady Stokes velocity. The Green equations (3.9) are satisfied by the steady stokeslets, which gives

$$-\nabla p^{s(m)} + \mu\nabla^2\mathbf{u}^{s(m)} = 0,\ \nabla\cdot\mathbf{u}^{s(m)} = 0. \tag{3.14}$$

The Lamb-Goldstein velocity decomposition suggests the form of the Green's functions is

$$u_j^{s(m)}(\mathbf{z}) = \frac{\partial\phi^{s(m)}}{\partial z_j} + \chi^s\delta_{jm} \tag{3.15}$$

where the velocity potential $\phi^{s(m)}(\mathbf{z})$ is associated with flow outside the wake and the velocity χ^s is associated with the wake velocity. As we are dealing with an incompressible fluid the potential $\phi^{s(m)}$ is a harmonic function satisfying $\nabla^2\phi^{s(m)} = 0$. Taking the divergence of the decomposition (3.15) and applying the continuity equation, we find that the potential χ^s satisfies the continuity equation; $\nabla\chi^s = 0$.

Substituting the decomposition (3.15) into the Green's steady Stokes equation (3.14), gives

$$\begin{aligned} \mu\frac{\partial^2}{\partial z_l \partial z_l}\left(\frac{\partial\phi^{s(m)}(\mathbf{z})}{\partial z_j} + \chi^s(\mathbf{z})\delta_{jm}\right) &= \frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} \\ \mu\frac{\partial}{\partial z_j}\frac{\partial^2\phi^{s(m)}(\mathbf{z})}{\partial z_l \partial z_l} + \mu\frac{\partial^2\chi^s(\mathbf{z})}{\partial z_l \partial z_l}\delta_{jm} &= \frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} \end{aligned} \tag{3.16}$$

using $\nabla^2\phi^{s(m)} = 0$ gives

$$\mu\frac{\partial^2\chi^s(\mathbf{z})}{\partial z_l \partial z_l}\delta_{jm} = \frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j}. \tag{3.17}$$

The pressure solution is given by Oseen [13] as

$$p^{s(m)}(\mathbf{z}) = \frac{1}{4\pi}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right), \tag{3.18}$$

where $R = |\mathbf{z}| = \sqrt{z_1^2 + z_2^2 + z_3^2}$ Substituting (3.18) into (3.17) and using $\frac{\partial^2}{\partial z_l \partial z_l}\delta_{jm} = \frac{\partial^2}{\partial z_m \partial z_j}$ give

$$\mu\frac{\partial^2\chi^s(\mathbf{z})}{\partial z_m \partial z_j} = \frac{1}{4\pi}\frac{\partial^2}{\partial z_m \partial z_j}\left(\frac{1}{R}\right). \tag{3.19}$$

Let

$$\chi^s(\mathbf{z}) = \frac{1}{4\pi\mu}\left(\frac{1}{R}\right). \tag{3.20}$$

Next, we turn our attention to compute the potential $\phi^{s(m)}$, which disappears from the flow equation (3.17). However, ϕ can be obtained as a harmonic function, a particular solution choice is

$$\phi^{s(m)}(\mathbf{z}) = -\frac{1}{8\pi\mu}\frac{\partial R}{\partial z_m} = -\frac{1}{8\pi\mu}\frac{z_m}{R}\,. \tag{3.21}$$

The complete steady stokeslets solutions are

$$\begin{aligned} u_j^{s(m)}(\mathbf{z}) &= \frac{\partial \phi^{s(m)}(\mathbf{z})}{\partial z_j} + \chi^s(\mathbf{z})\delta_{jm} \\ &= -\frac{1}{8\pi\mu}\frac{\partial^2 R}{\partial z_j \partial z_m} + \frac{1}{4\pi\mu}\frac{\delta_{jm}}{R}\,. \end{aligned} \tag{3.22}$$

Since $\frac{\partial^2 R}{\partial z_j \partial z_m} = \frac{\delta_{jm}}{R} - \frac{z_j z_m}{R^3}$, the stokeslets can be written as

$$\begin{aligned} u_j^{s(m)}(\mathbf{z}) &= \frac{1}{8\pi\mu}\{\frac{-\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + 2\frac{\delta_{jm}}{R}\} \\ &= \frac{1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}, \end{aligned} \tag{3.23}$$

which decay to zero at infinity and are the steady stokeslet solutions given by Oseen [13].

3.4.1 Green's Integral Representation of the Steady Stokes Velocity

The Green's surface integral representation of the steady Stokes flow has been given in (3.13) as

$$\begin{aligned} &\int\!\!\int_S \{u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j \\ &-\mu u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}n_l + \mu u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}n_l\}dS = 0. \end{aligned} \tag{3.24}$$

We consider the surface S consisting of a surface S_δ, a sphere radius $\delta \to 0$ around the point $z = 0$, a surface S_B enclosing the body, and a large spherical surface S_R of radius R extending to infinity, enclosing the body and centred at the point $\mathbf{z} = 0$, see figure (3.1).

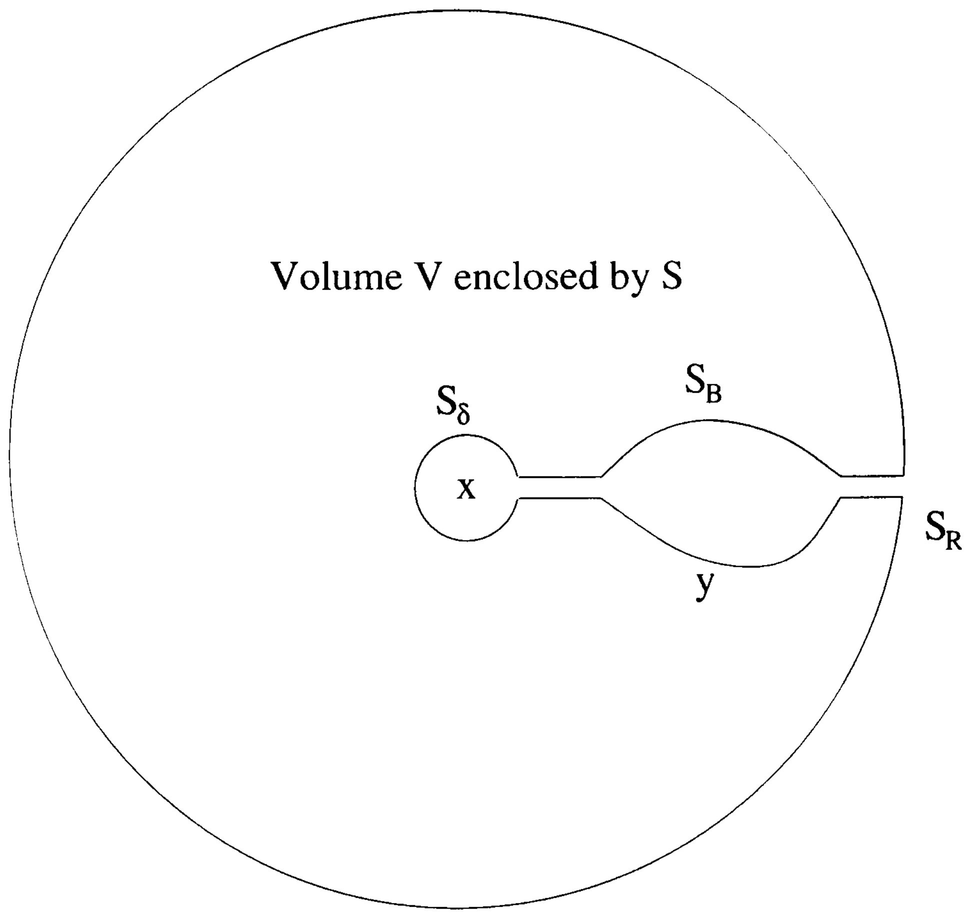

Figure 3.1: The surface S and the relation of the points **x** and **y**

We re-write the integral over the surface S as a sum of the integrals over the surfaces S_δ, S_B, and S_R,

$$\int\int_S = \int\int_{S_\delta} + \int\int_{S_B} + \int\int_{S_R} = 0. \tag{3.25}$$

Next, we calculate the contributions over the surface S_δ as $\delta \to 0$, and over S_R as $R \to \infty$, to give integral representation for the steady Stokes velocity $u_j^s(\mathbf{x})$.

The Contribution over the Surface S_δ as $\delta \to 0$.

The integral over the surface S_δ is denoted by I_{S_δ}, which is

$$\begin{aligned} I_{S_\delta} &= \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{y}) n_j dS + \int\int_{S_\delta} u_j^s(\mathbf{y}) p^{s(m)}(\mathbf{z}) n_j dS \\ &\quad -\mu \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) \frac{\partial u_j^s(\mathbf{y})}{\partial y_l} n_l dS + \mu \int\int_{S_\delta} u_j^s(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} n_l dS. \end{aligned} \tag{3.26}$$

Let us split I_{S_δ} into parts, I_1, I_2, I_3 and I_4, thus $I_{S_\delta} = I_1 + I_2 + I_3 + I_4$ and these are considered separately. This is done in order to simplify the work, where

$$\begin{aligned} I_1 &= \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{y}) n_j dS, \\ I_2 &= \int\int_{S_\delta} u_j^{s}(\mathbf{y}) p^{s(m)}(\mathbf{z}) n_j dS, \\ I_3 &= -\mu \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) \frac{\partial u_j^s(\mathbf{y})}{\partial y_l} n_l dS, \\ I_4 &= \mu \int\int_{S_\delta} u_j^{s}(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} n_l dS. \end{aligned} \tag{3.27}$$

Since $\mathbf{z} = \mathbf{x} - \mathbf{y}$, then $\mathbf{y} = \mathbf{x} - \mathbf{z}$ and $n_j = \frac{z_j}{R}$ $(R = \delta)$ points outward the control volume V, see figure (3.1).

Obtaining I_1

We can show that this integral vanishes as $\delta \to 0$,

$$I_1 = \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{y}) n_j dS = \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{x} - \mathbf{z}) n_j dS. \tag{3.28}$$

From the Taylor series

$$p^s(\mathbf{x} - \mathbf{z}) = p^s(\mathbf{x}) + \frac{\partial p^s(\mathbf{x})}{\partial z_k} z_k + O(R^2)$$

and using the steady stokeslets (3.23) the integral I_1 becomes

$$
\begin{aligned}
I_1 &= \int\int_{S_\delta} (\frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\})(p^s(\mathbf{x}) + \frac{\partial p^s(\mathbf{x})}{\partial z_k} z_k + O(R^2))\frac{z_j}{R} dS \\
&= p^s(\mathbf{x})(\frac{-1}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}\frac{z_j}{R} dS) + \frac{\partial p^s(\mathbf{x})}{\partial z_k}(\frac{-1}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}}{R} \\
&+ \frac{z_j z_m}{R^3}\}\frac{z_k z_j}{R} dS) + O(R^2) = O(R) \to 0 \qquad (3.29)
\end{aligned}
$$

where $z_m = O(R)$ and $dS = O(R^2)$.

Obtaining I_2

$$
I_2 = \int\int_{S_\delta} u_j^s(\mathbf{y}) p^{s(m)}(\mathbf{z}) n_j \, dS = \int\int_{S_\delta} u_j^s(\mathbf{x} - \mathbf{z}) p^{s(m)}(\mathbf{z}) n_j \, dS \qquad (3.30)
$$

Similarly to above

$$
\begin{aligned}
I_2 &= -\frac{1}{4\pi}\int\int_{S_\delta} (u_j^s(\mathbf{x}) + \frac{\partial u_j^s(\mathbf{x})}{\partial z_k} z_k + O(R^2))(\frac{z_m}{R^3})\frac{z_j}{R} dS \\
&= -\frac{u_j^s(\mathbf{x})}{4\pi}\int\int_{S_\delta} \frac{z_m z_j}{R^4} \, dS + O(R) \\
&= \frac{-u_j^s(\mathbf{x})}{4\pi}\frac{1}{R^4}\int\int_{S_\delta} z_j z_m \, dS + O(R) \qquad (3.31)
\end{aligned}
$$

Using $z_j = Rn_j$ and the divergence theorem we compute

$$\begin{aligned}\int\int_{S_\delta} z_j z_m dS &= R\int\int_{S_\delta} z_m\, n_j\; dS = R\int\int\int_V \frac{\partial z_m}{\partial z_j}\; dV \\ &= R\delta_{jm}\int\int\int_V dV = \frac{4\pi}{3}\delta_{jm}R^4. \end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) when $R\ (= \delta) \to 0$ we find

$$I_2 = \frac{-u_j^s(\mathbf{x})}{4\pi}\frac{1}{R^4}\frac{4\pi}{3}\delta_{jm}R^4 = \frac{-u_j^s(\mathbf{x})}{3} \tag{3.33}$$

Obtaining I_3

$$\begin{aligned} I_3 &= -\mu\int\int_{S_\delta} u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}\; n_l\; dS \\ &= -\mu\int\int_{S_\delta}\left(-\frac{1}{8\pi\mu}\right)\left\{\frac{\delta_{jm}}{R}+\frac{z_j z_m}{R^3}\right\}\frac{\partial u_j^s(\mathbf{x}-\mathbf{z})}{\partial y_l}\; n_l\; dS \\ &= \frac{1}{8\pi}\int\int_{S_\delta}\left\{\frac{\delta_{jm}}{R}+\frac{z_j z_m}{R^3}\right\}\left(\frac{\partial u_j^s(\mathbf{x})}{\partial y_l}+\frac{\partial^2 u_j^s(\mathbf{x})}{\partial y_l\partial z_k}z_k + O(R^2)\right)\frac{z_l}{R}dS \\ &= O(R) \to 0, \end{aligned} \tag{3.34}$$

as $\delta \to 0$. The Taylor series for $\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}$ around $z = 0$ and the approximation (4.59) has been used.

Obtaining I_4

In similar way to I_2, we can take $u_j^s(\mathbf{y})$ outside the integral to give

$$\begin{aligned} I_4 &= \mu \int\int_{S_\delta} u_j^s(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} \, n_l \, dS \\ &= \mu u_j^s(\mathbf{x}) \int\int_{S_\delta} \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} \, n_l \, dS. \end{aligned} \tag{3.35}$$

From (3.23) the steady Stokes solutions $u_j^{s(m)}(\mathbf{z})$, we can obtain the derivatives $\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}$, as follows since

$$u_j^{s(m)}(\mathbf{z}) = \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}. \tag{3.36}$$

Differentiating both sides with respect to y_l, gives

$$\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} = \frac{-1}{8\pi\mu}\frac{\partial}{\partial y_l}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}. \tag{3.37}$$

Using $\frac{\partial}{\partial y_l} = -\frac{\partial}{\partial z_l}$, gives

$$\begin{aligned} \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} &= \frac{1}{8\pi\mu}\frac{\partial}{\partial z_l}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\} \\ &= \frac{1}{8\pi\mu}\{\delta_{jm}\frac{\partial}{\partial z_l}(\frac{1}{R}) + \frac{\partial}{\partial z_l}(\frac{z_j z_m}{R^3})\} \\ &= \frac{1}{8\pi\mu}\{\delta_{jm}(-\frac{z_l}{R^3}) + (\frac{z_j}{R^3}\delta_{lm} + \frac{z_m}{R^3}\delta_{lj} - 3\frac{z_j z_m z_l}{R^5})\}. \end{aligned} \tag{3.38}$$

So

$$
\begin{aligned}
I_4 &= \frac{u_j^s(\mathbf{x})}{8\pi\mu}\int\int_{S_\delta}\{\delta_{jm}(-\frac{z_l}{R^3}) + (\frac{z_j}{R^3}\delta_{lm} + \frac{z_m}{R^3}\delta_{lj} - 3\frac{z_j z_m z_l}{R^5})\}\frac{z_l}{R}\,dS \\
&= \frac{u_j^s(\mathbf{x})}{8\pi\mu}\int\int_{S_\delta}\{\delta_{jm}(-\frac{z_l^2}{R^4}) + \frac{z_j z_l}{R^4}\delta_{lm} + \frac{z_m z_l}{R^4}\delta_{lj} - 3\frac{z_j z_m z_l^2}{R^6}\}\,dS \\
&= \frac{u_j^s(\mathbf{x})}{8\pi\mu}\int\int_{S_\delta}\{-\frac{\delta_{jm}}{R^2} + 2\frac{z_j z_m}{R^4} - 3\frac{z_j z_m}{R^4})\}\,dS \\
&= \frac{u_j^s(\mathbf{x})}{8\pi\mu}\int\int_{S_\delta}\{-\frac{\delta_{jm}}{R^2} - \frac{z_j z_m}{R^4}\}\,dS \qquad (3.39)
\end{aligned}
$$

this by the use of $n_l = \frac{z_l}{R}$ and $z_m = z_l\delta_{lm}$. Since $\int\int_S dS = 4\pi R^2$ and $\int\int_{S_\delta} z_j z_m dS = \frac{4\pi}{3}\delta_{jm}R^4$, the integral becomes

$$
\begin{aligned}
I_4 &= -\frac{u_j^s(\mathbf{x})}{8\pi\mu}(\frac{\delta_{jm}}{R^2}.4\pi R^2 + \frac{1}{R^4}.\frac{4\pi}{3}\delta_{jm}R^4) \\
&= -\frac{u_j^s(\mathbf{x})}{8\pi\mu}\delta_{jm}(4\pi + \frac{4\pi}{3}) = -\frac{2u_m(\mathbf{x})}{3}, \qquad (3.40)
\end{aligned}
$$

as $\delta \to 0$. From (3.29), (3.33), (3.34) and (3.40) we find that

$$
\begin{aligned}
I_{S_\delta} &= I_1 + I_2 + I_3 + I_4 = 0 - \frac{u_m^s(\mathbf{x})}{3} + 0 - \frac{2u_m^s(\mathbf{x})}{3} \\
&= -u_m^s(\mathbf{x}) \qquad (3.41)
\end{aligned}
$$

The Contribution over the Surface S_R as $R \to \infty$

Now, we determine the far field integral

$$
\begin{aligned}
I_{S_R} &= \int\int_{S_R} u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j dS + \int\int_{S_R} u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j dS \\
&\quad -\mu\int\int_{S_R} u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}n_l dS + \mu\int\int_{S_R} u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}n_l dS,
\end{aligned}
$$

(3.42)

where the surface S_R is a sphere with large radius R as shown in figure (3.1). From the definition of the steady stokeslets and their associated pressure we can see that both decay to zero at infinity,

$$u_j^{s(m)}(\mathbf{z}) = \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\} = O(R), \tag{3.43}$$

and

$$p^{s(m)}(\mathbf{z}) = \frac{1}{4\pi}(-\frac{z_m}{R^3}) = O(R^{-2}). \tag{3.44}$$

Also

$$\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} = \frac{1}{8\pi\mu}\{\delta_{jm}(-\frac{z_l}{R^3}) + (\frac{z_j}{R^3}\delta_{lm} + \frac{z_m}{R^3}\delta_{lj} - 3\frac{z_j z_m z_l}{R^5})\} = O(R^{-2}), \tag{3.45}$$

substituting the stokeslets, the pressure and (3.45) into the integral over S_R and letting R tends to infinity, lead to

$$I_{S_R} = O(R). \tag{3.46}$$

Thus the far field integral is non-zero, meaning that it is necessary to match Stokes flow to a far field Oseen flow to resolve this integral. In particular, the moment calculated in far-field Stokes flow is unbounded (Filon's paradox) [27] and is resolved by matching to a far field Oseen flow, see Imai [11]. With this matching in mind and using (3.41) and (3.46), we find

$$\begin{aligned} u_m^s(\mathbf{x}) &= \int\int_{S_B} \{u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j \\ &\quad -\mu u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_k}n_k + \mu u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_k}n_k\}dS \end{aligned} \tag{3.47}$$

which is the representation of the flow velocity in terms of the steady Stokes solutions.

3.5 Force Integral Representation in Steady Stokes Flow

The surface force on the body due to the fluid action, is given in equation (**??**) as

$$F_j^s = \int\int_{S_B} \tau_{jl}\, n_l \;\, dS, \tag{3.48}$$

where S_B is the body surface, **n** is the outward normal of S_B and τ is the stress tensor of steady Stokes flow, which is

$$\tau_{jl} = -p\delta_{jl} + \mu(\frac{\partial u_j^s}{\partial x_l} + \frac{\partial u_l^s}{\partial x_j}).$$

Applying the divergence theorem to (3.48) for the volume V which is bounded by the surfaces S_B and the far field spherical surface S_R of radius R, gives

$$F_j^s = \int\int_{S_R} \tau_{jl} n_l \;\, dS - \int\int\int_V \frac{\partial \tau_{jl}}{\partial x_l} dV, \tag{3.49}$$

where **n** is the unit normal vector pointing outside the volume V. From the steady Stokes equations (3.2) and the continuity equation (3.3) we find that

$$\begin{aligned} \frac{\partial \tau_{jl}}{\partial x_l} &= \frac{\partial}{\partial x_l}[-p^s\delta_{jl} + \mu(\frac{\partial u_j^s}{\partial x_l} + \frac{\partial u_l^s}{\partial x_j})] \\ &= -\frac{\partial p^s}{\partial x_j} + \mu(\frac{\partial^2 u_j^s}{\partial x_l \partial x_l} + \frac{\partial^2 u_l^s}{\partial x_l \partial x_j}) \\ &= -\frac{\partial p^s}{\partial x_j} + \mu\frac{\partial^2 u_j^s}{\partial x_l \partial x_l} = 0. \end{aligned} \tag{3.50}$$

Therefore, there is no contribution of the volume integral and the force can be written as the far field integral

$$F_j^s = \int\int_{S_R} \tau_{jl} n_l dS, \tag{3.51}$$

which is the same as the result given by Blake ([28], page 309) and Pozrikidis ([26], page 3).

3.5.1 Force Generated by steady Stokeslet

In this section we compute the forces generated by the steady stokeslet, using the force far field integral representation in terms of the stokeslets, as follows.

Recall the force representation (3.51), which is

$$F_j^s = \int\int_{S_R} \tau_{jl}\, n_l\, dS. \tag{3.52}$$

Substituting the steady stokeslets $u_j^{s(m)}$ gives

$$F_j^{s(m)} = \int\int_{S_R} \tau_{jl}^{s(m)} n_l\, dS, \tag{3.53}$$

$F_j^{s(m)}$ denotes the force generated by the steady stokeslets, where

$$\begin{aligned} \tau_{jl}^{s(m)} &= -p^{s(m)}\delta_{jl} + \mu\left(\frac{\partial u_j^{s(m)}}{\partial z_l} + \frac{\partial u_l^{s(m)}}{\partial z_j}\right), \\ u_j^{s(m)} &= \frac{-1}{8\pi\mu}\left\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\right\}, \\ p^{s(m)} &= \frac{-1}{4\pi}\left(\frac{z_m}{R}\right). \end{aligned} \tag{3.54}$$

To work out the force we expand the derivatives,

$$\begin{aligned}\frac{\partial u_j^{s(m)}}{\partial z_l} &= \frac{-1}{8\pi\mu}\{\delta_{jm}(\frac{-z_l}{R^3}) + \delta_{jl}\frac{z_m}{R^3} + \delta_{lm}\frac{z_j}{R^3} - 3\frac{z_j z_m z_l}{R^5}\} \\ \frac{\partial u_l^{s(m)}}{\partial z_j} &= \frac{-1}{8\pi\mu}\{\delta_{jm}(\frac{-z_j}{R^3}) + \delta_{jl}\frac{z_m}{R^3} + \delta_{jm}\frac{z_l}{R^3} - 3\frac{z_j z_m z_l}{R^5}\},\end{aligned} \tag{3.55}$$

so

$$\frac{\partial u_j^{s(m)}}{\partial z_l} + \frac{\partial u_l^{s(m)}}{\partial z_j} = \frac{-1}{8\pi\mu}\{2\delta_{jl}\frac{z_m}{R^3} - 6\frac{z_j z_m z_l}{R^5}\}. \tag{3.56}$$

The stress tensor in terms of stokeslets is

$$\begin{aligned}\tau_{jl}^{s(m)} &= -p^{s(m)}\delta_{jl} + \mu(\frac{\partial u_j^{s(m)}}{\partial z_l} + \frac{\partial u_l^{s(m)}}{\partial z_j}), \\ &= \frac{1}{4\pi}\delta_{jl}(\frac{z_m}{R}) - \frac{1}{8\pi\mu}\{2\delta_{jl}\frac{z_m}{R^3} - 6\frac{z_j z_m z_l}{R^5}\} \\ &= \frac{3}{4\pi}(\frac{z_j z_m z_l}{R^5}).\end{aligned} \tag{3.57}$$

Therefore, the forces are

$$\begin{aligned}F_j^{s(m)} &= \frac{3}{4\pi}\int\int_{S_R} \frac{z_j z_m z_l}{R^5}\, n_l\, dS = \frac{3}{4\pi}\int\int_{S_R} \frac{z_j z_m z_l^2}{R^6}\, dS \\ &= \frac{3}{4\pi R^4}\int\int_{S_R} z_j z_m\, dS.\end{aligned} \tag{3.58}$$

From (3.32)

$$F_j^{s(m)} = \frac{3}{4\pi R^4}\,\frac{4\pi}{3}\delta_{jm}R^4 = \delta_{jm}. \tag{3.59}$$

So the m-stokeslet gives a unit force in the m-direction.

3.6 Conclusion

The steady stokeslets are obtained using potentials, which are similar to known solutions for the steady Stokes flow. This is the first time in literature to represent the stokeslets in terms of potentials and results which are given in this chapter are similar to the existing results using the singular method. The Green integral representation of the flow is given and we show that the steady stokeslet generate a unit force in the direction of the point force. In chapter six we will show that at a particular limit the oscillatory oseenlet reduces to the steady stokeslet.

Chapter 4

Oscillatory Stokes Flow

In this chapter we consider the oscillatory Stokes flow. The Oscillatory stokeslets are first given using the singularity method by Pozrikidis [9], we obtain the oscillatory stokeslet in terms of potentials, using a similar approach which we used in chapter 2 for the steady stokeslets. The potentials representation will enable us later to show that the oscillatory stokeslets can be recovered from the oscillatory oseenlet at a particular limit. After introducing the oscillatory flow in section one, we give the Green's surface integral representation of the flow in the section two. In section three, we construct the oscillatory stokeslets using potentials representation, and then we show that they are similar to the oscillatory stokeslets given by Pozrikidis. Also, in section three we establish the behaviour of the flow in the far field and at high frequencies. The representation of the flow velocity in terms of the oscillatory Stokes solutions which requires to know the behaviour of the stokeslets close to the point force, is given in section four. In section five, we present the force integral representation and the force generated by the oscillatory stokeslets.

4.1 The Governing Equations

The time-dependent incompressible Navier Stokes equations are given by

$$\rho\frac{\partial u_j^\dagger(\mathbf{x})}{\partial t}+\rho u_l^\dagger(\mathbf{x})\frac{\partial u_j^\dagger(\mathbf{x})}{\partial x_l}=-\frac{\partial p^\dagger(\mathbf{x})}{\partial x_j}+\mu\frac{\partial^2 u_j^\dagger(\mathbf{x})}{\partial x_l\partial x_l} \tag{4.1}$$

where: $u_j^\dagger$ is the velocity component in the j direction of a Cartesian coordinate system x_j and $j,\ l=1,2,3$; $p^\dagger$ is the fluid pressure; t denotes time; ρ is the fluid density; and μ is the fluid viscosity.

In the near-field the Stokes approximation is valid, the inertial convective term on the left-hand side of (4.1) is small compared with the rest of the terms and thus may be neglected. The flow is governed by linearised Navier-Stokes equations

$$\rho\frac{\partial u_j^\dagger(\mathbf{x})}{\partial t}=-\frac{\partial p^\dagger(\mathbf{x})}{\partial x_j}+\mu\frac{\partial^2 u_j^\dagger(\mathbf{x})}{\partial x_l\partial x_l}. \tag{4.2}$$

We consider linearised oscillatory flow. Thus we seek time-periodic solutions of the form

$$\begin{aligned} u_j^\dagger(\mathbf{x}) &= \sum_{n=-\infty}^{\infty} u_j^{sn}(\mathbf{x})e^{i\omega_n t} \\ p^\dagger(\mathbf{x}) &= \sum_{n=-\infty}^{\infty} p^{sn}(\mathbf{x})e^{i\omega_n t} \end{aligned} \tag{4.3}$$

where i is the imaginary number $\sqrt{-1}$, $\omega_n=\frac{2n\pi}{T}$ and T is the time period of the motion. Since the left hand side of (4.3) represents real variables, then $u_j^{sn}=\bar{u}_j^{s(-n)}$ and $p^{sn}=\bar{p}^{s(-n)}$ where the bar denotes the complex conjugate and the superscripts s denotes the

Stokes solutions. Substituting (4.3) into (4.2) gives for each n

$$i\rho\omega_n u_j^{sn}(\mathbf{x}) = -\frac{\partial p^{sn}(\mathbf{x})}{\partial x_j} + \mu\frac{\partial^2 u_j^{sn}(\mathbf{x})}{\partial x_l \partial x_l}. \tag{4.4}$$

For simplicity we omit n from equation 4.5 and subsequent equations, and becomes

$$i\rho\omega u_j^{s}(\mathbf{x}) = -\frac{\partial p^{s}(\mathbf{x})}{\partial x_j} + \mu\frac{\partial^2 u_j^{s}(\mathbf{x})}{\partial x_l \partial x_l}. \tag{4.5}$$

This equation represent the oscillatory Stokes flow. Also one can show from the time-periodic representation (4.3) that the Stokes velocity satisfies the continuity equation $\nabla \cdot \mathbf{u}^s = 0$. Taking the divergence of the oscillatory Stokes equation (4.2) gives the Stokes pressure to satisfy the Laplace equation

$$\nabla^2 p^s = 0. \tag{4.6}$$

4.2 Green's Surface Integral Representation

In this section we give the Green's surface integral representation of the oscillatory Stokes equations, following the Green's integral formulation as given by Oseen [13], except apply it to the oscillatory rather than the steady or transient case. Consider four solutions for the velocity and pressure field given by $u_j^s(\mathbf{y})$, $p^s(\mathbf{y})$ and $u_j^{s(m)}(\mathbf{z})$, $p^{s(m)}(\mathbf{z})$ where $1 \leq m \leq 3$. The first solution refers to a general velocity and pressure field, and the subsequent three solutions refer to the specific Green's functions satisfying a Green's integral which we shall construct.

We consider distinct Cartesian coordinates y_j and $z_j = x_j - y_j$, the coordinate $\mathbf{y}$ parameterises a point on or within a fixed closed surface and the coordinate **x** shall refer to a general fluid point. The four fluid solutions then satisfy the equations

$$i\rho\omega u_j^s(\mathbf{y}) = -\frac{\partial p^s(\mathbf{y})}{\partial y_j} + \mu\frac{\partial^2 u_j^s(\mathbf{y})}{\partial y_l \partial y_l}, \tag{4.7}$$

and

$$i\rho\omega u_j^{s(m)}(\mathbf{z}) = -\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} + \mu\frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial z_l \partial z_l}. \tag{4.8}$$

Since $\mathbf{z} = \mathbf{x} - \mathbf{y}$, then the adjoint equation in **y** is satisfied as $\frac{\partial}{\partial z_j} = -\frac{\partial}{\partial y_j}$, which gives

$$i\rho\omega u_j^{s(m)}(\mathbf{z}) = \frac{\partial p^{s(m)}(\mathbf{z})}{\partial y_j} + \mu\frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial y_l \partial y_l}. \tag{4.9}$$

Following the method of Oseen [13] to get Oseen's Greens function representation, we dot product (4.7) with $u_j^{s(m)}$ and take it from the dot product of (4.9) with u_j^s.

$$\begin{aligned} - & \left[u_j^{s(m)}(\mathbf{z})\frac{\partial p^s(\mathbf{z})}{\partial y_j} + u_j^s(\mathbf{y})\frac{\partial p^{s(m)}(\mathbf{z})}{\partial y_j}\right] + \mu\left[u_j^{s(m)}(\mathbf{z})\frac{\partial^2 u_j^s(\mathbf{y})}{\partial y_l \partial y_l}\right. \\ & \left. -u_j^s(\mathbf{y})\frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial y_l \partial y_l}\right] = 0. \end{aligned} \tag{4.10}$$

Applying the continuity equation $\frac{\partial u_j^s}{\partial y_j} = 0$ then gives

$$\begin{aligned} - & \frac{\partial}{\partial y_j}\left[u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y}) + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})\right] \\ & +\mu\frac{\partial}{\partial y_l}\left[u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l} - u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}\right] = 0. \end{aligned} \tag{4.11}$$

This holds within a volume V of fluid bounded by the surface S where the Stokes approximation is valid, and parameterised by the coordinate system y. Then, applying the divergence theorem gives

$$\int\int_S \{u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j$$
$$-\mu u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}n_l + \mu u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}n_l\}dS = 0, \tag{4.12}$$

where $\mathbf{n}$ is the unit vector normal to the surface S pointing outwards from the control volume V. In this way, we have obtained the Green's surface integral representation of the time-harmonic Stokes equation, which is identical to the Green's representation for the steady Stokes flow (3.13), as the oscillatory part has canceled from the governing differential equation and we note that this can be obtained from the Green's surface representation of time-harmonic Oseen equations (which is identical to the steady Oseen representation), by letting the forward uniform flow U be zero.

4.3 Oscillatory Stokeslets

The form of the oscillatory stokeslets are given by Pozrikidis [9], but we need them here in terms of the potentials ϕ and χ which are the potentials introduced by Lamb [21] to represent the Oseen equations. This representation then enables us to infer the form of the oscillatory stokeslests from the oscillatory oseenlets. The oscillatory stokeslets must satisfy (4.8) for the oscillatory Stokes equation, which is

$$\rho i\omega u_j^{s(m)}(\mathbf{z}) = -\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} + \mu \frac{\partial^2 u_j^{s(m)}(\mathbf{z})}{\partial z_l \partial z_l}. \tag{4.13}$$

The velocity decomposition

$$u_j^{s(m)}(\mathbf{z}) = \frac{\partial \phi^{s(m)}(\mathbf{z})}{\partial z_j} + w_j^{s(m)}(\mathbf{z}) \tag{4.14}$$

is assumed for the stokeslets, which is the Lamb-Goldstein velocity decomposition used in Oseen flow [21], [29]. Where the velocity potential $\phi^{s(m)}(\mathbf{z})$ is associated with flow outside the wake and the velocity $w_j^{s(m)}(\mathbf{z})$ is associated with the wake velocity. But we shall use the Lamb-Goldstein velocity decomposition here as it is equally applicable for Stokes flow. The potential $\phi^{s(m)}$ has to satisfy Laplace equation as we consider viscous incompressible flow, so

$$\nabla^2 \phi^{s(m)} = 0 \tag{4.15}$$

and this potential is associated with flow outside the wake. In contrast, the velocity $w_j^{s(m)}$ is associated with the wake velocity. Taking the divergence of (4.14) and using $\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial z_j} = 0$ and $\frac{\partial^2 \phi^{s(m)}(\mathbf{z})}{\partial z_j \partial z_j} = 0$, shows that the wake velocity satisfies the continuity equation

$$\frac{\partial w_j^{s(m)}(\mathbf{z})}{\partial z_j} = 0. \tag{4.16}$$

Substituting the decomposition (4.14) into (4.13) gives

$$\rho i\omega\frac{\partial\phi^{s(m)}(\mathbf{z})}{\partial z_j}+\rho i\omega w_j^{s(m)}(\mathbf{z})=-\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j}+\mu\frac{\partial^2}{\partial z_l\partial z_l}(\frac{\partial\phi^{s(m)}(\mathbf{z})}{\partial z_j})+\mu\frac{\partial^2 w_j^{s(m)}(\mathbf{z})}{\partial z_l\partial z_l}. \tag{4.17}$$

From (4.15), $\frac{\partial^2\phi^{s(m)}(\mathbf{z})}{\partial z_l\partial z_l}=0$, then

$$\rho i\omega\frac{\partial\phi^{s(m)}(\mathbf{z})}{\partial z_j}+\rho i\omega w_j^{s(m)}(\mathbf{z})-\mu\frac{\partial^2 w_j^{s(m)}(\mathbf{z})}{\partial z_l\partial z_l}=-\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j}. \tag{4.18}$$

In section 4.1 we have shown that the oscillatory Stokes pressure satisfies the Laplace equation (4.6). Furthermore, in the steady case limit $\omega\longrightarrow 0$, then the solution for the oscillatory stokeslet must tend to the solution for the steady stokeslet. The steady Stokes pressure also satisfies the Laplace equation. So, the solution for the pressure for both steady and the amplitude oscillatory stokeslets must be the same. The steady stokeslet solution for pressure is given by Oseen [13], consequently, this is also the oscillatory stokeslet solution for pressure given by

$$p^{s(m)}(\mathbf{z})=\frac{1}{4\pi}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right) \tag{4.19}$$

where the radial distance from the stokeslet singularity is given by $R=|\mathbf{z}|$.

Now, since both the pressure $p^{s(m)}(\mathbf{z})$ and the potential $\phi^{s(m)}(\mathbf{z})$ are harmonic functions, the pressure term can be removed from the oscillatory Stokes equation (4.18) for the stokeslet by the particular choice

$$i\rho\omega\frac{\partial\phi^{s(m)}(\mathbf{z})}{\partial z_j}=-\frac{\partial p^{s(m)}(\mathbf{z})}{\partial z_j} \tag{4.20}$$

or

$$i\rho\omega\phi^{s(m)}(\mathbf{z}) = -p^{s(m)}(\mathbf{z}) \tag{4.21}$$

which gives

$$\phi^{s(m)}(\mathbf{z}) = \frac{i}{\rho\omega}p^{s(m)}(\mathbf{z}) = \frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right). \tag{4.22}$$

Substituting (4.20) into (4.18) implies that

$$\frac{\partial^2 w_j^{s(m)}(\mathbf{z})}{\partial z_l \partial z_l} - h^2 w_j^{s(m)}(\mathbf{z}) = 0 \tag{4.23}$$

where $h = \sqrt{\frac{i\rho\omega}{\mu}}$. The equation (4.23) can be re-written in operator form as

$$(\nabla^2 - h^2)w_j^{s(m)}(\mathbf{z}) = 0. \tag{4.24}$$

Following Lamb's approach for decomposing the Oseen equation, we introduce the potentials $\chi^{s(m)}$ and χ^{s*} such that

$$w_j^{s(m)}(\mathbf{z}) = \frac{\partial \chi^{s(m)}(\mathbf{z})}{\partial z_j} + \chi^{s*}(\mathbf{z})\delta_{jm}. \tag{4.25}$$

In order to obtain the steady results, we must choose $\chi^{s(m)}$ such that in the steady limit $\omega \longrightarrow 0$ the oscillatory dependence disappears hence this term cancels with the potential term $\phi^{s(m)}$. Therefore we must have

$$\chi^{s(m)}(\mathbf{z}) = -\frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right), \tag{4.26}$$

and consequently

$$\chi^{s*}(\mathbf{z}) = \frac{ih^2}{4\pi\rho\omega}\left(\frac{e^{-hR}}{R}\right). \tag{4.27}$$

Therefore, the oscillatory stokeslet solutions are

$$\begin{aligned} u_j^{s(m)}(\mathbf{z}) = {} & \tfrac{i}{4\pi\rho\omega}\tfrac{\partial^2}{\partial z_j \partial z_m}\left(\tfrac{1}{R}\right) - \tfrac{i}{4\pi\rho\omega}\tfrac{\partial^2}{\partial z_j \partial z_m}\left(\tfrac{e^{-hR}}{R}\right) \\ & + \tfrac{ih^2}{4\pi\rho\omega}\tfrac{e^{-hR}}{R}\delta_{jm}. \end{aligned} \tag{4.28}$$

4.3.1 Pozrikidis' form of Oscillatory stokeslets

In this section we show that the oscillatory stokeslets (4.28) are similar to the oscillatory stokeslets given by Pozrikidis in [9]. Expanding the partial derivatives in (4.28) and collecting like terms produces Pozrikidis' form, as follows. First expand the partial derivatives, such that

$$\begin{aligned} \frac{\partial^2}{\partial z_j \partial z_m}\left(\frac{1}{R}\right) &= -\frac{\delta_{jm}}{R^3} + 3\frac{z_m z_j}{R^5} \\ &= \frac{\delta_{jm}}{R}\left(\frac{-1}{R^2}\right) + \frac{z_j z_m}{R^3}\left(\frac{3}{R^2}\right) \end{aligned} \tag{4.29}$$

and

$$\frac{\partial^2}{\partial z_j \partial z_m}\left(\frac{e^{-hR}}{R}\right) = -h\frac{\delta_{jm}}{R^2}e^{-hR} + h^2\frac{z_m z_j}{R^3} + 2h\frac{z_j z_m}{R^4}e^{-hR}$$
$$-\frac{\delta_{jm}}{R^3}e^{-hR} - h\frac{z_m z_j}{R^4}e^{-hR} + 3\frac{z_m z_j}{R^5}e^{-hR}$$
$$= \frac{\delta_{jm}}{R}\left(-\frac{h}{R} - \frac{1}{R^2}\right)e^{-hR} + \frac{z_j z_m}{R^3}\left(h^2 + \frac{3h}{R} + \frac{3}{R^2}\right)e^{-hR}. \tag{4.30}$$

The coefficient $\frac{i}{4\pi\rho\omega}$ can be written as

$$\frac{1}{8\pi\mu}.\frac{-2}{h^2} \tag{4.31}$$

where $h^2 = \frac{i\rho\omega}{\mu}$. Working out the terms in the oscillatory stokeslets (4.28), gives the first term

$$\frac{i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j \partial z_m}\left(\frac{1}{R}\right) = \frac{1}{8\pi\mu}\ \frac{-2}{h^2}\{\frac{\delta_{jm}}{R}\left(\frac{-1}{R^2}\right) + \frac{z_j z_m}{R^3}\left(\frac{3}{R^2}\right)\}$$
$$= \frac{1}{8\pi\mu}\{\frac{\delta_{jm}}{R}\left(\frac{2}{h^2R^2}\right) + \frac{z_j z_m}{R^3}\left(\frac{-6}{h^2R^2}\right)\}, \tag{4.32}$$

the second term

$$\frac{-i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j \partial z_m}\left(\frac{e^{-hR}}{R}\right) = \frac{1}{8\pi\mu}.\frac{2}{h^2}[\frac{\delta_{jm}}{R}\left(-\frac{h}{R} - \frac{1}{R^2}\right)e^{-hR}$$
$$+\frac{z_j z_m}{R^3}\left(h^2 + \frac{3h}{R} + \frac{3}{R^2}\right)e^{-hR}]$$
$$= \frac{1}{8\pi\mu}[\frac{\delta_{jm}}{R}\left(-\frac{2}{hR} - \frac{2}{h^2R^2}\right)e^{-hR}$$
$$+\frac{z_j z_m}{R^3}\left(2 + \frac{6}{hR} + \frac{6}{h^2R^2}\right)e^{-hR}], \tag{4.33}$$

and the third term

$$\frac{ih^2}{4\pi\rho\omega}\left(\frac{e^{-hR}}{R}\right)\delta_{jm} = \frac{h^2}{8\pi\mu}\frac{-2}{h^2}\left(\frac{e^{-hR}}{R}\right)\delta_{jm} = \frac{1}{8\pi\mu}\left(-2e^{-hR}\right)\frac{\delta_{jm}}{R}. \quad (4.34)$$

Substituting (4.32), (4.33), and (4.34) into the oscillatory stokeslets 4.28, gives

$$\begin{aligned} u_j^{s(m)}(\mathbf{z}) &= \frac{1}{8\pi\mu}\{-\frac{\delta_{jm}}{R}\left[\frac{-2}{h^2R^2} + 2e^{-hR}\left(1+\frac{1}{hR}+\frac{1}{h^2R^2}\right)\right] \\ &\quad -\frac{z_j z_m}{R^3}\left[\frac{6}{h^2R^2} - 2e^{-hR}\left(1+\frac{3}{hR}+\frac{3}{h^2R^2}\right)\right]\} \end{aligned} \quad (4.35)$$

which can be written as

$$u_j^{s(m)}(\mathbf{z}) = \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R}A(r) + \frac{z_j z_m}{R^3}B(r)\} \quad (4.36)$$

which is the form given by Pozrikidis [9], where $r = hR$ and the functions $A(r)$ and $B(r)$ are defined as

$$A(r) = 2e^{-r}\left(1+\frac{1}{r}+\frac{1}{r^2}\right) - \frac{2}{r^2} \quad (4.37)$$

$$B(r) = -2e^{-r}\left(1+\frac{3}{r}+\frac{3}{r^2}\right) + \frac{6}{r^2}\,. \quad (4.38)$$

However, the form we presented here is more beneficial for us, as we seek later to use oscillatory stokeslets in order to infer the form of the oscillatory oseenlets in terms of the potentials $\phi^{s(m)}$, $\chi^{s(m)}$ and χ^{s*}.

It is also noted that letting $h \to 0$ or $R \to 0$ in (4.36), the steady stokeslet solution is recovered, as $A(0) = B(0) = 1$.

4.3.2 Stokeslets in far field and at high frequencies

To establish the behaviour of the flow in the far field and at high frequencies, we expand the oscillatory stokeslet $u_j^{s(m)}$ in an asymptotic series for large R and for large ω, obtaining the result given in [9],[25]. In order to obtain the asymptotic series we expand the oscillatory stokeslet in a Taylor series for small $\frac{1}{R}$ in the far field and for small $\frac{1}{h}$ at high frequencies.

For simplicity, we start with the equation (4.35)

$$\begin{aligned} u_j^{s(m)}(\mathbf{z}) &= \frac{1}{8\pi\mu}\{-\frac{\delta_{jm}}{R}\left[\frac{-2}{h^2R^2} + 2e^{-hR}\left(1 + \frac{1}{hR} + \frac{1}{h^2R^2}\right)\right] \\ &\quad -\frac{z_j z_m}{R^3}\left[\frac{6}{h^2R^2} - 2e^{-hR}\left(1 + \frac{3}{hR} + \frac{3}{h^2R^2}\right)\right]\} \end{aligned} \tag{4.39}$$

where R is a radius of a large sphere encloses the oscillating body and $h = \sqrt{\frac{i\omega\rho}{\mu}}$.

The asymptotic series for large R

Expanding the oscillatory stokeslets in a Taylor series for small $R^* = \frac{1}{R}$, gives

$$u_j^{s(m)}(\mathbf{z}) = u_j^{s(m)}|_{R^*=0} + \frac{\partial u_j^{s(m)}}{\partial R^*}R^* + \frac{\partial^2 u_j^{s(m)}}{\partial R^* \partial R^*}R^{*2} + ... \tag{4.40}$$

where

$$u_j^{s(m)}(R^*) = -\frac{1}{8\pi\mu}[\delta_{jm}\{-2\frac{R^{*3}}{h^2} + 2e^{-h/R^*}(R^* + \frac{R^{*2}}{h} + \frac{R^{*3}}{h^2})\} + z_m z_j\{6\frac{R^{*5}}{h^2} - 2e^{-h/R^*}(R^{*3} + 3\frac{R^{*4}}{h} + 3\frac{R^{*5}}{h^2})\}]. \tag{4.41}$$

The derivatives of the oscillatory stokeslet are

$$\frac{\partial u_j^{s(m)}}{\partial R^*} = -\frac{1}{8\pi\mu}[\delta_{jm}\{-6\frac{R^{*2}}{h^2} + 2e^{-h/R^*}(2 + \frac{h}{R^*} + 3\frac{R^*}{h} + 3\frac{R^{*2}}{h^2})\} + z_j z_m\{30\frac{R^{*4}}{h^2} - 2e^{-h/R^*}(hR^* + 6R^{*2} + 15\frac{R^{*3}}{h} + 15\frac{R^{*4}}{h^2})\}] \tag{4.42}$$

then the second term becomes

$$\frac{\partial u_j^{s(m)}}{\partial R^*}R^* = -\frac{1}{8\pi\mu}[\delta_{jm}\{-6\frac{R^{*3}}{h^2} + 2e^{-h/R^*}(2R^* + h + 3\frac{R^{*2}}{h} + 3\frac{R^{*3}}{h^2})\} + z_j z_m\{30\frac{R^{*5}}{h^2} - 2e^{-h/R^{*2}}(hR^* + 6R^{*3} + 15\frac{R^{*4}}{h} + 15\frac{R^{*5}}{h^2})\}] \tag{4.43}$$

and the second derivatives is

$$\frac{\partial^2 u_j^{s(m)}}{\partial R^* \partial R^*} = -\frac{1}{8\pi\mu}[\delta_{jm}\{-12\frac{R^*}{h^2} + 2e^{-h/R^*}(\frac{h}{R^{*2}} + \frac{6}{h} + \frac{3}{R^*} + \frac{h^2}{R^{*3}} + 6\frac{R^*}{h^2})\} + z_j z_m\{120\frac{R^{*3}}{h2} - 2e^{-h/R^*}(h^2 + 8hR^* + 32R^{*2} + 75\frac{R^{*3}}{h} + 75\frac{R^{*4}}{h^2})\}] \tag{4.44}$$

then the third term becomes

$$\begin{aligned}\frac{\partial^2 u_j^{s(m)}}{\partial R^* \partial R^*} R^* &= -\frac{1}{8\pi\mu}\big[\delta_{jm}\{-12\frac{R^{*3}}{h^2} + 2e^{-h/R^*}(h + 6\frac{R^{*2}}{h} + 3R^* + \frac{h^2}{R^*} + 6\frac{R^{*3}}{h^2})\} \\ &+ z_j z_m\{120\frac{R^{*5}}{h2} - 2e^{-h/R^*}(h^2R^{*2} + 8hR^{*3} + 32R^{*4} + 75\frac{R^{*5}}{h} + 75\frac{R^{*6}}{h^2})\}\big].\end{aligned} \tag{4.45}$$

Therefore the Taylor series around $R^* = 0$ is

$$u_j^{s(m)} = -\frac{1}{8\pi\mu}\big[\frac{2}{h^2}(-\delta_{jm}R^{*3} + 3z_j z_m R^{*5}) + ...\big] \tag{4.46}$$

writing R^* in terms of R we obtain the asymptotic series of the oscillatory stokeslets in the far field as

$$u_j^{s(m)} = -\frac{1}{8\pi\mu}\big[\frac{2}{h^2}(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5}) + ...\big] \tag{4.47}$$

which is the same as the series given by Pozrikidis in [9] and [25].

The asymptotic series for large frequencies

Similarly, we expand the oscillatory stokeslets in the Taylor series for small $h^* = \frac{1}{h}$, when h is large

$$u_j^{s(m)}(\mathbf{z}) = u_j^{s(m)}|_{h^*=0} + \frac{\partial u_j^{s(m)}}{\partial h^*}h^* + \frac{\partial^2 u_j^{s(m)}}{\partial h^* \partial h^*}h^{*2} + ... \tag{4.48}$$

where $u_j^{s(m)}$ can be written in terms of h^* as

$$\begin{aligned}u_j^{s(m)} &= -\frac{1}{8\pi\mu}\big[\frac{\delta_{jm}}{R}\{-2\frac{h^{*2}}{R^2} + 2e^{-R/h^*}(1 + \frac{h^*}{R} + \frac{h^{*2}}{R^2})\} \\ &+ \frac{z_j z_m}{R^3}\{6\frac{h^{*2}}{R^2} - 2e^{-R/h^*}(1 + 3\frac{h^*}{R} + 3\frac{h^{*2}}{R^2})\}\big]\end{aligned} \tag{4.49}$$

Obtaining the derivatives and the series terms

$$\frac{\partial u_j^{s(m)}}{\partial h^*} = -\frac{1}{8\pi\mu}\left[\frac{\delta_{jm}}{R}\{-4\frac{h^*}{R^2} + 2e^{-R/h^*}(\frac{R}{h^{*2}} + \frac{1}{h^*} + \frac{2}{R} + 2\frac{h^*}{R^2})\}\right.$$
$$\left. + \frac{z_j z_m}{R^3}\{12\frac{h^*}{R^2} - 2e^{-R/h^*}(\frac{R}{h^{*2}} + \frac{3}{h^*} + \frac{6}{R} + 6\frac{h^*}{R^2})\}\right], \quad (4.50)$$

the first term becomes

$$\frac{\partial u_j^{s(m)}}{\partial h^*}h^* = -\frac{1}{8\pi\mu}\left[\frac{\delta_{jm}}{R}\{-4\frac{h^{*2}}{R^2} + 2e^{-R/h^*}(\frac{R}{h^*} + 1 + 2\frac{h^*}{R} + 2\frac{h^{*2}}{R^2})\}\right.$$
$$\left. + \frac{z_j z_m}{R^3}\{12\frac{h^{*2}}{R^2} - 2e^{-R/h^*}(\frac{R}{h^*} + 3 + 6\frac{h^*}{R} + 6\frac{h^{*2}}{R^2})\}\right], \quad (4.51)$$

the second derivative

$$\frac{\partial^2 u_j^{s(m)}}{\partial h^* \partial h^*} = -\frac{1}{8\pi\mu}\left[\frac{\delta_{jm}}{R}\{-\frac{4}{R^2} + 2e^{-R/h^*}(\frac{R^2}{h^{*4}} - \frac{R}{h^{*3}} + \frac{1}{h^{*2}} + \frac{2}{Rh^*} + \frac{2}{R^2})\}\right.$$
$$\left. \frac{z_j z_m}{R^3}\{\frac{12}{R^2} - 2e^{-R/h^*}(\frac{R^2}{h^{*4}} + \frac{R}{h^{*3}} + \frac{3}{h^{*2}} + \frac{6}{Rh^*} + \frac{6}{R^2})\}\right], \quad (4.52)$$

and the second term is

$$\frac{\partial^2 u_j^{s(m)}}{\partial h^* \partial h^*}h^{*2} = -\frac{1}{8\pi\mu}\left[\frac{\delta_{jm}}{R}\{-4\frac{h^{*2}}{R^2} + 2e^{-R/h^*}(\frac{R^2}{h^{*2}} - \frac{R}{h^*} + 1 + 2\frac{h^*}{R} + 2\frac{h^{*2}}{R^2})\}\right.$$
$$\left. \frac{z_j z_m}{R^3}\{12\frac{h^{*2}}{R^2} - 2e^{-R/h^*}(\frac{R^2}{h^{*2}} + \frac{R}{h^*} + 3 + 6\frac{h^*}{R} + 6\frac{h^{*2}}{R^2})\}\right]. \quad (4.53)$$

Then the Taylor series around $h^* = 0$ is

$$u_j^{s(m)} = -\frac{1}{8\pi\mu}\left[2h^{*2}(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5}) + ...\right] \quad (4.54)$$

letting $h^* = 1/h$ gives the asymptotic series

$$u_j^{s(m)} = -\frac{1}{8\pi\mu}\left[\frac{2}{h^2}\left(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5}\right) + ...\right] \tag{4.55}$$

From the series (4.47) and (4.55) we can see that the flow behaves in the same way both at high frequencies and away from the point force. The first term on the right side of the series (4.55) is known as the steady potential dipole, which means that at both the high frequency or large distance, the oscillatory stokeslets produce irrotational flow, see [9], [25].

4.4 The Integral Representation of the Oscillatory Stokes Velocity

In order to obtaining the integral representation of the time-harmonic Stokes velocity, we first approximate the oscillatory stokeslets around the point $z = 0$, to enable us to obtain the representation.

4.4.1 The Approximate formula of Oscillatory Stokeslets near $R = 0$

Recall the oscillatory stokeslets (4.28)

$$\begin{aligned} u_j^{(m)}(\mathbf{z}) &= \frac{i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j z_m}\left(\frac{1}{R}\right) - \frac{i}{4\pi\rho\omega}\frac{\partial^2}{\partial z_j z_m}\left(\frac{e^{-hR}}{R}\right) \\ &+ \frac{ih^2}{4\pi\rho\omega}\frac{e^{-hR}}{R}\delta_{jm}. \end{aligned} \tag{4.56}$$

Considering S_δ as a sphere radius δ around the point $z = 0$. We approximate $u_j^{s(m)}$ around the point $z = 0$ as $\delta \to 0$ $(R \to 0)$, so $R^n \ll 1,\ n > 2$. From the Taylor series

$$
\begin{aligned}
e^{-hR} &= 1 - hR + \frac{h^2R^2}{2!} - \frac{h^3R^3}{3!} + ... \\
&= 1 - hR + \frac{h^2R^2}{2} + O(h^3R^3).
\end{aligned} \tag{4.57}
$$

Substituting (4.57) into (4.28), gives

$$
\begin{aligned}
u_j^{s(m)}(\mathbf{z}) &\approx \frac{i}{4\pi\rho\omega}\{\frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R}) - \frac{\partial^2}{\partial z_j \partial z_m}(\frac{1 - hR + \frac{h^2R^2}{2}}{R}) + h^2(\frac{1 - hR + \frac{h^2R^2}{2}}{R})\delta_{jm}\} \\
&\approx \frac{i}{4\pi\rho\omega}\{\frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R}) - \frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R} - h + \frac{h^2R}{2}) + h^2(\frac{1}{R} - h + \frac{h^2R}{2})\delta_{jm}\} \\
&\approx \frac{i}{4\pi\rho\omega}\{-\frac{\partial^2}{\partial z_j \partial z_m}(\frac{h^2R}{2}) + h^2(\frac{1}{R} - h + \frac{h^2R}{2})\delta_{jm}\}
\end{aligned} \tag{4.58}
$$

$$
\begin{aligned}
&\approx \frac{-1}{8\pi\mu}\frac{2}{h^2}\{-\frac{\partial^2}{\partial z_j \partial z_m}(\frac{h^2R}{2}) + h^2(\frac{1}{R} - h + \frac{h^2R}{2})\delta_{jm}\} \\
&\approx \frac{-1}{8\pi\mu}\{-\frac{\partial^2}{\partial z_j \partial z_m}(R) + 2(\frac{1}{R} - h + \frac{h^2R}{2})\delta_{jm}\} \\
&\approx \frac{-1}{8\pi\mu}\{-\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + 2\frac{\delta_{jm}}{R} - 2h\delta_{jm} + h^2R\delta_{jm}\} \\
&\approx \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + 2h\delta_{jm}\} \\
&\approx \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}
\end{aligned} \tag{4.59}
$$

which are the steady stokeslet solutions associated with the pressure

$$p^{s(m)}(\mathbf{z}) = \frac{-z_m}{4\pi R^3}. \tag{4.60}$$

4.4.2 Green's Integral Representation of the Oscillatory Velocity

The Green's surface integral representation of the oscillatory Stokes flow has been given in (4.12) as

$$\int\int_S \{u_j^{s(m)}(\mathbf{z})p^s(\mathbf{y})n_j + u_j^s(\mathbf{y})p^{s(m)}(\mathbf{z})n_j$$
$$-\mu u_j^{s(m)}(\mathbf{z})\frac{\partial u_j^s(\mathbf{y})}{\partial y_l}n_l + \mu u_j^s(\mathbf{y})\frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l}n_l\}dS = 0. \tag{4.61}$$

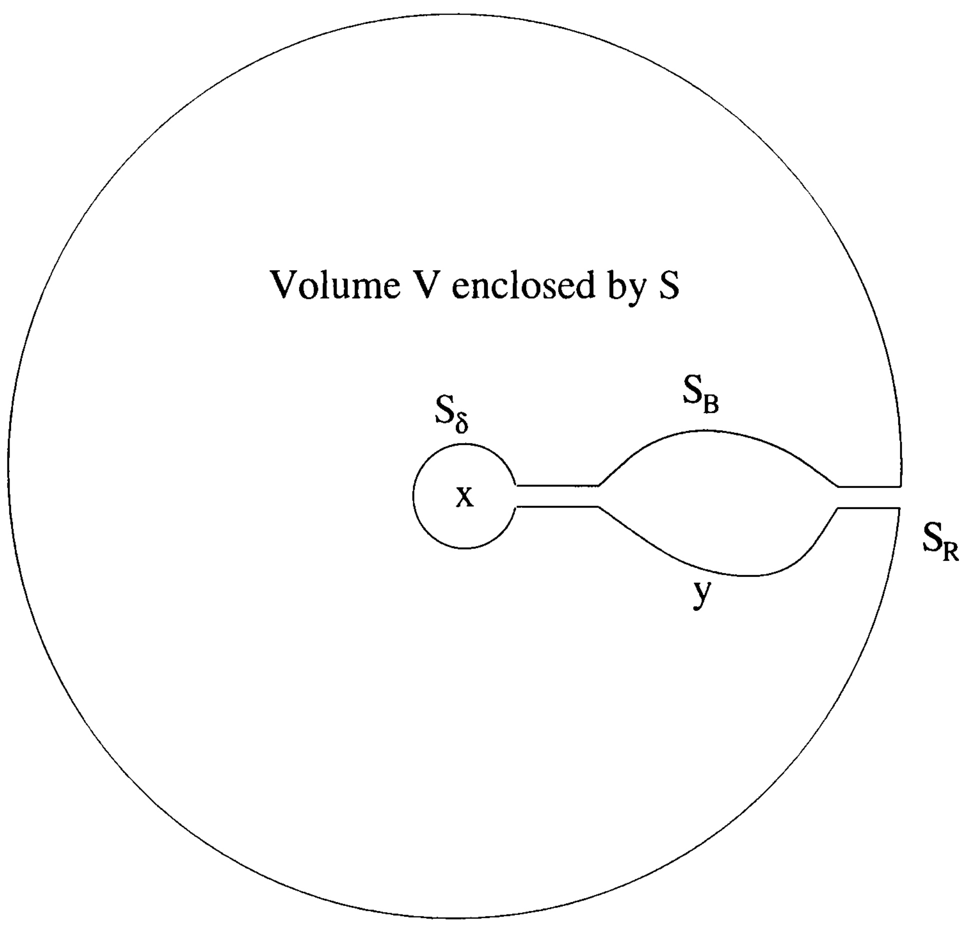

Figure 4.1: The surface S and the relation of the points **x** and **y**

We consider the surface S consisting of a surface S_δ, a sphere radius $\delta \to 0$ around the point $z = 0$, a surface S_B enclosing the oscillating body, and a large spherical surface S_R extending to infinity, enclosing the body and centred at the point $\mathbf{z} = 0$. See figure (4.1).

We re-write the integral over the surface S as a sum of the integrals over the surfaces S_δ, S_B, and S_R,

$$\int\int_S = \int\int_{S_\delta} + \int\int_{S_B} + \int\int_{S_R} = 0. \tag{4.62}$$

Next, we calculate the contributions over the surface S_δ as $\delta \to 0$, and over S_R as $R \to \infty$, to give integral representation for the oscillatory Stokes velocity $u_j^s(\mathbf{x})$.

The Contribution over the Surface S_δ as $\delta \to 0$.

The integral over the surface S_δ is denoted by I_{S_δ}, which is

$$\begin{aligned} I_{S_\delta} &= \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{y}) n_j dS + \int\int_{S_\delta} u_j^s(\mathbf{y}) p^{s(m)}(\mathbf{z}) n_j dS \\ &- \mu \int\int_{S_\delta} u_j^{s(m)}(\mathbf{z}) \frac{\partial u_j^s(\mathbf{y})}{\partial y_l} n_l dS + \mu \int\int_{S_\delta} u_j^s(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} n_l dS. \end{aligned} \tag{4.63}$$

Since the Green's surface integral representation of time-harmonic Stokes equation (4.12) is identical to the Green's representation of steady Stokes equation (3.13), then the above integral (4.63) is identical to I_{S_δ} (3.26) for the steady Stokes flow. Also as the oscillatory stokeslet tends to the steady stokeslet around zero, then the integral over the surface S_δ for oscillatory flow tends to the the integral over the surface S_δ for steady Stokes flow, which gives the contribution $-u_m^s(\mathbf{x})$. Hence

$$I_{S_\delta} = -u_m^s(\mathbf{x}). \tag{4.64}$$

The Contribution over the Surface S_R as $R \to \infty$

Now we determine

$$
\begin{aligned}
I_{S_R} &= \int\int_{S_R} u_j^{s(m)}(\mathbf{z}) p^s(\mathbf{y}) n_j dS + \int\int_{S_R} u_j^s(\mathbf{y}) p^{s(m)}(\mathbf{z}) n_j dS \\
&-\mu \int\int_{S_R} u_j^{s(m)}(\mathbf{z}) \frac{\partial u_j^s(\mathbf{y})}{\partial y_l} n_l dS + \mu \int\int_{S_R} u_j^s(\mathbf{y}) \frac{\partial u_j^{s(m)}(\mathbf{z})}{\partial y_l} n_l dS,
\end{aligned}
\tag{4.65}
$$

where the surface S_R is a sphere with large radius as shown in figure (4.1). We use here the asymptotic series of oscillatory stokeslets $u_j^{s(m)}$ for large R which has been given in (4.47), as

$$
\begin{aligned}
u_j^{s(m)}(\mathbf{z}) &= \frac{-1}{8\pi\mu}\{\frac{2}{h^2}(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5}) + ...\} \\
&= O(R^{-3})
\end{aligned}
\tag{4.66}
$$

and its associated pressure

$$
p^{s(m)}(\mathbf{z}) = \frac{i}{4\pi\omega\rho}(-\frac{z_m}{R^3}) = O(R^{-2}).
$$

Substituting the asymptotic series and the pressure into the integral over S_R and letting R tends to infinity, lead to

$$
I_{S_R} \to 0, \tag{4.67}
$$

as both the oscillatory stokeslets and its associated pressure vanish in infinity [9]. Since $I_{S_B} + I_{S_\delta} + I_{S_R} = 0$, and from (4.64) and (4.67), we obtain

$$\begin{aligned} u^s_m(\mathbf{x}) &= \int\int_{S_B} \{u^{s(m)}_j(\mathbf{z})p^s(\mathbf{y})n_j + u^s_j(\mathbf{y})p^{s(m)}(\mathbf{z})n_j \\ &\quad -\mu u^{s(m)}_j(\mathbf{z})\frac{\partial u^s_j(\mathbf{y})}{\partial y_k}n_k + \mu u^s_j(\mathbf{y})\frac{\partial u^{s(m)}_j(\mathbf{z})}{\partial y_k}n_k\}dS \end{aligned} \tag{4.68}$$

which is the representation of the flow velocity in terms of the oscillatory Stokes solutions.

4.5 Force Integral Representation in Oscillatory Stokes Flow

Denote the surface of the oscillating body by $S_t(\mathbf{x}, t)$ relative to fixed coordinate system on the body. The force on the body due to the action of the fluid is then

$$F^s_j = \int\int_{S_t} \tau^\dagger_{jl} n_l \; dS \tag{4.69}$$

where for an incompressible fluid

$$\tau^\dagger_{jl} = -p^\dagger \delta_{jl} + \mu\left(\frac{\partial u^\dagger_j}{\partial x_l} + \frac{\partial u^\dagger_l}{\partial x_j}\right)$$

is the symmetric Navier-Stokes stress tensor. The Fourier series for time periodic motion is

$$u^\dagger_j = \sum_{n=-\infty}^{\infty} u^n_j e^{i\omega_n t} \tag{4.70}$$

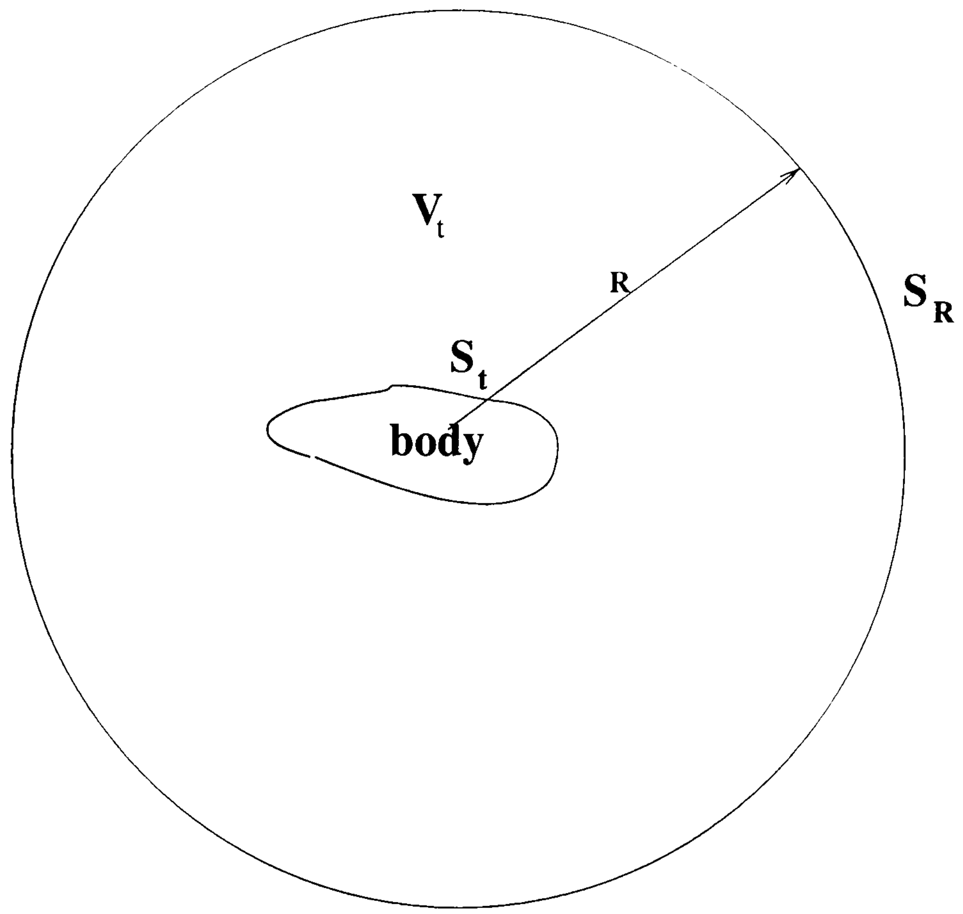

Figure 4.2: Volume V_t

where $\omega_n = \frac{2n\pi}{T}$, and T is the time period Then the Stokes force becomes

$$F_j^s = \int\int_{S_t} \sum_{n=-\infty}^{\infty} \tau_{jl}^{sn} e^{i\omega_n t} n_l \ dS \tag{4.71}$$

where $\tau_{jl}^{sn} = -p^{sn}\delta_{jl} + \mu(\frac{\partial u_j^{sn}}{\partial x_l} + \frac{\partial u_l^{sn}}{\partial x_j})$, $u_j^s = \sum_{n=-\infty}^{\infty} u_j^{sn} e^{i\omega_n t}$ and p^{sn} is the associated pressure. By changing the order of the summation and the integral in (4.71) the force F_j^s becomes

$$F_j^s = \sum_{n=-\infty}^{\infty} [\int\int_{S_t} \tau_{jl}^{sn} e^{i\omega_n t} n_l \ \ dS] \tag{4.72}$$

Considering a volume V_t, which is bounded by the body surface S_t and by a large spherical surface S_R of radius R enclosing the body, figure (4.2), and using the divergence theorem, the force on the body becomes

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \tau_{jl}^{sn} n_l \ dS - \int\int\int_{V_t} \frac{\partial}{\partial x_l}(\tau_{jl}^{sn}) dV \right] e^{i\omega_n t}. \quad (4.73)$$

However,

$$\begin{aligned} \frac{\partial}{\partial x_l}(\tau_{jl}^{sn}) &= \frac{\partial}{\partial x_l}(-p^{sn}\delta_{jl} + \mu(\frac{\partial u_j^{sn}}{\partial x_l} + \frac{\partial u_l^{sn}}{\partial x_j})) \\ &= -\frac{\partial p^{sn}}{\partial x_l}\delta_{jl} + \mu(\frac{\partial^2 u_j^{sn}}{\partial x_l \partial x_l} + \frac{\partial^2 u_l^{sn}}{\partial x_l \partial x_j}). \end{aligned} \quad (4.74)$$

Using the continuity equation $\frac{\partial^2 u_l^{sn}}{\partial x_l \partial x_j} = 0$ we can write

$$\frac{\partial}{\partial x_l}(\tau_{jl}^{sn}) = -\frac{\partial p^{sn}}{\partial x_l}\delta_{jl} + \mu \frac{\partial^2 u_j^{sn}}{\partial x_l \partial x_l}. \quad (4.75)$$

From the oscillatory Stokes equation (4.5), we obtain

$$\frac{\partial}{\partial x_l}(\tau_{jl}^{sn}) = i\rho\omega_n u_j^{sn}. \quad (4.76)$$

Therefore

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \tau_{jl}^{sn} n_l \ dS - \int\int\int_{V_t} i\rho\omega_n u_j^{sn} dV \right] e^{i\omega_n t}. \quad (4.77)$$

Also, since

$$\frac{\partial}{\partial y_k}(y_j u_k^{sn}) = \delta_{jk} u_k^{sn} + y_j \frac{\partial u_k^{sn}}{\partial y_k} = u_j^{sn},$$

by using the continuity equation, the volume integral can be re-written as

$$\int\int\int_V u_j^{sn} dV = \int\int\int_V \frac{\partial}{\partial y_k}(y_j u_k^{sn}) dV$$
$$= \int\int_{S_R} y_j u_k^{sn} n_k dS - \int\int_{S_t} y_j u_k^{sn} n_k dS. \quad (4.78)$$

So

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \tau_{jl}^{sn} n_l \, dS - \int\int_{S_R} i\rho\omega_n y_j u_k^{sn} n_k dS + \int\int_{S_t} i\rho\omega_n y_j u_k^{sn} n_k dS \right] e^{i\omega_n t}. \quad (4.79)$$

However

$$\sum_{n=-\infty}^{\infty} \left[\int\int_{S_t} i\rho\omega_n y_j u_k^{sn} n_k dS \right] e^{i\omega_n t} = \int\int_{S_t} \sum_{n=-\infty}^{\infty} \rho y_j (i\omega_n u_k^{sn} e^{i\omega_n t}) n_k \; dS$$
$$= \int\int_{S_t} \sum_{n=-\infty}^{\infty} \rho y_j \frac{\partial}{\partial t}(u_k^{sn} e^{i\omega_n t}) n_k \; dS$$
$$= \int\int_{S_t} \rho y_j \frac{\partial}{\partial t}\left(\sum_{n=-\infty}^{\infty} u_k^{sn} e^{i\omega_n t}\right) n_k \; dS$$
$$= \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k \; dS. \quad (4.80)$$

Then

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \{\tau_{jl}^{sn} n_l \; dS - i\rho\omega_n y_j u_k^{sn} n_k\} dS \right] e^{i\omega_n t} + \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k dS. \quad (4.81)$$

So we can write the total force as

$$F_j^s = \sum_{n=-\infty}^{\infty} f_j^{sn} e^{i\omega_n t} + f_j^B, \tag{4.82}$$

where

$$f_j^{sn} = \int\int_{S_R} \{\tau_{jl}^{sn} n_l - i\rho\omega_n y_j u_k^{sn} n_k\} dS \tag{4.83}$$

and

$$f_j^B = \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k \, dS. \tag{4.84}$$

4.6 Force Generated by the Oscillatory Stokeslet

Recall the force integral representation (4.79), which is

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \tau_{jl}^{sn} n_l \, dS - \int\int_{S_R} i\rho\omega_n y_j u_k^{sn} n_k dS + \int\int_{S_t} i\rho\omega_n y_j u_k^{sn} n_k dS \right] e^{i\omega_n t},$$

where S_R is a large spherical surface enclosed the oscillatory body and S_t is the body surface, figure (4.2).

Substituting the oscillatory stokeslets $u_j^{s(m)}$ into the force F_j gives $F_j^{s(m)}$ the force generated by $u_j^{s(m)}$,

$$\begin{aligned} F_j^{s(m)} &= \sum_{n=-\infty}^{\infty} \{ \int\int_{S_R} \tau_{jl}^{ns(m)} n_l \, dS - \int\int_{S_R} i\rho\omega_n z_j u_k^{ns(m)} n_k dS \\ &+ \int\int_{S_t} i\rho\omega_n z_j u_k^{ns(m)} n_k dS \} e^{i\omega_n t}. \end{aligned} \tag{4.85}$$

Using the asymptotic series of oscillatory stokeslets $u_j^{s(m)}$ for large R which has been given in (4.47), as

$$u_j^{s(m)}(\mathbf{z}) = \frac{-1}{8\pi\mu}\{\frac{2}{h^2}(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5}) + ...\}, \tag{4.86}$$

and its associated pressure

$$p^{s(m)}(\mathbf{z}) = \frac{i}{4\pi\omega\rho}(-\frac{z_m}{R^3}).$$

The contribution of the first integral

$$\int\int_{S_R} \tau_{jl}^{ns(m)} n_l \, dS, \tag{4.87}$$

where $\tau_{jl}^{ns(m)} = -p^{ns(m)}\delta_{jl} + \mu(\frac{\partial u_j^{s(m)}}{\partial z_l} + \frac{\partial u_l^{s(m)}}{\partial z_j})$. From the asymptotic series (4.47)

$$\begin{aligned}\frac{\partial u_j^{s(m)}}{\partial z_l} &= \frac{-1}{8\pi\mu}\{\frac{2}{h^2}(\frac{\partial}{\partial z_l}(\frac{-\delta_{jm}}{R^3}) + 3\frac{\partial}{\partial z_l}(\frac{z_j z_m}{R^5})) + ...\} \\ &= \frac{-1}{8\pi\mu}\{\frac{2}{h^2}\left[-\delta_{jm}\frac{\partial}{\partial z_l}(R^{-3}) + 3\frac{\partial}{\partial z_l}(\frac{z_j z_m}{R^5}) + ...\right]\} \\ &= \frac{-1}{8\pi\mu}\{\frac{2}{h^2}\left[3\delta_{jm}\frac{z_l}{R^5} + 3\delta_{jl}\frac{z_m}{R^5} + 3\delta_{ml}\frac{z_j}{R^5} - 15\frac{z_l z_j z_m}{R^7} + ...\right]\},\end{aligned} \tag{4.88}$$

and

$$\frac{\partial u_j^{s(m)}}{\partial z_l} = \frac{-1}{8\pi\mu}\{\frac{2}{h^2}\left[3\delta_{lm}\frac{z_j}{R^5} + 3\delta_{jl}\frac{z_m}{R^5} + 3\delta_{jm}\frac{z_l}{R^5} - 15\frac{z_l z_j z_m}{R^7} + ...\right]\}. \tag{4.89}$$

So

$$\mu(\frac{\partial u_j^{s(m)}}{\partial z_l} + \frac{\partial u_j^{s(m)}}{\partial z_l}) = \frac{-1}{4\pi h^2}\left[6\delta_{lm}\frac{z_j}{R^5} + 6\delta_{jl}\frac{z_m}{R^5} + 6\delta_{jm}\frac{z_l}{R^5} - 30\frac{z_l z_j z_m}{R^7} + ...\right],$$

and

$$\tau_{jl}^{ns(m)} = \frac{1}{4\pi}(\frac{z_m}{R^3})\delta_{jl} - \frac{1}{4\pi h^2}\left[6\delta_{lm}\frac{z_j}{R^5} + 6\delta_{jl}\frac{z_m}{R^5} + 6\delta_{jm}\frac{z_l}{R^5} - 30\frac{z_l z_j z_m}{R^7} + ...\right]. \tag{4.90}$$

Then

$$\begin{aligned}\int\int_{S_R} \tau_{jl}^{ns(m)} n_l\, dS &= \frac{-1}{4\pi}\{\int\int_{S_R}\frac{-z_m}{R^3}n_l\, dS + \frac{1}{h^2}(6\delta_{lm}\int\int_{S_R}\frac{z_j}{R^5}n_l\, dS \\ &+ 6\delta_{jl}\int\int_{S_R}\frac{z_m}{R^5}n_l\, dS + 6\delta_{jm}\int\int_{S_R}\frac{z_l}{R^5}n_l\, dS \\ &- 30\int\int_{S_R}\frac{z_l z_j z_m}{R^7}n_l\, dS + ...)\}.\end{aligned} \tag{4.91}$$

Since $n_l = \frac{z_l}{R}$ and $z_l = \delta_{lj}z_j$.

$$\begin{aligned}\int\int_{S_R} \tau_{jl}^{ns(m)} n_l\, dS &= \frac{-1}{4\pi}\{\int\int_{S_R}\frac{-z_m z_j}{R^4}\, dS + \frac{1}{h^2}(6\int\int_{S_R}\frac{z_j z_m}{R^6}\, dS \\ &+ 6\int\int_{S_R}\frac{z_m z_j}{R^6}\, dS + 6\delta_{jm}\int\int_{S_R}\frac{z_l^2}{R^6}\, dS \\ &- 30\int\int_{S_R}\frac{z_l^2 z_j z_m}{R^8}\, dS + ...)\}. \\ &= \frac{-1}{4\pi}\{-\frac{1}{R^4}\int\int_{S_R} z_j z_m\, dS + \frac{1}{h^2}(\frac{12}{R^6}\int\int_{S_R} z_j z_m\, dS \\ &+\frac{6\delta_{jm}}{R^4}\int\int_{S_R} dS - \frac{30}{R^6}\int\int_{S_R} z_j z_m\, dS + ...)\},\end{aligned} \tag{4.92}$$

since $\int\int_{S_R} z_j z_m \, dS = \frac{4\pi}{3}\delta_{jm}R^4$ and $\int\int_{S_R} dS = \frac{4\pi}{3}R^3$, we have

$$\begin{aligned}
\int\int_{S_R} \tau_{jl}^{ns(m)} n_l \, dS &= \frac{-1}{4\pi}\{-\frac{1}{R^4}\frac{4\pi}{3}\delta_{jm}R^4 + \frac{1}{h^2}(\frac{12}{R^6}\frac{4\pi}{3}\delta_{jm}R^4 \\
&\quad + \frac{6\delta_{jm}}{R^4}\frac{4\pi}{3}R^3 - \frac{30}{R^6}\frac{4\pi}{3}\delta_{jm}R^4) + ...\} \\
&= \frac{-1}{4\pi}\{-\frac{4\pi}{3}\delta_{jm} + \frac{8\pi\delta_{jm}}{h^2}(\frac{1}{R} - \frac{3}{R^2}) + ...\} \\
&= \frac{\delta_{jm}}{3},
\end{aligned} \tag{4.93}$$

when $R \to \infty$.

The contribution of the second integral $(-\int\int_{S_R} i\rho\omega_n z_j u_k^{ns(m)} n_k dS)$

$$\begin{aligned}
-\int\int_{S_R} i\rho\omega_n z_j u_k^{ns(m)} n_k dS &= \frac{1}{8\pi\mu} \, i\rho\omega_n \in \int_{S_R} z_j[\frac{1}{h^2}(-\frac{\delta_{jm}}{R^3} + 3\frac{z_j z_m}{R^5} + ...)] \, n_k \, dS \\
&= \frac{1}{4\pi}\int\int_{S_R}\left[(-\frac{z_j}{R^3}\delta_{jm} + 3\frac{z_j^2 z_m}{R^5}) + ...\right] \, n_k \, dS \\
&= \frac{1}{4\pi}\int\int_{S_R}(-\frac{z_m}{R^3}\frac{z_k}{R} + 3\frac{z_j^2 z_m z_l}{R^6}) \, dS + ... \\
&= \frac{1}{4\pi}\int\int_{S_R}(-\frac{z_m z_k}{R^4} + 3\frac{z_m z_l}{R^4}) \, dS + ... \\
&= \frac{1}{4\pi}\int\int_{S_R}\frac{z_m z_k}{R^4} \, dS + ... \\
&= \frac{1}{4\pi}\frac{1}{R^4}\int\int_{S_R} z_m z_k \, dS + ... \\
&= \frac{2}{3}\delta_{jm}.
\end{aligned} \tag{4.94}$$

Therefore from (4.93) and (4.94), the total force (4.85) becomes

$$\begin{aligned}
F_j^{s(m)} &= \sum_{n=-\infty}^{\infty}\left[\frac{\delta_{jm}}{3} + \frac{2}{3}\delta_{jm} + \int\int_{S_t} i\rho\omega_n z_j u_k^{ns(m)} n_k dS\right] e^{i\omega_n t} \\
&= \sum_{n=-\infty}^{\infty}\left[\delta_{jm} + \int\int_{S_t} i\rho\omega_n z_j u_k^{ns(m)} n_k dS\right] e^{i\omega_n t},
\end{aligned} \tag{4.95}$$

At the limit $\omega \to 0$, $F_j^{s(m)}$ tends to the force generated by steady Stokes flow. Considering a stokeslet inside the body gives the same result as follows

Recall the integral representation of the force (4.79) in terms the surface S_δ, which is small sphere central at the stokeslet point inside the body and the body surface S_t, as in figure (4.3), and that is

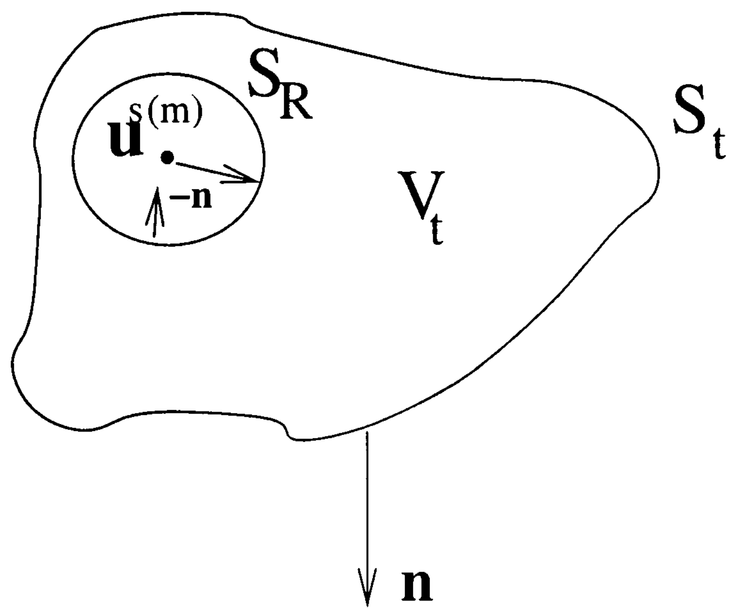

Figure 4.3: stokeslet inside the body, $R = \delta$

$$F_j^s = \sum_{n=-\infty}^{\infty} \left[\int\int_{S_\delta} \tau_{jl}^{sn} n_l \, dS - \int\int_{S_\delta} i\rho\omega_n y_j u_k^{sn} n_k dS + \int\int_{S_t} i\rho\omega_n y_j u_k^{sn} n_k dS \right] e^{i\omega_n t}.$$

Substituting the oscillatory stokeslets into the force, gives

$$\begin{aligned} F_j^{s(m)} &= \sum_{n=-\infty}^{\infty} \{ \int\int_{S_\delta} \tau_{jl}^{ns(m)} n_l \, dS - i\rho\omega_n \int\int_{S_\delta} z_j u_k^{ns(m)} n_k dS \\ &+ i\rho\omega_n \int\int_{S_t} z_j u_k^{ns(m)} n_k dS \} e^{i\omega_n t}. \end{aligned} \quad (4.96)$$

When $\delta \longrightarrow 0$ the oscillatory stokeslets tend to steady stokeslets, so that the oscillatory stress tensor $\tau_{jl}^{ns(m)}$ tends to the steady stokes stress tensor. Therefore from (3.59) the first integral gives

$$\int\int_{S_\delta} \tau_{jl}^{ns(m)} n_l \, dS = \delta_{jm}. \tag{4.97}$$

And the second integral of (4.96) can be computed using the asymptotic series (4.59) of the oscillatory stokeslet around zero, as follows

$$\begin{aligned}
-i\rho\omega_n \int\int_{S_\delta} z_j u_k^{ns(m)} n_k dS &= \frac{i\rho\omega_n}{8\pi\mu} \int\int_{S_\delta} z_j \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\} \, n_k \, dS \\
&= \frac{i\rho\omega_n}{8\pi\mu} \int\int_{S_\delta} \{\frac{z_m z_k}{R^2} + \frac{z_j^2 z_m z_k}{R^4} + O(R^{-1})\} \, dS \\
&= \frac{i\rho\omega_n}{8\pi\mu} \int\int_{S_\delta} \{\frac{z_m z_k}{R^2} + \frac{z_m z_k}{R^2} + O(R^{-1})\} \, dS \\
&= \frac{i\rho\omega_n}{8\pi\mu} \int\int_{S_\delta} \{2\frac{z_m z_k}{R^2} + O(R^{-1})\} \, dS \\
&= \frac{i\rho\omega_n}{8\pi\mu} \{\frac{2}{R^2} \int\int_{S_\delta} z_m z_k \, dS + O(R^2)\} \\
&= \frac{i\rho\omega_n}{8\pi\mu} \{\frac{2}{R^2} \frac{4\pi}{3} R^4 \delta_{mk} + O(R^2)\} \\
&= O(R^2) \to 0
\end{aligned} \tag{4.98}$$

as $\delta \to 0$. By (4.97) and (4.98), the force generated by the oscillatory stokeslet is

$$F_j^{s(m)} = \sum_{n=-\infty}^{\infty} \{\delta_{jm} + i\rho\omega_n \int\int_{S_t} z_j u_k^{ns(m)} n_k dS\} e^{i\omega_n t} \tag{4.99}$$

which is identical to (4.95).

4.7 Conclusion

In this chapter we represent the oscillatory Stokes flow and obtain the stokeslet using a different approach to the singular method which is used by Pozrikidis. The new result is that the stokeslet can be obtained using potentials approach and we show that this approach gives identical result to known one, using the singular method, see [9], [?]. The benefit of using potentials approach is to enable us to use the oscillatory stokeslets to infer the form of the oscillatory oseenlets. The behaviour of the flow far from the point force and at high frequencies is presented by the asymptotic series for large R and large ω. We show that close to the point force the oscillatory stokeslets reduce to the steady stokeslets and then the integral representation is given for the oscillatory velocity. Forces are calculated and we show that our results are identical to known results and that considering the point force inside or outside the body gives the similar results which reduce to the force generated by the steady stokeslet when the frequency is zero.

Chapter 5

Steady Oseen Flow

The steady Oseen flow has been studied exclusively, started from the improvement of stokes flow that has been made by Oseen to resolve the failure of Stokes approximation in the far field, see [23] [13]. Lamb and Goldstein, in [21] and [29], used the potentials decomposition for antisysmetric flow to obtain the steady oseenlets, and Chadwick in [30] presents the behaviour of the flow close to oseenlet (point force) for general steady flow. Fishwick and Chadwick give the Green's integral representation of Oseen velocity and it is shown that there is no contribution over the far field surface. In this chapter we repeat the known results in some details, and later in chapter 6 we show that the oscillatory oseenlets reduces to the steady oseenlet when the frequency tends to zero. The problem of uniform flow past a steady body in unbound region and Oseen's approximation are given in section 1. In section 2 the Green's integral representation of Oseen flow is given, and following Lamb and Goldstein decomposition the steady oseenlets are presented in section 3. We show how to obtain the asymptotic series of the oseenlet around zero in section 4, which then used to obtain the Green's integral representation of Oseen velocity. Finally, the force is given as a far field integral in more detail.

5.1 Governing Equations

Steady Oseen equations are obtained by applying Oseen 's approximation to (2.20) the steady Navier-Stokes equations [21], which are

$$\rho(\mathbf{u}^\dagger \cdot \nabla)\mathbf{u}^\dagger = -\nabla p^\dagger + \mu \nabla^2 \mathbf{u}^\dagger \tag{5.1}$$

and the continuity equation

$$\nabla \cdot \mathbf{u}^\dagger = 0 \tag{5.2}$$

where $\mathbf{u}^\dagger$ and $p^\dagger$ are the Navier-Stokes velocity and pressure,respectively. ρ is the fluid density, μ is the fluid viscosity and $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ is the gradient operator.

We consider a uniform stream U in the x_1-direction past a closed body in an unbounded domain. The Oseen approximation $|\frac{\mathbf{u}}{U}|, |\frac{p}{p_0}| = O(\varepsilon)$ and $\varepsilon \ll 1$ holds , where the notation "O "means 'of order of '. Hence, we consider Oseen's linearisation

$$\mathbf{u}^\dagger = U\widehat{x} + \mathbf{u} + O(\varepsilon^2), \;\; p^\dagger = p + O(\varepsilon^2), \tag{5.3}$$

where $\widehat{x}$ is the unit vector in x_1 direction in the Cartesian coordinates system (x_1, x_2, x_3). Substituting (5.3) into (5.1) gives

$$\begin{aligned} \rho(U\delta_{j1} + u_j + O(\varepsilon^2))\frac{\partial}{\partial x_l}(U\delta_{j1} + u_j + O(\varepsilon^2)) &= -\frac{\partial}{\partial x_j}(p + O(\varepsilon^2)) \\ &\quad + \mu\frac{\partial^2}{\partial x_l \partial x_l}(U\delta_{j1} + u_j + O(\varepsilon^2)) \\ \rho(U\delta_{j1} + u_j + O(\varepsilon^2))\frac{\partial u_j}{\partial x_l} &= -\frac{\partial p}{\partial x_j} + \mu\frac{\partial^2 u_j}{\partial x_l \partial x_l}, \end{aligned} \tag{5.4}$$

where δ_{ij} is the Kronecker delta ($\delta_{jl} = 1$ when $j = l$ and zero otherwise), and u_j is the Oseen velocity component in the j direction of a Cartesian coordinate system x_j and

$j, \ l = 1, \ 2, \ 3$. By considering only the terms of order ε we find

$$\rho U \frac{\partial u_j(\mathbf{x})}{\partial x_1} = -\frac{\partial p(\mathbf{x})}{\partial x_j} + \mu \frac{\partial^2 u_j(\mathbf{x})}{\partial x_l \partial x_l}, \tag{5.5}$$

and

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{5.6}$$

which is the steady Oseen equation, and the velocity satisfies the condition $\mathbf{u} \longrightarrow 0$ as $R \longrightarrow \infty$. taking the divergence of the equation (5.5) and applying the continuity equation (2.1), gives

$$\nabla^2 p = 0, \tag{5.7}$$

which means that the Oseen pressure is harmonic function, where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplacian operator [17]. Applying the infinity condition $\mathbf{u} \longrightarrow 0$ as $R \longrightarrow \infty$ to the Oseen equation (5.5) yields

$$\nabla p = 0, \tag{5.8}$$

as $R \longrightarrow \infty$, hence we may choose the solution $p = 0$ in the far field. Also, the condition of no fluid outflow from any surface enclosing the body,

$$\int\int_S \mathbf{u} \cdot \mathbf{n} \, dS = 0, \tag{5.9}$$

is satisfied, where S is a surface enclosing the body.

5.2 The Green's surface Integral Representation

In this section, we give the Green's surface integral representation of the steady Oseen equation, see [30]. Considering four solutions given by $\mathbf{u}, p, \mathbf{u}^{(m)}$ and $p^{(m)}$; $m = 1, \ 2, \ 3$

where $\mathbf{u}$ and p are the Oseen velocity and Oseen pressure, respectively; $\mathbf{u}^{(m)}$ and $p^{(m)}$ are the fundamental solutions of steady Oseen equations acting at the origin and each yields a unit force in the m-direction which is the direction of the force point. see [30].

From (5.5), we find that

$$\rho U \frac{\partial u_j(\mathbf{y})}{\partial y_1} = -\frac{\partial p(\mathbf{y})}{\partial y_j} + \mu \frac{\partial^2 u_j(\mathbf{y})}{\partial y_l \partial y_l}, \tag{5.10}$$

which is the Oseen equations in variable y, and

$$\rho U \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial z_1} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j} + \mu \frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l}, \tag{5.11}$$

where $\mathbf{z} = \mathbf{x} - \mathbf{y}$, $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$. By using the fact $\frac{\partial}{\partial z_j} = -\frac{\partial}{\partial y_j}$, we get the adjoint equation in y_j, which is

$$\rho U \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_1} = \frac{\partial p^{(m)}(\mathbf{z})}{\partial y_j} + \mu \frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial y_l \partial y_l}. \tag{5.12}$$

Following the method of Oseen [13] to obtain the Green's functions representation, we dot product (5.10) with $u_j^{(m)}(\mathbf{z})$ and take it from the dot product of (5.12) with $u_j(\mathbf{y})$, and find

$$\begin{aligned} -\rho U u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_1} - \rho U u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_1} &= u_j(\mathbf{y}) \frac{\partial p^{(m)}(\mathbf{z})}{\partial y_j} + u_j^{(m)}(\mathbf{z}) \frac{\partial p(\mathbf{z})}{\partial y_j} \\ &+ \mu [u_j(\mathbf{y}) \frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial y_l \partial y_l} - u_j^{(m)}(\mathbf{z}) \frac{\partial^2 u_j(\mathbf{y})}{\partial y_l \partial y_l}]. \end{aligned} \tag{5.13}$$

Using the continuity equation (5.6) enables us to write

$$-\rho U \frac{\partial}{\partial y_1}[u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})] = \frac{\partial}{\partial y_j}[u_j(\mathbf{y})p^{(m)}(\mathbf{z}) + u_j^{(m)}(\mathbf{z})p(\mathbf{y})] + \mu \frac{\partial}{\partial y_l}[u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} - u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l}], \quad (5.14)$$

then

$$\rho U \frac{\partial}{\partial y_1}[u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})] = -\frac{\partial}{\partial y_j}[u_j(\mathbf{y})p^{(m)}(\mathbf{z}) + u_j^{(m)}(\mathbf{z})p(\mathbf{y})] + \mu \frac{\partial}{\partial y_l}[u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l} - u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l}]. \quad (5.15)$$

This holds within a volume V of the fluid bounded by the surface S where the Oseen approximation is valid, and parameterised by the coordinate y. Applying the divergence theorem leads to

$$\int\int_S \{\rho U u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})n_1 + u_j^{(m)}(\mathbf{z})p(\mathbf{y})n_j + u_j(\mathbf{y})p^{(m)}(\mathbf{z})n_j - \mu u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l}n_l + \mu u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l}n_l\}dS = 0 \quad (5.16)$$

where $\mathbf{n}$ is the unit vector normal to the surface S pointing outwards from the control volume V. This gives the Green's surface integral representation of the steady Oseen equations, see [30].

5.3 Steady Oseenlets

Lamb and Goldstein used the velocity decomposition to obtain the steady oseenlets in [21] and [29] which decompose the fluid velocity into a potential $\phi(\mathbf{z})$ and a wake velocity

$\mathbf{w}(\mathbf{z})$, such that

$$u_j^{(m)}(\mathbf{z}) = \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_j} + w_j^{(m)}(\mathbf{z}). \tag{5.17}$$

As we consider incompressible flow, the velocity potential ϕ has to satisfy the Laplace equation

$$\nabla^2 \phi^{(m)}(\mathbf{z}) = 0. \tag{5.18}$$

Taking the divergence vector of the decomposition (5.17), using the continuity equation and then $\nabla^2 \phi^{(m)}(\mathbf{z}) = 0$, shows that the wake velocity satisfies the continuity equation

.

$$\nabla \cdot \mathbf{w}^{(m)} = 0. \tag{5.19}$$

Now to obtain $\phi^{(m)}$ and $\mathbf{w}^{(m)}$ we substitute (5.17) into (5.11),

$$\rho U \frac{\partial^2 \phi^{(m)}(\mathbf{z})}{\partial z_1 \partial z_j} + \rho U \frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j} + \mu \frac{\partial}{\partial z_j}(\frac{\partial^2 \phi^{(m)}(\mathbf{z})}{\partial z_l \partial z_l}) + \mu \frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} \tag{5.20}$$

and using the continuity equation $\frac{\partial^2 \phi^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} = 0$, gives

$$\rho U \frac{\partial^2 \phi^{(m)}(\mathbf{z})}{\partial z_1 \partial z_j} + \rho U \frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} - \mu \frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j}. \tag{5.21}$$

Since both $p^{(m)}$ and $\phi^{(m)}$ are harmonic functions, a particular solution is found if we choose

$$\rho U \frac{\partial^2 \phi^{(m)}(\mathbf{z})}{\partial z_1 \partial z_j} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j}. \tag{5.22}$$

Integrating for z_j gives

$$p^{(m)}(\mathbf{z}) = -\rho U \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_1}. \tag{5.23}$$

The pressure is given by Lamb [21] as

$$p^{(m)}(\mathbf{z}) = \frac{1}{4\pi} \frac{\partial}{\partial z_m} (\frac{1}{R}), \tag{5.24}$$

from (5.23)

$$\begin{aligned}\frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_1} &= -\frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}(\frac{1}{R}) \\ &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}(\frac{-1}{R}) = \frac{1}{4\pi\rho U}\frac{\partial^2}{\partial z_m \partial z_1}(\ln(R - z_1))\end{aligned} \tag{5.25}$$

this by the use of $\frac{\partial}{\partial z_1}(\ln(R - z_1)) = \frac{-1}{R}$. So

$$\phi^{(m)}(\mathbf{z}) = \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}(\ln(R - z_1)). \tag{5.26}$$

as stated by Goldstein in [29]. Now we consider the wake velocity $\mathbf{w}^{(m)}$, since the pressure balances with the potential $\phi^{(m)}$ in (5.21), we find

$$\rho U \frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} - \mu \frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} = 0, \tag{5.27}$$

which we re-write as

$$\frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} - \frac{\rho U}{\mu}\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} = 0. \tag{5.28}$$

In operator form

$$(\nabla^2 - 2k\frac{\partial}{\partial z_1})w_j^{(m)}(\mathbf{z}) = 0, \tag{5.29}$$

where the constant $k = \frac{\rho U}{2\mu}$. Since the velocity $\mathbf{u}$ must be finite and continuous in the fluid and satisfies the condition $u_j^{(m)} \to 0$ at infinity , Goldstein divided $w_j^{(m)}$ into two parts, the first part cancels out the discontinuities in $\frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_j}$ and the second part is continuous and tends to zero at infinity and both parts satisfy the equations (5.19) and (5.29), see[29] page 202. Following Goldstein we write

$$w_j^{(m)}(\mathbf{z}) = \frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_j} + \chi^*(\mathbf{z})\delta_{jm}, \tag{5.30}$$

since χ^* is a continuous solution of (5.29), which is

$$(\nabla^2 - 2k\frac{\partial}{\partial z_1})\chi^*(\mathbf{z}) = 0. \tag{5.31}$$

we can write

$$(\nabla^2 - k^2)e^{kz_1}\chi^*(\mathbf{z}) = 0, \tag{5.32}$$

which is the Heat equation, see [31], and its solution given in [21] as

$$e^{kz_1}\chi^*(\mathbf{z}) = \frac{2k}{4\pi\rho U}\frac{e^{kR}}{R} \tag{5.33}$$

therefore

$$\chi^*(\mathbf{z}) = \frac{2k}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R}. \tag{5.34}$$

Now we consider $\chi^{(m)}$, since

$$(\nabla^2 - 2k\frac{\partial}{\partial z_1})\chi^{(m)}(\mathbf{z}) = 0, \tag{5.35}$$

then

$$\nabla^2\chi^{(m)}(\mathbf{z}) = 2k\frac{\partial\chi^{(m)}(\mathbf{z})}{\partial z_1}, \tag{5.36}$$

and from the continuity equation $\frac{\partial w_j^{(m)}}{\partial z_j}$, we have

$$\frac{\partial\chi^{(m)}}{\partial z_j\partial z_j} = -\frac{\partial\chi^*}{\partial z_j}\delta_{jm} = -\frac{\partial\chi^*}{\partial z_m}. \tag{5.37}$$

So, from (5.36)

$$2k\frac{\partial\chi^{(m)}(\mathbf{z})}{\partial z_1} = -\frac{\partial\chi^*}{\partial z_m} \tag{5.38}$$

then

$$
\begin{aligned}
\frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_1} &= \frac{-1}{2k}\frac{\partial \chi^*}{\partial z_m} \\
&= \frac{-1}{2k}\frac{\partial}{\partial z_m}\{\frac{2k}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R}\} \\
&= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}\{e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1)\} \\
&= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_1}\{e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\} \qquad (5.39)
\end{aligned}
$$

hence

$$
\chi^{(m)}(\mathbf{z}) = \frac{1}{4\pi\rho U}\{e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\}. \qquad (5.40)
$$

Above we used the fact that

$$
\frac{\partial}{\partial z_m}\{e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1)\} = \frac{\partial}{\partial z_1}\{e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\}
$$

which is shown next, we have

$$
\begin{aligned}
\frac{\partial}{\partial z_m}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1)\right) &= \frac{\partial}{\partial z_m}(e^{-k(R-z_1)})\frac{\partial}{\partial z_1}\ln(R-z_1) \\
&\quad + e^{-k(R-z_1)}\frac{\partial^2}{\partial z_m \partial z_1}\ln(R-z_1) \\
&= -k(\frac{z_m}{R}-\delta_{m1})e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1) \\
&\quad + e^{-k(R-z_1)}\frac{\partial}{\partial z_m}(-\frac{1}{R}) \\
&= e^{-k(R-z_1)}\{-k(\frac{z_m-\delta_{m1}R}{R})\frac{1}{R-z_1}(\frac{z_1}{R}-1) \\
&\quad + \frac{\partial}{\partial z_m}(-\frac{1}{R})\} \\
&= e^{-k(R-z_1)}\{k(\frac{z_m-\delta_{m1}R}{R^2}) + \frac{\partial}{\partial z_m}(-\frac{1}{R})\},
\end{aligned}
\qquad (5.41)
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial z_1}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right) &= \frac{\partial}{\partial z_1}(e^{-k(R-z_1)})\frac{\partial}{\partial z_1}\ln(R-z_1) \\
&+ e^{-k(R-z_1)}\frac{\partial^2}{\partial z_1 \partial z_m}\ln(R-z_1) \\
&= -k(\frac{z_1}{R}-1)e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1) \\
&+ e^{-k(R-z_1)}\frac{\partial}{\partial z_m}(-\frac{1}{R})
\end{aligned}
\tag{5.42}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z_1}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right) &= e^{-k(R-z_1)}\{k(\frac{z_1-R}{R})\frac{1}{R-z_1}(\frac{z_m}{R}-\delta_{m1}) \\
&+ \frac{\partial}{\partial z_m}(-\frac{1}{R})\}. \\
&= e^{-k(R-z_1)}\{k\frac{(z_m-\delta_{m1}R)}{R}+\frac{\partial}{\partial z_m}(\frac{-1}{R})\}.
\end{aligned}
\tag{5.43}
$$

Therefore, from (5.41) and (5.43)

$$
\frac{\partial}{\partial z_1}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right) = \frac{\partial}{\partial z_m}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1)\right). \tag{5.44}
$$

The complete oseenlet solutions are

$$
u_j^{(m)}(\mathbf{z}) = \frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_j} + \frac{\partial\chi^{(m)}(\mathbf{z})}{\partial z_j} + \chi^*(\mathbf{z})\delta_{jm} \tag{5.45}
$$

where

$$\begin{aligned}
\phi^{(m)}(\mathbf{z}) &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_m}\ln(R - z_1) \\
\chi^{(m)}(\mathbf{z}) &= \frac{-1}{4\pi\rho U}(e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R - z_1)) \\
\chi^{*}(\mathbf{z}) &= \frac{2k}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R} \qquad (5.46)
\end{aligned}$$

as given by Chadwick in [30], which can be written as

$$\begin{aligned}
u_j^{(m)}(\mathbf{z}) &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_j}\{(1 - e^{-k(R-z_1)})\frac{\partial}{\partial z_m}\ln(R - z_1)\} \\
&- \frac{1}{4\pi\mu}(\frac{e^{-k(R-z_1)}}{R})\delta_{jm}. \qquad (5.47)
\end{aligned}$$

5.4 Steady Oseenlet around the point $z = 0$

Chadwick [30] gives the asymptotic series of the oseenlet for general steady flow. Here, we show how to obtain the series in more details, which we will use in next section for integral representation of the velocity. The series is obtained by substituting the Taylor series expansion of the exponential $e^{-k(R-z_1)}$ into the oseenlets, and neglect terms of order $O(R^2)$.

$$\begin{aligned}
e^{-k(R-z_1)} &= 1 - k(R - z_1) + \frac{(k(R - z_1))^2}{2!} + ... \\
&= 1 - k(R - z_1) + O(R^2). \qquad (5.48)
\end{aligned}$$

Substitute this into (5.47) gives

$$\begin{aligned} u_j^{(m)}(\mathbf{z}) &= \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_j}\{(1-(1-k(R-z_1)+O(R^2))\frac{\partial}{\partial z_m}\ln(R-z_1)\} \\ &\quad -\frac{1}{4\pi\mu}(\frac{1-k(R-z_1)+O(R^2)}{R})\delta_{jm} \\ &\approx \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_j}\{k(R-z_1)\frac{\partial}{\partial z_m}\ln(R-z_1)\} \\ &\quad -\frac{1}{4\pi\mu}(\frac{1-k(R-z_1)}{R})\delta_{jm} \\ &\approx \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_j}\{k(R-z_1)\frac{1}{(R-z_1)}(\frac{z_m}{R}-\delta_{m1})\} \\ &\quad -\frac{2k}{4\pi\rho U}(\frac{1-k(R-z_1)}{R})\delta_{jm}, \end{aligned} \tag{5.49}$$

where $k = \frac{\rho U}{2\mu}$

$$\begin{aligned} u_j^{(m)}(\mathbf{z}) &\approx \frac{1}{4\pi\rho U}\frac{\partial}{\partial z_j}\{k(\frac{z_m}{R}-\delta_{m1})\} - \frac{2k}{4\pi\rho U}(\frac{1-k(R-z_1)}{R})\delta_{jm} \\ &\approx \frac{k}{4\pi\rho U}\{\frac{\partial}{\partial z_j}(\frac{z_m}{R}-\delta_{m1}) - 2(\frac{1}{R}-\frac{k(R-z_1)}{R})\} \\ &\approx \frac{k}{4\pi\rho U}\{\frac{R\delta_{jm}-z_m z_j/R}{R^2} - 2\frac{\delta_{jm}}{R} + 2k(\frac{R-z_1}{R})\delta_{jm}\}. \end{aligned} \tag{5.50}$$

As $\lim_{R\to 0}\frac{R-z_1}{R} = 1$, we can write

$$\begin{aligned} u_j^{(m)}(\mathbf{z}) &\approx \frac{1/2\mu}{4\pi}\{\frac{\delta_{jm}}{R} - \frac{z_m z_j}{R^3} - 2\frac{\delta_{jm}}{R} + O(1)\} \\ &\approx \frac{1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_m z_j}{R^3} + O(1)\}. \end{aligned} \tag{5.51}$$

which means that the steady oseenlet tends to the steady stokeslets close to zero. And pressure is

$$p^{(m)}(\mathbf{z}) = \frac{1}{4\pi}(-\frac{z_m}{R}). \tag{5.52}$$

These are the steady stokeslet solutions [13].

5.5 Green's Integral Representation of Oseen velocity

To obtain the Green's integral representation for the Oseen velocity. we use the approximation formula of the oseenlets (5.51) with the Green surface integral representation (5.16), then calculate the contributions over the surfaces. This representation has been given by Fishwick and Chadwick in [32], where it is shown that the contribution over far field surface S_R is zero.

First recall the Green's surface integral representation (5.16), which is

$$\int\int_S \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j \\ -\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l \} dS = 0, \tag{5.53}$$

where the surface S consisting of a surface S_δ, a sphere radius $\delta \longrightarrow 0$ around the point $z = 0$, a surface S_B enclosing the body and a large spherical surface S_R extending to infinity, enclosing the body and centred at the point $z = 0$, see figure (5.1). The integral over the surface S can be rewritten as a sum of integrals over the surface S_δ, S_B and S_R, as

$$\int\int_S = \int\int_{S_\delta} + \int\int_{S_B} + \int\int_{S_R}. \tag{5.54}$$

Next, we calculate the contributions over the surface S_δ as $\delta \longrightarrow 0$ and then over the surface S_R as $R \longrightarrow \infty$.

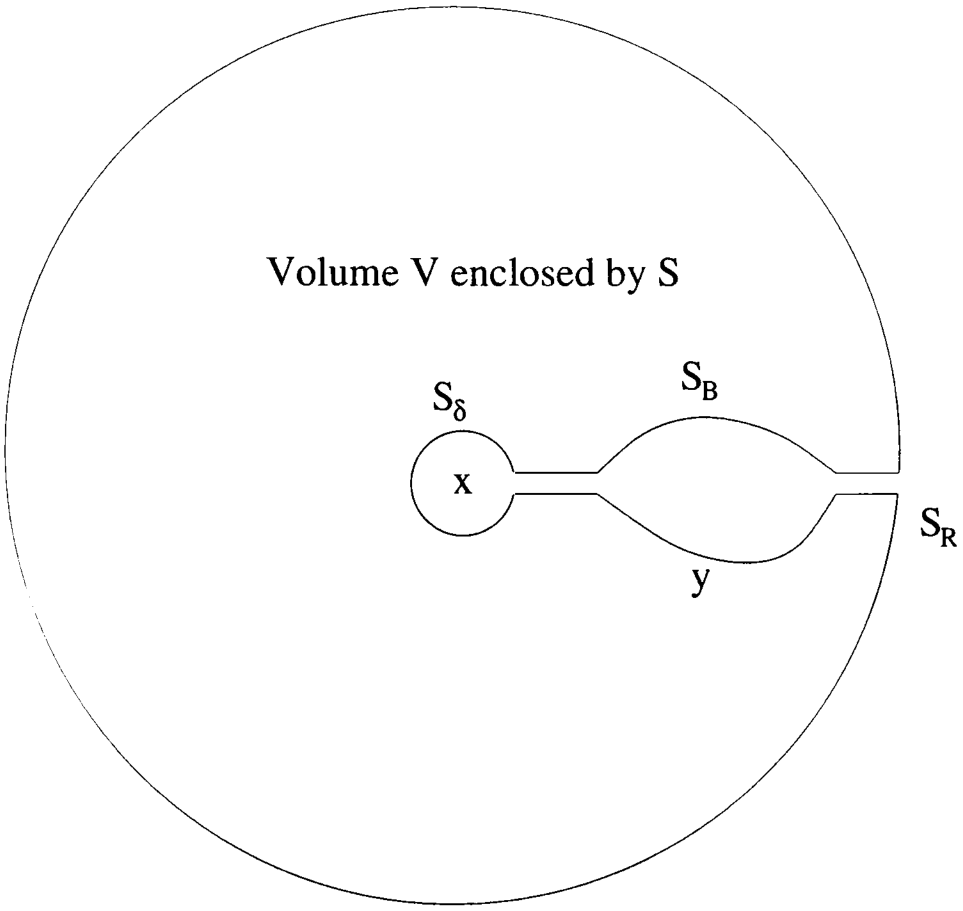

Figure 5.1: The surface S

The Contribution over the Surface S_δ as $\delta \to 0$.

Here, we work out the contribution from the integral over the surface S_δ, which will be denoted by I_{S_δ},

$$\begin{aligned} I_{S_\delta} &= \int\int_{S_\delta} \rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 dS + \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j dS + \int\int_{S_\delta} u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j dS \\ &\quad -\mu \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l dS + \mu \int\int_{S_\delta} u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l dS. \end{aligned} \tag{5.55}$$

For simplicity , we write

$$I_{S_\delta} = I_1 + I_2 + I_3 + I_4 + I_5$$

where

$$\begin{aligned}
I_1 &= \int\int_{S_\delta} \rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 dS, \\
I_2 &= \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j dS, \\
I_3 &= \int\int_{S_\delta} u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j dS, \\
I_4 &= -\mu \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l dS, \\
I_5 &= \mu \int\int_{S_\delta} u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l dS.
\end{aligned} \tag{5.56}$$

Since $\mathbf{z} = \mathbf{x} - \mathbf{y}$, then $\mathbf{y} = \mathbf{x} - \mathbf{z}$ and $n_j = \frac{z_j}{R}$ $(R = \delta)$ points outward the control volume V.

Obtaining I_1

we can show that this integral vanishes as $\delta \to 0$,

$$I_1 = \int\int_{S_\delta} \rho U u_j^{(m)}(\mathbf{z}) u_j(\mathbf{y}) n_1 dS = \rho U u_j(\mathbf{x}) \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) n_1 dS, \tag{5.57}$$

this is because

$$\mathbf{u}(\mathbf{y}) = \mathbf{u}(\mathbf{x} - \mathbf{z}) = \mathbf{u}(\mathbf{x}) + \frac{\partial \mathbf{u}(\mathbf{x})}{\partial z_k} z_k + O(R^2) \tag{5.58}$$

$\mathbf{z} \longrightarrow 0$ as $R \longrightarrow 0$, then $\mathbf{u}(\mathbf{y}) \longrightarrow \mathbf{u}(\mathbf{x})$. And using the asymptotic series of $u_j^{(m)}$ which given in (5.51), we can write

$$\begin{aligned} I_1 &= \rho U u_j(\mathbf{x}) \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) n_1 dS \\ &\approx \rho U u_j(\mathbf{x}) \int\int_{S_\delta} \frac{-1}{8\pi\mu} \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\} n_1 \, dS \\ &\approx \frac{-\rho U u_j(\mathbf{x})}{8\pi\mu} \int\int_{S_\delta} \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\} \frac{z_1}{R} \, dS \\ &\approx \frac{-\rho U u_j(\mathbf{x})}{8\pi\mu} \int\int_{S_\delta} \{\frac{z_1}{R^2} \delta_{jm} + \frac{z_j z_m z_1}{R^4}\} \, dS \end{aligned} \tag{5.59}$$

since $n_1 = \frac{z_1}{R}$. Also, we have $\frac{z_1}{R^2} \, dS = O(R)$ and $\frac{z_j z_m z_1}{R^4} \, dS = O(R)$, thus

$$\rho U u_j(\mathbf{x}) \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) n_1 dS \approx O(R). \tag{5.60}$$

Consequently, as $R \longrightarrow 0$ this integral vanishes.

Obtaining I_2

$$I_2 = \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j dS = \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) p(\mathbf{x} - \mathbf{z}) n_j dS. \tag{5.61}$$

From the Taylor series

$$p(\mathbf{x} - \mathbf{z}) = p(\mathbf{x}) + \frac{\partial p(\mathbf{x})}{\partial z_k} z_k + O(R^2),$$

and from the approximation (5.51) the integral I_2 becomes

$$\begin{aligned} I_2 &= \int\int_{S_\delta} (\frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\})(p(\mathbf{x}) + \frac{\partial p(\mathbf{x})}{\partial z_k} z_k + O(R^2))\frac{z_j}{R} dS \\ &= p(\mathbf{x})(\frac{-1}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}\frac{z_j}{R}dS) + \frac{\partial p(\mathbf{x})}{\partial z_k}(\frac{-1}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}}{R} \\ &+ \frac{z_j z_m}{R^3}\}\frac{z_k z_j}{R}dS) + O(R^2) = O(R) \to 0, \end{aligned} \tag{5.62}$$

where $z_m = O(R)$ and $dS = O(R^2)$.

Obtaining I_3

$$I_3 = \int\int_{S_\delta} u_j(\mathbf{y})p^{(m)}(\mathbf{z})n_j \; dS = \int\int_{S_\delta} u_j(\mathbf{x}-\mathbf{z})p^{(m)}(\mathbf{z})n_j \; dS. \tag{5.63}$$

Similarly to above and using $z_j = R\, n_j$ and the divergence theorem we find

$$\begin{aligned} I_3 &= -\frac{1}{4\pi}\int\int_{S_\delta}(u_j(\mathbf{x}) + \frac{\partial u_j(\mathbf{x})}{\partial z_k} z_k + O(R^2))(\frac{z_m}{R^3})\frac{z_j}{R}dS \\ &= -\frac{u_j(\mathbf{x})}{4\pi}\int\int_{S_\delta}\frac{z_m z_j}{R^4}\; dS + O(R) \\ &= \frac{-u_j(\mathbf{x})}{4\pi}\frac{1}{R^4}\int\int_{S_\delta} z_j z_m \; dS + O(R) \\ &= \frac{-u_j(\mathbf{x})}{4\pi}\frac{1}{R^4}\int\int_{S_\delta} z_m R\, n_j \; dS + O(R) \\ &= \frac{-u_j(\mathbf{x})}{4\pi}\frac{1}{R^3}\int\int\int_V \frac{\partial z_m}{\partial z_j}\; dV + O(R) \\ &= \frac{-u_j(\mathbf{x})}{4\pi}\frac{1}{R^3}\delta_{jm}\frac{4\pi}{3}R^3 + O(R) \\ &= -\frac{u_m(\mathbf{x})}{3}, \end{aligned} \tag{5.64}$$

as $\delta \longrightarrow 0$.

Obtaining I_4

$$\begin{aligned} I_4 &= -\mu \int\int_{S_\delta} u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} \; n_l \; dS \\ &= -\mu \int\int_{S_\delta} (-\frac{1}{8\pi\mu})\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\} \frac{\partial u_j(\mathbf{x}-\mathbf{z})}{\partial y_l} \; n_l \; dS \\ &= \frac{1}{8\pi} \int\int_{S_\delta} \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}(\frac{\partial u_j(\mathbf{x})}{\partial y_l} + \frac{\partial^2 u_j(\mathbf{x})}{\partial y_l \partial z_k} z_k + O(R^2))\frac{z_l}{R} dS \\ &= O(R) \rightarrow 0. \end{aligned} \quad (5.65)$$

The Taylor series for $\frac{\partial u_j(\mathbf{y})}{\partial y_l}$ around $z = 0$ and the approximation (5.51) has been used.

Obtaining I_5

In similar way to I_3, we can take $u_j(y)$ outside the integral to give

$$\begin{aligned} I_5 &= \mu \int\int_{S_\delta} u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} \; n_l \; dS \\ &= \mu u_j(\mathbf{x}) \int\int_{S_\delta} \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} \; n_l \; dS. \end{aligned} \quad (5.66)$$

The approximation formula of $\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l}$ can be obtained from (5.51) the approximation formula of $u_j^{(m)}(\mathbf{z})$, as following

$$u_j^{(m)}(\mathbf{z}) = \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}. \tag{5.67}$$

Differentiating both sides with respect to y_l, gives

$$\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} = \frac{-1}{8\pi\mu}\frac{\partial}{\partial y_l}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}. \tag{5.68}$$

Using $\frac{\partial}{\partial y_l} = -\frac{\partial}{\partial z_l}$, gives

$$\begin{aligned}\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} &= \frac{1}{8\pi\mu}\frac{\partial}{\partial z_l}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\} \\ &= \frac{1}{8\pi\mu}\{\delta_{jm}\frac{\partial}{\partial z_l}(\frac{1}{R}) + \frac{\partial}{\partial z_l}(\frac{z_j z_m}{R^3})\} \\ &= \frac{1}{8\pi\mu}\{\delta_{jm}(-\frac{z_l}{R^3}) + (\frac{z_j}{R^3}\delta_{lm} + \frac{z_m}{R^3}\delta_{lj} - 3\frac{z_j z_m z_l}{R^5})\}.\end{aligned} \tag{5.69}$$

So

$$\begin{aligned} I_5 &= \frac{u_j(\mathbf{x})}{8\pi}\int\int_{S_\delta}\{\delta_{jm}(-\frac{z_l}{R^3}) + (\frac{z_j}{R^3}\delta_{lm} + \frac{z_m}{R^3}\delta_{lj} - 3\frac{z_j z_m z_l}{R^5})\}\frac{z_l}{R}\,dS \\ &= \frac{u_j(\mathbf{x})}{8\pi}\int\int_{S_\delta}\{\delta_{jm}(-\frac{z_l^2}{R^4}) + \frac{z_j z_l}{R^4}\delta_{lm} + \frac{z_m z_l}{R^4}\delta_{lj} + -3\frac{z_j z_m z_l^2}{R^6}\}\,dS \\ &= \frac{u_j(\mathbf{x})}{8\pi}\int\int_{S_\delta}\{-\frac{\delta_{jm}}{R^2} + 2\frac{z_j z_m}{R^4} - 3\frac{z_j z_m}{R^4})\}\,dS \\ &= \frac{u_j(\mathbf{x})}{8\pi}\int\int_{S_\delta}\{-\frac{\delta_{jm}}{R^2} - \frac{z_j z_m}{R^4}\}\,dS \end{aligned} \tag{5.70}$$

this by the use of $n_l = \frac{z_l}{R}$ and $z_m = z_l \delta_{lm}$.

$$\begin{aligned} I_5 &= -\frac{u_j(\mathbf{x})}{8\pi}\{\frac{\delta_{jm}}{R^2}.\int\int_{S_\delta} dS + \frac{1}{R^4}.\int\int_{S_\delta} z_j z_m \, dS\} \\ &= -\frac{u_j(\mathbf{x})}{8\pi}\{\frac{\delta_{jm}}{R^2}.4\pi R^2 + \frac{1}{R^3}.\delta_{jm}\frac{4\pi}{3}R^3\} \\ &= -\frac{u_j(\mathbf{x})}{8\pi}\delta_{jm}(4\pi + \frac{4\pi}{3}) \\ &\rightarrow -\frac{2u_m(\mathbf{x})}{3}, \end{aligned} \tag{5.71}$$

as $R \rightarrow 0$. From (5.60), (5.62), (5.64), (5.65) and (5.71) we find that

$$\begin{aligned} I_{S_\delta} &= I_1 + I_2 + I_3 + I_4 + I_5 = 0 + 0 - \frac{u_m(\mathbf{x})}{3} + 0 - \frac{2u_m(\mathbf{x})}{3} \\ &= -u_m(\mathbf{x}). \end{aligned} \tag{5.72}$$

The Contribution over the Surface S_R as $R \rightarrow \infty$.

Now we consider the far field integral, which is

$$\begin{aligned} I_{S_R} = & \int\int_{S_\delta} \rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 dS + \int\int_{S_R} u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j dS + \int\int_{S_R} u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j dS \\ & -\mu \int\int_{S_R} u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l dS + \mu \int\int_{S_R} u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l dS, \end{aligned} \tag{5.73}$$

where the surface S_R is a sphere with large radius R, as shown in the figure (5.1). Chadwick shows in [30] that the far field integral I_{S_R} is zero since the velocity **u** and the pressure p tend to zero for large R. More calculations are given by Fishwick in [32], [33] to demonstrate that there is no contribution from the integral I_{S_R}. Therefore

$$I_{S_R} = 0. \tag{5.74}$$

Hence, the integral representation of the velocity is

$$u_m(\mathbf{x}) = \int\int_{S_B} \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS, \quad (5.75)$$

as given in [30].

5.6 Integral representation of the force

Here, we give the force on the body as a far field integral, which given by Chadwick in [30]. Letting S_B be the surface of the body and $\mathbf{n}$ be the normal vector pointing outward the surface. Then, the force on the body due to the action of the fluid is

$$F_j = \int\int_{S_B} \tau_{jl}^{\dagger} \, n_l \, dS, \quad (5.76)$$

where for an incompressible fluid

$$\tau_{jl}^{\dagger} = -p^{\dagger}\delta_{jl} + \mu \left(\frac{\partial u_j^{\dagger}}{\partial x_l} + \frac{\partial u_l^{\dagger}}{\partial x_j} \right),$$

is the symmetric Navier-Stokes stress tensor, $\mathbf{u}^{\dagger}$ and $p^{\dagger}$ are the Navier-Stokes velocity and pressure, respectively. So

$$F_j = \int\int_{S_B} \{-p^{\dagger}\delta_{jl} + \mu \left(\frac{\partial u_j^{\dagger}}{\partial x_l} + \frac{\partial u_l^{\dagger}}{\partial x_j} \right) \} n_l \, dS \quad (5.77)$$

The last term can be shown is zero as following

$$\mu \int\int_{S_B} \frac{\partial u_l^\dagger}{\partial x_j} n_l dS = \mu \int\int\int_V \frac{\partial}{\partial x_l}(\frac{\partial u_l^\dagger}{\partial x_j}) dV$$
$$= \mu \int\int\int_V \frac{\partial}{\partial x_j}(\frac{\partial u_l^\dagger}{\partial x_l})\, dV = 0, \quad (5.78)$$

this by applying the divergence theorem, then using the continuity equation $\frac{\partial u_l^\dagger}{\partial x_l} = 0$. Hence

$$F_j = \int\int_{S_B} \{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l}\} n_l\, dS. \quad (5.79)$$

Since on the body the fluid velocity $\mathbf{u}^\dagger$ is equal to the body velocity $\mathbf{u}^B$, which is zero ($\mathbf{u}^\dagger = \mathbf{u}^B = 0$). Therefore, we can add the term $\rho u_l^\dagger u_j^\dagger$,

$$\begin{aligned} F_j &= \int\int_{S_B} \{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l} - \rho u_l^\dagger u_j^\dagger\} n_l dS \\ &= \int\int_{S_R} \{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l} - \rho u_l^\dagger u_j^\dagger\} n_l\, dS \\ &\quad - \int\int\int_V \frac{\partial}{\partial x_l}\{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l} - \rho u_l^\dagger u_j^\dagger\}\, dV, \end{aligned} \quad (5.80)$$

where S_R is a surface in far field region, encloses the body. And the divergence theorem used. Since

$$\begin{aligned} \frac{\partial}{\partial x_l}\{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l} - \rho u_l^\dagger u_j^\dagger\} &= -\frac{\partial p^\dagger}{\partial x_j} + \mu \frac{\partial^2 u_j^\dagger}{\partial x_l \partial x_l} - \rho \frac{\partial u_l^\dagger}{\partial x_l} u_j^\dagger - \rho u_l^\dagger \frac{\partial u_j^\dagger}{\partial x_l} \\ &= -\frac{\partial p^\dagger}{\partial x_j} + \mu \frac{\partial^2 u_j^\dagger}{\partial x_l \partial x_l} - \rho u_l^\dagger \frac{\partial u_j^\dagger}{\partial x_l} = 0 \end{aligned} \quad (5.81)$$

from the Navier-Stokes equations (5.1) and the force becomes

$$F_j = \int\int_{S_R} \{-p^\dagger \delta_{jl} + \mu \frac{\partial u_j^\dagger}{\partial x_l} - \rho u_l^\dagger u_j^\dagger\} n_l\, dS. \quad (5.82)$$

Substitute the Oseen linearisation (5.3) into (5.82) and consider only the terms that contribute to the force and neglecting terms of order ε^2,

$$
\begin{aligned}
F_j &= \int\int_{S_R} \{-p\delta_{jl} + O(\varepsilon) + \mu\frac{\partial}{\partial x_l}(U\delta_{j1} + u_j + O(\varepsilon)) - \rho(U\delta_{l1} + u_l + O(\varepsilon)) \\
&\quad (U\delta_{j1} + u_j + O(\varepsilon))\}n_l dS \\
&= \int\int_{S_R} \{-p\delta_{jl} + \mu\frac{\partial u_j}{\partial x_l} - \rho U u_j \delta_{l1}\}\ n_l\ dS \qquad (5.83)
\end{aligned}
$$

this by the use of $\int\int_{S_R} u_l n_l dS = 0$ which is the no outflow condition (5.9). The force (5.83) is the same result as the result given in [30]

5.7 Force Generated by the Steady Oseenlet

Considering a single oseenlet inside the body and S_δ is a sphere that central at the force point (the oseenlet) and whose radius $\delta \longrightarrow 0$, see figure (5.2). Recalling the force integral representation,

$$
\begin{aligned}
F_j &= \int\int_{S_B} \{-p\delta_{jl} + \mu(\frac{\partial u_j}{\partial x_l} + \frac{\partial u_l}{\partial x_j}) - \rho U u_j \delta_{l1}\}n_l\ dS \\
&= \int\int_{S_B} \{\tau_{jl} - \rho U u_j \delta_{l1}\}n_l\ dS, \qquad (5.84)
\end{aligned}
$$

where u_j and p are the steady Oseen velocity and pressure, τ_{jl} is steady Oseen stress tensor and S_B is the body surface. Applying the divergence theorem for the volume V which is bounded by the surface S_δ and body surface S_B, gives

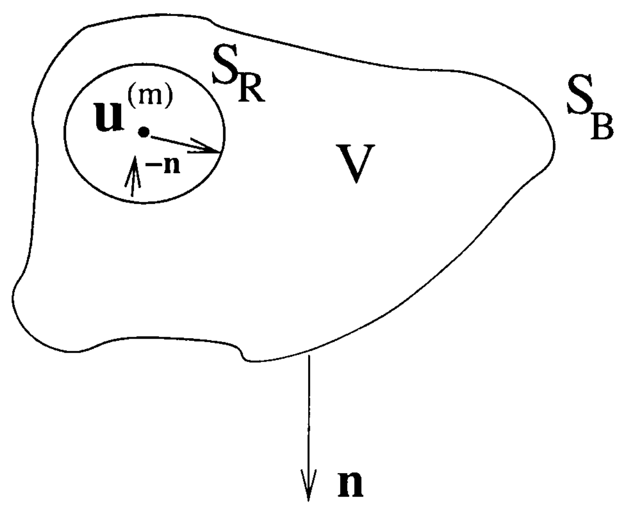

Figure 5.2: Oseenlet inside the body and $R = \delta$

$$F_j = \int\int_{S_\delta} \{\tau_{jl} - \rho U u_j \delta_{l1}\} n_l \, dS + \int\int\int_V \frac{\partial}{\partial x_l}\{\tau_{jl} - \rho U u_j \delta_{l1}\} dV. \tag{5.85}$$

Since

$$\begin{aligned} \frac{\partial}{\partial x_l}\{\tau_{jl} - \rho U u_j \delta_{l1}\} &= \frac{\partial \tau_{jl}}{\partial x_l} - \rho U \frac{\partial u_j}{\partial x_l}\delta_{l1} \\ &= -\frac{\partial p}{\partial x_l}\delta_{jl} + \mu \frac{\partial^2 u_j}{\partial x_l \partial x_l} + \mu \frac{\partial^2 u_l}{\partial x_l \partial x_j} - \rho U \frac{\partial u_j}{\partial x_l}\delta_{l1} \\ &= -\frac{\partial p}{\partial x_l}\delta_{jl} + \mu \frac{\partial^2 u_j}{\partial x_l \partial x_l} - \rho U \frac{\partial u_j}{\partial x_l}\delta_{l1} \\ &= \rho U \frac{\partial u_j}{\partial x_1} - \rho U \frac{\partial u_j}{\partial x_1} = 0, \end{aligned} \tag{5.86}$$

this from the continuity equation and the steady Oseen equation

$$\rho U \frac{\partial u_j}{\partial x_1} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_l \partial x_l}.$$

Therefore, the volume integral vanishes and the total force becomes

$$F_j = \int\int_{S_\delta} \{\tau_{jl} - \rho U u_j \delta_{l1}\}\, n_l\, dS. \tag{5.87}$$

Substituting the steady oseenlets $u_j^{(m)}$ and $p^{(m)}$ into (5.87) gives the force

$$F_j^{(m)} = \int\int_{S_\delta} \{\tau_{jl}^{(m)} - \rho U u_j^{(m)} \delta_{l1}\}\, n_l\, dS, \tag{5.88}$$

where

$$\tau_{jl}^{(m)} = -p^{(m)}\delta_{jl} + \mu(\frac{\partial u_j^{(m)}}{\partial x_l} + \frac{\partial u_l^{(m)}}{\partial x_j}).$$

Using the approximation series of $u_j^{(m)}$ around zero, which is

$$u_j^{(m)}(\mathbf{z}) = \frac{1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}, \tag{5.89}$$

gives

$$F_j^{(m)} = \int\int_{S_\delta} \tau_{jl}^{s(m)}\, n_l\, dS - \frac{\rho U}{8\pi\mu}\int\int_{S_\delta} \{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3} + O(1)\}\, n_1\, dS. \tag{5.90}$$

Since oseenlet tends to the steady stokeslet around zero. Hence, at the limit $\delta \to 0$, Oseen stress tensor tends to steady Stokes stress tensor $\tau_{jl}^{(m)} \to \tau_{jl}^{s(m)}$. The last integral in (5.90) is of order $O(\delta)$, which gives zero contribution when $\delta \to 0$. And in section 3.5.1, we demonstrate that

$$\int\int_{S_\delta} \tau_{jl}^{s(m)}\, n_l\, dS = \delta_{jm}.$$

Therefore

$$F_j^{(m)} = \delta_{jm}. \tag{5.91}$$

5.8 Conclusion

The results in this chapter are well-known and we give them in more details. We seek to use this chapter to infer a form of the oscillatory oseenlets in next chapter. Force generated by the steady oseenlets is shown to give unit force in the direction of the point force as accepted.

Chapter 6

Oscillatory Oseen Flow

6.1 Introduction

The steady and transient oseenlets are currently available in the literature [30], [16]. The omission of the oscillatory oseenlets representation within the literature is significant. In this chapter, the oscillatory oseenlet solution for velocity and pressure are presented. Furthermore, the force generated by them is presented and the reduction to the steady oseenlets and oscillatory stokeslets in appropriate limits are given. We consider a uniform flow U past an oscillating body with velocity $\mathbf{u}$, see figure (6.1).

The far-field is assumed to consist of both steady and time periodic components $\mathbf{u} = \mathbf{u}_s(\mathbf{x}) + \mathbf{u}_t(\mathbf{x}, t)$. The time periodic component $\mathbf{u}_t(\mathbf{x}, t)$ can be decomposed into a Fourier expansion series of time-harmonic components. The steady component in terms of the steady oseenlets is well-known [13]. However, the time-harmonic components in terms of the oscillatory oseenlets do not yet appear to be in the literature.

This chapter is structured as follows. In the following section two the statement of the problem and the governing equations of oscillatory Oseen flow are given. The Green's surface integral representation of the oscillatory Oseen equation is placed in the third section.

In section four, the Lamb-Goldstein decomposition of the velocity in terms of the potentials ϕ and χ , which is introduced by Lamb [21], is used to obtain the oscillatory oseenlets for the velocity and pressure. Then, we demonstrate that the new oscillatory oseenlets reduce to the steady oseenlets, which are given in [30], and the oscillatory stokeslets, which are given in [9]. Furthermore, the Pozrikidis' form of the oscillatory oseenlets is given.

In section five, we present the integral representation of the oscillatory Oseen velocity and we expand the oscillatory oseenlets around zero. In section six, the force generated by the oscillatory oseenlets is given in terms of the velocity, pressure and the frequency.

6.2 Governing Equations

The time-harmonic Oseen equations are obtained by applying Oseen's approximation to the time-dependent Navier-Stokes equations, which are

$$\rho\frac{\partial u_j^\dagger}{\partial t} + \rho u_l^\dagger\frac{\partial u_j^\dagger}{\partial x_l} = -\frac{\partial p^\dagger}{\partial x_j} + \mu\frac{\partial^2 u_j^\dagger}{\partial x_l \partial x_l}, \tag{6.1}$$

where $u_j^\dagger$ is the velocity component in the j direction of a Cartesian coordinate system x_j, $j, l = 1,\ 2,\ 3$; $p^\dagger$ is the fluid pressure; t denotes time; ρ is the fluid density; and μ is the fluid viscosity.

Assuming that in the far-field the flow tends toward a uniform stream U in the x_1 direction, and that the fluid velocity and pressure can be represented by a steady component together with a time-periodic component, then applying a Fourier expansion suppose the velocity and pressure have the form

$$\begin{aligned} u_j^\dagger &= U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t} \\ p^\dagger &= p_0 + \sum_{n=-\infty}^{\infty} p^n e^{i\omega_n t}, \end{aligned} \tag{6.2}$$

where i is the imaginary number $\sqrt{-1}$, the frequency $\omega_n = \frac{2n\pi}{T}$, T is the time period of the motion and δ_{ij} is the Kronecker delta ($\delta_{jl} = 1$ when $j = l$ and zero otherwise). Since (6.1) represents real variables, then $u_j^n = \bar{u}_j^{(-n)}$ and $p^n = \bar{p}^{(-n)}$ where the bar denotes the complex conjugate. Linearising the Navier-Stokes equation to a uniform stream U by using the form (6.2) and assuming that the Oseen approximation $|\frac{\mathbf{u}}{U}|$, $|\frac{p}{p_0}| = O(\varepsilon)$ and $\varepsilon \ll 1$ holds, where the notation "O 'means 'of order of ', yield the time-harmonic Oseen equations which are

$$i\omega\rho u_j + \rho U \delta_{l1} \frac{\partial u_j}{\partial x_l} = -\frac{\partial p}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_l \partial x_l}, \tag{6.3}$$

where $\mathbf{u}$ and p are the Oseen velocity and pressure, respectively. Since the Navier-Stokes velocity $\mathbf{u}^\dagger$ satisfies the continuity equation $\nabla \cdot \mathbf{u}^\dagger = 0$, then the oscillatory Oseen velocity $\mathbf{u}$ does as well. Also, the pressure is a harmonic function which is seen by taking the divergence of the oscillatory Oseen's equation (6.3), given

$$i\omega\rho \frac{\partial u_j}{\partial x_j} + \rho U \frac{\partial}{\partial x_j} \frac{\partial u_j}{\partial x_1} = -\frac{\partial}{\partial x_j} \frac{\partial p}{\partial x_j} + \mu \frac{\partial}{\partial x_j} \frac{\partial^2 u_j}{\partial x_l \partial x_l}.$$

Re-writing in operator form, gives

$$i\omega\rho\nabla \cdot \mathbf{u} + \rho U \frac{\partial}{\partial x_1} \nabla \cdot \mathbf{u} = -\nabla^2 p + \mu \nabla^2 . \nabla \cdot \mathbf{u}. \tag{6.4}$$

However $\nabla \cdot \mathbf{u} = 0$ and so

$$\nabla^2 p = 0, \tag{6.5}$$

where ∇^2 is the Laplacian operator, therefore the pressure is harmonic and satisfies the Laplace equation. As we move further away from the disturbance created by the oscillating body we assume that $\mathbf{u}$ tends to zero, taking the oscillatory Oseen's equation (6.3) to infinity and applying the assumption $\mathbf{u} \longrightarrow 0$ at infinity, yields

$$\nabla p = 0. \tag{6.6}$$

Thus, we may choose $p \longrightarrow 0$ at infinity.

6.3 Integral Representation of the Oseen Equations

Following the Green's integral formulation as given by Oseen in [13], except applying it to the oscillatory rather than steady or transient case and in the similar way to the Green's integral representation of the oscillatory Stokes equation in chapter 4, we obtain the integral representation for the oscillatory Oseen equations. We consider four solutions for the velocity and pressure field given by $(\mathbf{u}(\mathbf{x}), p(\mathbf{x}))$ and $(\mathbf{u}^{(m)}(\mathbf{z}), p^{(m)}(\mathbf{z}))$, where $m = 1,\ 2,\ 3$. The first solution refers to a general velocity and pressure field, and the subsequent solutions refer to the specific Green's functions that satisfy a Green's integral which we shall construct. As in the previous chapter, consider distinct Cartesian coordinates y_j and $z_j = x_j - y_j$. The coordinate **y** parameterises a point on or within a fixed closed surface and the coordinate **x** refers to a general fluid point.

The four solutions then satisfy the oscillatory Oseen equation

$$i\omega\rho u_j(\mathbf{y}) + \rho U\delta_{l1}\frac{\partial u_j(\mathbf{y})}{\partial y_l} = -\frac{\partial p(\mathbf{y})}{\partial y_j} + \mu\frac{\partial^2 u_j(\mathbf{y})}{\partial y_l \partial y_l}, \tag{6.7}$$

which is the Oseen equation in variable y, and

$$i\omega\rho u_j^{(m)}(\mathbf{z}) + \rho U\delta_{l1}\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial z_l} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j} + \mu\frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l}. \tag{6.8}$$

Since $\mathbf{z} = \mathbf{x} - \mathbf{y}$, then the adjoint equation in **y** gives

$$i\omega\rho u_j^{(m)}(\mathbf{z}) - \rho U\delta_{l1}\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} = \frac{\partial p^{(m)}(\mathbf{z})}{\partial y_j} + \mu\frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial y_l \partial y_l}, \tag{6.9}$$

since $\frac{\partial}{\partial z_j} = -\frac{\partial}{\partial y_j}$. Following the method of Oseen [13] to obtain the Green's functions representation, we dot product (6.7) with $u_j^{(m)}(\mathbf{z})$ and take it from the dot product of (6.9) with $u_j(\mathbf{y})$, and find

$$
\begin{aligned}
-\rho U u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_1} - \rho U u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_1} &= u_j(\mathbf{y})\frac{\partial p^{(m)}(\mathbf{z})}{\partial y_j} + u_j^{(m)}(\mathbf{z})\frac{\partial p(\mathbf{y})}{\partial y_j} \\
&+\mu\big[u_j(\mathbf{y})\frac{\partial^2 u_j^{(m)}(\mathbf{z})}{\partial y_l \partial y_l} - u_j^{(m)}(\mathbf{z})\frac{\partial^2 u_j(\mathbf{y})}{\partial y_l \partial y_l}\big].
\end{aligned}
\tag{6.10}
$$

Using the continuity equation enables us to write

$$
\begin{aligned}
-\rho U\frac{\partial}{\partial y_1}(u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})) &= \frac{\partial}{\partial y_j}[u_j(\mathbf{y})p^{(m)}(\mathbf{z}) + u_j^{(m)}(\mathbf{z})p(\mathbf{y})] \\
&+\mu\frac{\partial}{\partial y_l}[u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} - u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l}] = 0.
\end{aligned}
\tag{6.11}
$$

So

$$
\begin{aligned}
\rho U\frac{\partial}{\partial y_1}(u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})) &= -\frac{\partial}{\partial y_j}[u_j(\mathbf{y})p^{(m)}(\mathbf{z}) + u_j^{(m)}(\mathbf{z})p(\mathbf{y})] \\
&+\mu\frac{\partial}{\partial y_l}[u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l} - u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l}] = 0.
\end{aligned}
\tag{6.12}
$$

This holds within a volume V of fluid bounded by the surface S where the Oseen approximation is vaild, and parameterised by the coordinate $\mathbf{y}$. Applying the divergence theorem leads to

$$\int\int_S \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS = 0. \quad (6.13)$$

This is the Green's surface integral representation of the time-harmonic Oseen equations.

We can see that the integral representation of the oscillatory Stokes equations (4.12) can be obtained by letting $U \to 0$ in (6.13), also we note that the representation of the steady Oseen equation (5.16) is identical to (6.13), since the oscillatory part of the governing differential equation which includes the frequency has cancelled.

6.4 Oscillatory Oseenlets

In this section, we obtain the Green's functions $u_j^{(m)}(\mathbf{z})$ and $p^{(m)}(\mathbf{z})$ for the oscillatory Oseen equations by using the Lamb-Goldstein velocity decomposition [21], which decompose the fluid velocity into a potential $\phi(\mathbf{z})$ and a wake velocity $w(\mathbf{z})$, such that

$$u_j^{(m)}(\mathbf{z}) = \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_j} + w_j^{(m)}(\mathbf{z}). \quad (6.14)$$

As we consider incompressible flow, the velocity potential ϕ has to satisfy the Laplace equation

$$\nabla^2 \phi(\mathbf{z}) = 0. \quad (6.15)$$

Applying the divergence vector to the decomposition (6.14) and using $\nabla \cdot \mathbf{u}^{(m)}(\mathbf{z}) = 0$ and $\nabla^2 \phi(\mathbf{z}) = 0$ shows the wake velocity satisfies the continuity equation

$$\nabla \cdot \mathbf{w}^{(m)}(\mathbf{z}) = 0. \tag{6.16}$$

Substituting (6.14) into (6.8) gives

$$\begin{aligned}(i\rho\omega\frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_j} + \rho U\frac{\partial^2\phi^{(m)}(\mathbf{z})}{\partial z_1 \partial z_j} - \mu\frac{\partial^3\phi^{(m)}(\mathbf{z})}{\partial z_l \partial z_l \partial z_j}) + (i\rho\omega w_j^{(m)}(\mathbf{z})\\ + \rho U\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} - \mu\frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l}) = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j}.\end{aligned} \tag{6.17}$$

From (6.15) $\frac{\partial^2\phi^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} = 0$, then

$$\begin{aligned}(i\rho\omega\frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_j} + \rho U\frac{\partial^2\phi^{(m)}(\mathbf{z})}{\partial z_j \partial z_1}) + (i\rho\omega w_j^{(m)}(\mathbf{z}) + \rho U\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1}\\ - \mu\frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l}) = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j}.\end{aligned} \tag{6.18}$$

Since $\nabla^2 p^{(m)}(\mathbf{z}) = 0$, a particular solution is obtained if we choose

$$i\rho\omega\frac{\partial\phi^{(m)}(\mathbf{z})}{\partial z_j} + \rho U\frac{\partial^2\phi^{(m)}(\mathbf{z})}{\partial z_j \partial z_1} = -\frac{\partial p^{(m)}(\mathbf{z})}{\partial z_j}. \tag{6.19}$$

Integrating gives the pressure to be

$$p^{(m)}(\mathbf{z}) = -(i\rho\omega\phi^{(m)}(\mathbf{z}) + \rho U \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_1}). \tag{6.20}$$

This choice enables us to remove the pressure term from the equation (6.18), which becomes

$$i\rho\omega w_j^{(m)}(\mathbf{z}) + \rho U \frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} - \mu \frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} = 0. \tag{6.21}$$

From (6.20), the potential $\phi^{(m)}(\mathbf{z})$ satisfies the equation

$$\rho U \frac{\partial}{\partial z_1}\left(\phi^{(m)}(\mathbf{z}) e^{i\omega z_1/U}\right) = -p^{(m)}(\mathbf{z}) e^{i\omega z_1/U}. \tag{6.22}$$

Since the pressure satisfies the Laplace equation, and in the low Reynold number limit the pressure associated to the oscillatory oseenlets must tend to pressure associated with the oscillatory stokeslets, then we infer they must be the same. Therefore, the pressure for the oscillatory oseenlet is given by (4.19) which is

$$p^{(m)}(\mathbf{z}) = \frac{1}{4\pi} \frac{\partial}{\partial z_m} \left(\frac{1}{R}\right), \tag{6.23}$$

where the radial distance from the oseenlet singularity is given by $R = |\mathbf{z}|$. Near the point $z = 0$ the pressure $p^{(m)}(\mathbf{z})$ is unbounded and integrating directly the equation (6.22) produces an indeterminate integral. However, we may remove the term producing the singularity within the integrand in order to represent $\phi^{(m)}$ by a determinate integral. This gives

$$
\begin{aligned}
\phi^{(m)}(\mathbf{z}) &= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}(\frac{1}{R'})dz_1' \\
&= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1}\left\{e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}(\frac{1}{R'}) - \frac{\partial}{\partial z_m'}(\frac{1}{R'}) + \frac{\partial}{\partial z_m'}(\frac{1}{R'})\right\}dz_1' \\
&= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\left\{\int_{-\infty}^{z_1}(e^{i\omega z_1'/U}-1)\frac{\partial}{\partial z_m'}(\frac{1}{R'})dz_1' + \int_{-\infty}^{z_1}\frac{\partial}{\partial z_m'}(\frac{1}{R'})dz_1'\right\}.
\end{aligned}
\tag{6.24}
$$

Since $\frac{\partial}{\partial z_1}\ln(R-z_1) = \frac{-1}{R}$,

$$
\phi^{(m)}(\mathbf{z}) = -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\left\{\int_{-\infty}^{z_1}(e^{i\omega z_1'/U}-1)\frac{\partial}{\partial z_m'}(\frac{1}{R'})dz_1' - \frac{\partial}{\partial z_m}\ln(R-z_1)\right\},
\tag{6.25}
$$

where $z_2' = z_2$, $z_3' = z_3$, z_1' is the dummy integration variable, and $R' = |\mathbf{z}'|$. For brevity, we represent this integral by

$$
\phi^{(m)}(\mathbf{z}) = -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\,⨎_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{1}{R'}\right)dz_1', \tag{6.26}
$$

where the integral sign $⨎$ implies the removal of the singularity in the integration. The wake velocity satisfies the equation (6.21), which can be re-written as

$$
\frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} - \frac{i\rho\omega}{\mu}w_j^{(m)}(\mathbf{z}) - \frac{\rho U}{\mu}\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} = 0. \tag{6.27}
$$

Letting $w_j^{(m)}(\mathbf{z}) = w_j^{*(m)}(\mathbf{z})e^{kz_1}$ where $k = \rho U/(2\mu)$ gives

$$\begin{aligned}
\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_1} &= \frac{\partial w_j^{*(m)}(\mathbf{z})}{\partial z_1} e^{kz_1} + k w_j^{*(m)}(\mathbf{z}) e^{kz_1} \\
\frac{\partial w_j^{(m)}(\mathbf{z})}{\partial z_l} &= \frac{\partial w_j^{*(m)}(\mathbf{z})}{\partial z_l} e^{kz_1} + k\delta_{l1} w_j^{*(m)}(\mathbf{z}) e^{kz_1} \\
\frac{\partial^2 w_j^{(m)}(\mathbf{z})}{\partial z_l \partial z_l} &= \frac{\partial^2 w_j^{*(m)}(\mathbf{z})}{\partial z_l \partial z_l} e^{kz_1} + 2k\delta_{l1} \frac{\partial w_j^{*(m)}(\mathbf{z})}{\partial z_l} e^{kz_1} + k^2 \delta_{l1} w_j^{*(m)}(\mathbf{z}) e^{kz_1}.
\end{aligned} \tag{6.28}$$

Substituting the derivatives into (6.27) yields

$$\frac{\partial^2 w_j^{*(m)}(\mathbf{z})}{\partial z_l \partial z_l} - k^{*2} w_j^{*(m)} = 0. \tag{6.29}$$

Where $k^{*2} = k^2 + h^2$, and h is defined in section 4.3 as $h = \sqrt{\frac{i\rho\omega}{\mu}}$. Solutions to this equation are given by [31]. We look for a solution that reduces to the oscillatory stokeslet in the limit as $\mathbf{U} \longrightarrow 0$, given by solutions of the type

$$w_j^{(m)}(\mathbf{z}) = \frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_j} - \chi^*(\mathbf{z})\delta_{jm} \tag{6.30}$$

where $\chi^*(\mathbf{z})e^{-kz_1}$ satisfies the heat conduction equation (6.29). We pick a solution for χ^* in [31] such that it reduces to the oscillatory stokeslet in the limit as $\mathbf{z} \longrightarrow 0$. This is given by

$$\chi^*(\mathbf{z}) = \frac{2k}{4\pi\rho U} \frac{e^{-k^* R} e^{kz_1}}{R}. \tag{6.31}$$

From the continuity equation $\frac{\partial w_j^{(m)}}{\partial z_j} = 0$, it follows that $\frac{\partial^2 \chi^{(m)}(\mathbf{z})}{\partial z_j \partial z_j} - \frac{\partial \chi^*(\mathbf{z})}{\partial z_m} = 0$, and from (6.27),

$$\frac{\rho i\omega}{\mu}\chi^{(m)}(\mathbf{z}) + 2k\frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_1} = \frac{\partial^2 \chi^{(m)}(\mathbf{z})}{\partial z_j \partial z_j} = \frac{\partial \chi^*(\mathbf{z})}{\partial z_m}, \tag{6.32}$$

then $\chi^{(m)}$ is given by integrating the equation

$$2k\frac{\partial}{\partial z_1}\left(\chi^{(m)}(\mathbf{z})e^{i\omega z_1/U}\right) = e^{i\omega z_1/U}\frac{\partial \chi^*(\mathbf{z})}{\partial z_m}. \tag{6.33}$$

However, in a similar way as for $\phi^{(m)}(\mathbf{z})$, integrating this equation directly produces an indeterminate integral. Removing the singularity term gives the determinate integral

$$\chi^{(m)}(\mathbf{z}) = \frac{e^{-i\omega z_1/U}}{4\pi\rho U}\left\{\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^*R'}e^{kz_1'}}{R'}\right) - \frac{\partial}{\partial z_m'}\left(\frac{e^{-k(R'-z_1')}}{R'}\right)dz_1' - e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right\}. \tag{6.34}$$

For brevity, we represent this integral by

$$\chi^{(m)}(\mathbf{z}) = \frac{e^{-i\omega z_1/U}}{4\pi\rho U}\,⨍_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^*R'}e^{kz_1'}}{R'}\right)dz_1', \tag{6.35}$$

where in a similar way to the representation for $\phi^{(m)}$, the integral sign ⨍ implies the removal of the singularity in the integration.

So the complete solution for the harmonic oscillatory oseenlets is given by

$$\begin{aligned}
u_j^{(m)}(\mathbf{z}) &= \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_j} + \frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_j} - \chi^*(\mathbf{z})\delta_{jm} \,, \\
\phi^{(m)}(\mathbf{z}) &= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U} \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial}{\partial z_m'} \left(\frac{1}{R'}\right) dz_1' \,, \\
\chi^{(m)}(\mathbf{z}) &= \frac{e^{-i\omega z_1/U}}{4\pi\rho U} \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial}{\partial z_m'} \left(\frac{e^{-k^*R'} e^{kz_1'}}{R'}\right) dz_1' \,, \\
\chi^*(\mathbf{z}) &= \frac{2k}{4\pi\rho U} \frac{e^{-k^*R} e^{kz_1}}{R} \,, \\
p^{(m)}(\mathbf{z}) &= \frac{1}{4\pi} \frac{\partial}{\partial z_m} \left(\frac{1}{R}\right),
\end{aligned} \tag{6.36}$$

where $k = \rho U/(2\mu)$, and $k^* = \sqrt{k^2 + h^2}$. Expanding the derivatives enables us to re-write the oscillatory oseenlets in the following form

$$u_j^{(m)}(\mathbf{z}) = \frac{-1}{4\pi\rho U} \{ e^{-i\omega z_1/U} [\fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial^2}{\partial z_j' \partial z_m'} \left(\frac{1}{R'}\right) dz_1$$
$$- \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial^2}{\partial z_j' \partial z_m'} \left(\frac{e^{-k^*R' + kz_1'}}{R'}\right) dz_1] + 2k \left(\frac{e^{-k^*R + kz_1}}{R}\right) \delta_{jm} \}. \tag{6.37}$$

6.4.1 Oscillatory Oseenlets and known solutions

We can check the oscillatory oseenlets in the two limiting cases, as $\omega \to 0$ which reduces to the steady oseenlets, and as $U \to 0$ which reduces to the oscillatory stokeslets.

Case $\omega \to 0$

When $\omega \to 0$, then $k^* \to k$, $\phi^{(m)}$ reduces to

$$\begin{aligned}
\phi^{(m)}\big|_{\omega \to 0} &= -\frac{1}{4\pi\rho U} \fint_{-\infty}^{z_1} \frac{\partial}{\partial z_m'} \left(\frac{1}{R'}\right) dz_1' \\
&= \frac{1}{4\pi\rho U} \frac{\partial}{\partial z_m} \ln(R - z_1)
\end{aligned} \tag{6.38}$$

since $1/R = -(\partial/\partial z_1)\ln(R - z_1)$, which is the steady oseenlet solution for $\phi^{(m)}$, which given in (5.26).

Similarly, $\chi^{(m)}$ reduces to

$$\begin{aligned} \chi^{(m)}\big|_{\omega\to 0} &= \frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k(R'-z_1')}}{R'}\right)dz_1' \\ &= -\frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_m'}\left(e^{-k(R'-z_1')}\frac{\partial}{\partial z_1'}\ln(R'-z_1')\right)dz_1' \\ &= -\frac{1}{4\pi\rho U}\int_{-\infty}^{z_1}\frac{\partial}{\partial z_1'}\left(e^{-k(R'-z_1')}\frac{\partial}{\partial z_m'}\ln(R'-z_1')\right)dz_1' \\ &= -\frac{1}{4\pi\rho U}e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1), \end{aligned} \tag{6.39}$$

which is the steady oseenlet solution for $\chi^{(m)}$, as in (5.40) and [13], and we used (5.44) which is

$$\frac{\partial}{\partial z_m}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_1}\ln(R-z_1)\right) = \frac{\partial}{\partial z_1}\left(e^{-k(R-z_1)}\frac{\partial}{\partial z_m}\ln(R-z_1)\right). \tag{6.40}$$

Finally, χ^* reduces to

$$\chi^*\big|_{\omega\to 0} = \frac{2k}{4\pi\rho U}\frac{e^{-k(R-z_1)}}{R} \tag{6.41}$$

which is (5.34) the steady oseenlet solution for χ^*. Therefore, the oscillatory oseenlet solution reduces to the steady oseenlet solution in the limit as $\omega \to 0$.

Case $U \to 0$

When $U \to 0$, then $k \to 0$ and $k^* \to h$, $\phi^{(m)}$ reduces to

$$\begin{aligned}
\phi^{(m)}\big|_{U\to 0} = &= -\lim_{U\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\,⨍_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{1}{R'}\right)dz_1' \\
&= -\lim_{U\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right)\frac{e^{i\omega z_1/U}}{i\omega/U} \\
&= -\lim_{U\to 0}\frac{1}{4\pi i\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right) \\
&= \frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{1}{R}\right)
\end{aligned} \tag{6.42}$$

which is the oscillatory stokeslet solution for $\phi^{s(m)}$, given in (4.22). Similarly, $\chi^{(m)}$ reduces to

$$\begin{aligned}
\chi^{(m)}\big|_{U\to 0} = &= \lim_{U\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\,⨍_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-hR'}}{R'}\right)dz_1' \\
&= \lim_{U\to 0}\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right)\frac{e^{i\omega z_1/U}}{i\omega/U} \\
&= \lim_{U\to 0}\frac{1}{4\pi i\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right) \\
&= -\frac{i}{4\pi\rho\omega}\frac{\partial}{\partial z_m}\left(\frac{e^{-hR}}{R}\right)
\end{aligned} \tag{6.43}$$

which is the oscillatory stokeslet solution for $\chi^{s(m)}$, given in (4.26). Finally, χ^* reduces to

$$\begin{aligned}
\chi^*|_{U\to 0} &= \lim_{U\to 0}\frac{2k}{4\pi\rho U}\frac{e^{-hR}}{R} \\
&= \frac{ih^2}{4\pi\rho\omega}\frac{e^{-hR}}{R}
\end{aligned} \tag{6.44}$$

which is the oscillatory stokeslet solution for χ^{s*}, given in (4.27). Therefore, the oscillatory oseenlet solution reduces to the oscillatory stokeslet solution in the limit as $U \to 0$.

6.4.2 Oscillatory Oseenlets in Pozrikidis' form

In order to re-write the oscillatory oseenlets in similar form to that Pozirkidis has given for the oscillatory stokeslets [9] [25], we first expand the derivatives,

$$
\begin{aligned}
&\frac{\partial}{\partial z_m}(\frac{1}{R}) = -\frac{z_m}{R^3} \\
&\frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R}) = \frac{\delta_{jm}}{R}(-\frac{1}{R^2}) + \frac{z_m z_j}{R^3}(\frac{3}{R^2}) \\
&\frac{\partial}{\partial z_m}(\frac{e^{-k^*R+kz_1}}{R}) = \{-k^*\frac{z_m}{R^2} + k\frac{\delta_{m1}}{R} - \frac{z_m}{R^3}\}e^{-k^*R+kz_1} \\
&\frac{\partial^2}{\partial z_j \partial z_m}(\frac{e^{-k^*R+kz_1}}{R}) = \{\frac{\delta_{jm}}{R}(\frac{-k^*}{R} - \frac{1}{R^2} + k^2) + \frac{z_m z_j}{R^3}(\frac{3k^*}{R} + \frac{3}{R^2} + k^{*2}) \\
&+k(\delta_{m1}\frac{z_j}{R^2} + \delta_{j1}\frac{z_m}{R^2})(-\frac{1}{R} - k^*)\}e^{-k^*R+kz_1},
\end{aligned}
\tag{6.45}
$$

$$
\begin{aligned}
\frac{\partial \phi^{(m)}}{\partial z_j} &= -\frac{1}{4\pi\rho U}\{\frac{\partial}{\partial z_j}(e^{-i\omega z_1/U}) \int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{1}{R'}\right) dz_1' \\
&+e^{-i\omega z_1/U}\frac{\partial}{\partial z_j}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{1}{R'}\right) dz_1'\} \\
&= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\{\int_{-\infty}^{z_1} e^{i\omega z_1'/U}[(-\frac{i\omega}{U})\delta_{j1}\frac{\partial}{\partial z_m'}\left(\frac{1}{R'}\right) \\
&+(\frac{i\omega}{U})\delta_{j1}\frac{\partial}{\partial z_m'}(\frac{1}{R'}) + \frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R})]dz_1'\} \\
&= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial^2}{\partial z_j \partial z_m}(\frac{1}{R})]dz_1' \\
&= -\frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\{\frac{\delta_{jm}}{R'}(\frac{-1}{R'^2}) + \frac{z_m' z_j'}{R'^3}(\frac{3}{R'^2})\}dz_1',
\end{aligned}
\tag{6.46}
$$

and

$$
\begin{aligned}
\frac{\partial \chi^{(m)}}{\partial z_j} &= \frac{1}{4\pi\rho U}\{\frac{\partial}{\partial z_j}(e^{-i\omega z_1/U}) \int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^* R' + k z_1'}}{R'}\right) dz_1' \\
&+e^{-i\omega z_1/U}\frac{\partial}{\partial z_j}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^* R' + k z_1'}}{R'}\right) dz_1'\} \\
&= \frac{1}{4\pi\rho U}\{(-\frac{i\omega}{U})\delta_{j1}e^{-i\omega z_1/U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^* R' + k z_1'}}{R'}\right) dz_1' \\
&+e^{-i\omega z_1/U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}[(\frac{i\omega}{U})\delta_{j1}\frac{\partial}{\partial z_m'}\left(\frac{e^{-k^* R' + k z_1'}}{R'}\right) + \frac{\partial^2}{\partial z_j \partial z_m}(\frac{e^{-k^* R + k z_1}}{R})]dz_1'\} \\
&= \frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\frac{\partial^2}{\partial z_j \partial z_m}(\frac{e^{-k^* R + k z_1}}{R})dz_1' \\
&= \frac{e^{-i\omega z_1/U}}{4\pi\rho U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}\{\frac{\delta_{jm}}{R}(\frac{-k^*}{R} - \frac{1}{R^2} + k^2) + \frac{z_m z_j}{R^3}(\frac{3k^*}{R} + \frac{3}{R^2} + k^{*2}) \\
&+k(\delta_{m1}\frac{z_j}{R^2} + \delta_{j1}\frac{z_m}{R^2})(-\frac{1}{R} - k^*)\}e^{-k^* R + k z_1}dz_1'.
\end{aligned}
\tag{6.47}
$$

Substituting the above derivatives into (6.37) gives

$$
\begin{aligned}
u_j^{(m)}(\mathbf{z}) &= \frac{\partial \phi^{(m)}(\mathbf{z})}{\partial z_j} + \frac{\partial \chi^{(m)}(\mathbf{z})}{\partial z_j} - \chi^*(\mathbf{z})\delta_{jm} \\
&= -\frac{1}{4\pi\rho U}\{2k\frac{\delta_{jm}}{R}e^{-k^*R+kz_1} + e^{-i\omega z_1/U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\delta_{jm}}{R'}(\frac{-1}{R'^2} + e^{-k^*R'+kz_1} \\
& (\frac{k^*}{R'} + \frac{1}{R'^2} - k^2)) + \frac{z_m'z_j'}{R'^3}(\frac{3}{R'^2} - e^{-k^*R'+kz_1}(\frac{3k^*}{R'} + \frac{3}{R'^2} + k^{*2})) \\
& -ke^{-k^*R'+kz_1'}(\delta_{m1}\frac{z_j'}{R'^2} + \delta_{j1}\frac{z_m'}{R'^2})(-\frac{1}{R'} - k^*)]dz_1'\}.
\end{aligned}
\tag{6.48}
$$

The coefficient $\frac{1}{4\pi\rho U} = \frac{1}{8\pi\mu k}$ as $\rho U = 2\mu k$.

$$
\begin{aligned}
u_j^{(m)}(\mathbf{z}) &= \frac{-1}{8\pi\mu}\{2\frac{\delta_{jm}}{R}e^{-k^*R+kz_1} + e^{-i\omega z_1/U}\int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\delta_{jm}}{kR'}(\frac{-1}{R'^2} + \\
& (\frac{k^*}{R'} + \frac{1}{R'^2} - k^2)e^{-k^*R'+kz_1}) + \frac{z_m'z_j'}{kR'^3}(\frac{3}{R'^2} - (\frac{3k^*}{R'} + \frac{3}{R'^2} + k^{*2})e^{-k^*R'+kz_1}) \\
& -(\delta_{m1}\frac{z_j'}{R'^2} + \delta_{j1}\frac{z_m'}{R'^2})(-\frac{1}{R'} - k^*)e^{-k^*R'+kz_1'}]dz_1'\}.
\end{aligned}
\tag{6.49}
$$

Taking the limit $U \to 0$, reduces it to the oscillatory stokeslets in Pozrikidis' form (4.36).

6.5 Integral Representation of oscillatory Oseen Velocity

In order to obtain the integral representation of the time-harmonic Oseen velocity, we first approximate the oscillatory oseenlets around the point $z = 0$.

6.5.1 Green's Integral of Oscillatory Oseenlet around point $z = 0$

The oscillatory oseenlets are

$$u_j^{(m)}(\mathbf{z}) = \frac{-1}{4\pi\rho U}\{e^{-i\omega z_1/U} \int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{1}{R'}\right)$$
$$-\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{e^{-k^*R'+kz_1'}}{R'}\right)]dz_1 + 2k\left(\frac{e^{-k^*R+kz_1}}{R}\right)\delta_{jm}\}. \qquad (6.50)$$

The Taylor series of $e^{-k^*R+kz_1}$ around the point $z = 0$, is

$$\begin{aligned} e^{-k^*R+kz_1} &= 1-(-k^*R+kz_1)+\frac{(-k^*R+kz_1)^2}{2!}+\ldots. \\ &= 1-k^*R+\frac{k^{*2}R^2}{2}+O(R^3). \end{aligned} \qquad (6.51)$$

By substituting this series into the oscillatory oseenlets we find

$$\begin{aligned} u_j^{(m)}(\mathbf{z}) &\approx \frac{-1}{4\pi\rho U}\{e^{-i\omega z_1/U} \int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{1}{R'}\right) \\ &-\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{1-k^*R'+\frac{k^{*2}R'^2}{2}}{R'}\right)]dz_1 + 2k\left(\frac{1-k^*R+\frac{k^{*2}R'^2}{2}}{R}\right)\delta_{jm}\} \\ &\approx \frac{-1}{4\pi\rho U}\{e^{-i\omega z_1/U} \int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{1}{R'}\right) \\ &-\frac{\partial^2}{\partial z_j' \partial z_m'}\left(\frac{1}{R'}-k^*+\frac{k^{*2}R'}{2}\right)]dz_1 + 2k\left(\frac{1}{R'}-k^*+\frac{k^{*2}R}{2}\right)\delta_{jm}\} \\ &\approx \frac{-1}{4\pi\rho U}\{e^{-i\omega z_1/U} \int_{-\infty}^{z_1} e^{i\omega z_1'/U}[\frac{\partial^2}{\partial z_j' \partial z_m'}\left(k^*-\frac{k^{*2}R'}{2}\right)]dz_1 \\ &+2k\left(\frac{1}{R}-k^*+\frac{k^{*2}R}{2}\right)\delta_{jm}\}. \end{aligned} \qquad (6.52)$$

The coefficient $\frac{-1}{4\pi\rho U}$ can be written as $\frac{-1}{8\pi\mu k}$, then

$$
\begin{aligned}
u_j^{(m)}(\mathbf{z}) &\approx \frac{-1}{8\pi\mu}\{\frac{e^{-i\omega z_1/U}}{k}(-\frac{k^{*2}}{2}) \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial^2}{\partial z_j' \partial z_m'}(R')\, dz_1' \\
&+ 2(\frac{\delta_{jm}}{R} - k^*\delta_{jm} + \frac{k^{*2}R}{2}\delta_{jm})\}.
\end{aligned} \tag{6.53}
$$

Taking the limit $R \to 0$ is equivalent to taking the two limits $k \to 0$ and $\omega \to 0$, therefore

$$
\begin{aligned}
u_j^{(m)}(\mathbf{z}) &\approx \frac{-1}{8\pi\mu} \lim_{k\to 0}\{\frac{e^{-i\omega z_1/U}}{k}(-\frac{k^{*2}}{2}) \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial^2}{\partial z_j' \partial z_m'}(R')\, dz_1' \\
&+ 2(\frac{\delta_{jm}}{R} - k^*\delta_{jm} + \frac{k^{*2}R}{2}\delta_{jm})\} \\
&\approx \frac{-1}{8\pi\mu} \lim_{k\to 0}\{\frac{e^{-i\omega z_1/U}}{k}(\frac{-k^{*2}}{2}) \frac{\partial^2}{\partial z_j' \partial z_m'}(R')\,\frac{e^{i\omega z_1'/U}}{i\omega/U} \\
&+ 2(\frac{\delta_{jm}}{R} - k^*\delta_{jm} + \frac{k^{*2}R}{2}\delta_{jm})\} \\
&\approx \frac{-1}{8\pi\mu} \lim_{k\to 0}\{\frac{1}{k}(\frac{-k^{*2}}{2}) \frac{\partial^2}{\partial z_j' \partial z_m'}(R')\,\frac{2k}{h^2} \\
&+ 2(\frac{\delta_{jm}}{R} - k^*\delta_{jm} + \frac{k^{*2}R}{2}\delta_{jm})\},
\end{aligned} \tag{6.54}
$$

where $\frac{i\omega}{U} = \frac{h^2}{2k}$, $h^2 = \frac{i\rho\omega}{\mu}$ and $k = \frac{\rho U}{2\mu}$. So

$$
\begin{aligned}
u_j^{(m)}(\mathbf{z}) &\approx \frac{-1}{8\pi\mu} \lim_{k\to 0}\{\frac{-k^{*2}}{h^2} \frac{\partial^2}{\partial z_j' \partial z_m'}(R') + 2\frac{\delta_{jm}}{R} - 2k^*\delta_{jm} + k^{*2}R\delta_{mj}\} \\
&\approx \frac{-1}{8\pi\mu}\{-\frac{\partial^2}{\partial z_j' \partial z_m'}(R') + 2\frac{\delta_{jm}}{R} - 2h\delta_{mj} + h^2 R\delta_{mj}\},
\end{aligned} \tag{6.55}
$$

and taking the limit $\omega \longrightarrow 0$ gives

$$\begin{aligned}
u_j^{(m)}(\mathbf{z}) &\approx \frac{-1}{8\pi\mu}\lim_{\omega\to 0}\{-\frac{\partial^2}{\partial z_j'\partial z_m'}(R') + 2\frac{\delta_{jm}}{R} - 2h\delta_{mj} + h^2R\delta_{mj}\} \\
&\approx \frac{-1}{8\pi\mu}\{-\frac{\partial^2}{\partial z_j'\partial z_m'}(R') + 2\frac{\delta_{jm}}{R}\} \\
&\approx \frac{-1}{8\pi\mu}\{-(\frac{\delta_{jm}}{R} - \frac{z_j z_m}{R^3}) + 2\frac{\delta_{jm}}{R}\} \\
&\approx \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}. \qquad (6.56)
\end{aligned}$$

We obtain that the oscillatory oseenlets approximate to the steady stokeslets around the point $z = 0$ ($R \to 0$). This is a similar result as given by Chadwick [30] for the steady oseenlets and by Pozrikidis [9] for the oscillatory stokeslets.

6.5.2 Green's Integral Representation of the velocity

The Green's surface integral representation of the oscillatory Oseen flow has been given in section (6.3) as

$$\begin{aligned}
&\int\int_S \{\rho U u_j(\mathbf{y})u_j^{(m)}(\mathbf{z})n_1 + u_j^{(m)}(\mathbf{z})p(\mathbf{y})n_j + u_j(\mathbf{y})p^{(m)}(\mathbf{z})n_j \\
&-\mu u_j^{(m)}(\mathbf{z})\frac{\partial u_j(\mathbf{y})}{\partial y_l}n_l + \mu u_j(\mathbf{y})\frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l}n_l\}dS = 0. \qquad (6.57)
\end{aligned}$$

We consider the surface S consisting of a surface S_δ a sphere radius $\delta \longrightarrow 0$, around the point $z = 0$, a surface S_B enclosing the oscillating body, and a large spherical surface S_R extending to infinity, enclosing the body and centred at the point $\mathbf{z} = 0$, see figure (6.2).

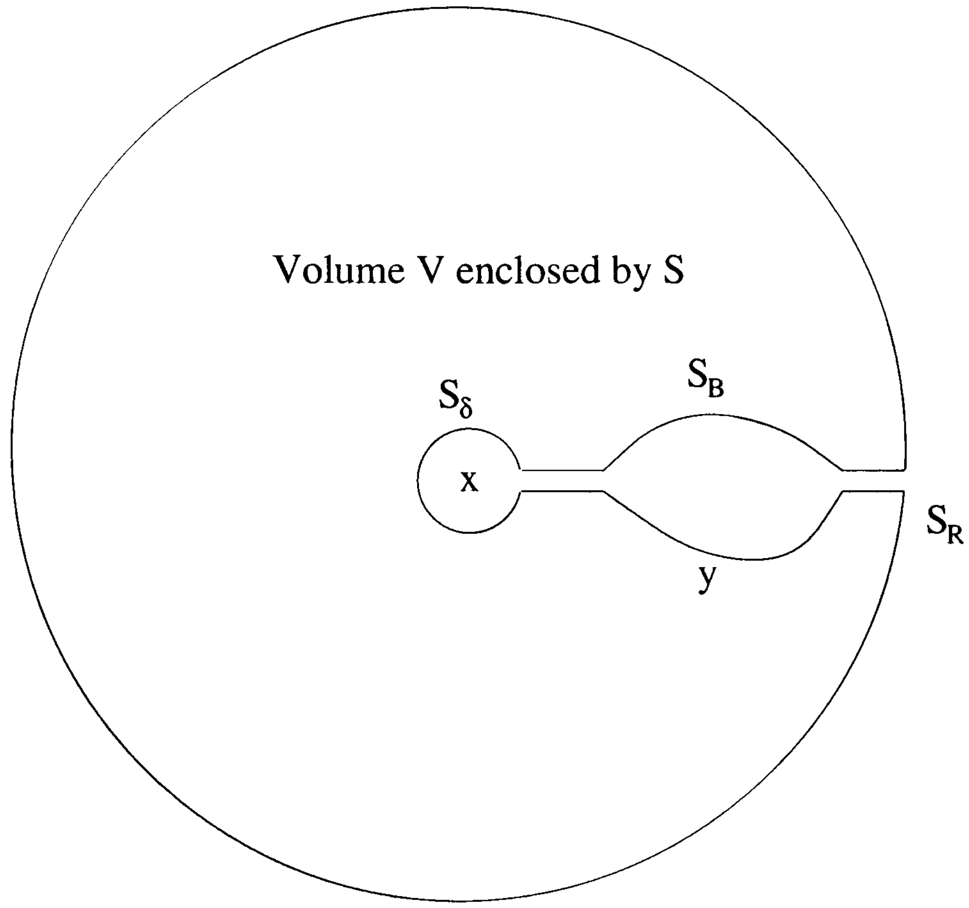

Figure 6.2: The surface S and the relation of the points **x** and **y**

We re-write the integral over the surface S as a sum of the integrals over the surfaces S_δ, S_B, and S_R,

$$\int\int_S = \int\int_{S_\delta} + \int\int_{S_B} + \int\int_{S_R} = 0. \qquad (6.58)$$

Then we calculate the contributions over the surface S_δ as $\delta \to 0$, and over S_R as $R \to \infty$, to give integral representation for the Oscillatory Oseen velocity $u_j(\mathbf{x})$.

The Contribution over the Surface S_δ as $\delta \to 0$

Comparing the oscillatory oseen case with the steady oseen, we can see that the integral representation (6.13) is identical to the integral representation of steady Oseen equations (5.16). Aslo, the asymptotic series (6.56) of oscillatory oseenlets around zero is identical to the steady oseen case (5.51), both solutions approximate to the steady stokeslets around the point $z = 0$. Therefore, the integral over the surface S_δ for oscillatory Oseen case, gives the same contribution given by the steady Oseen, which is

$$\int\int_{S_\delta} \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS = -u_m(\mathbf{x}). \tag{6.59}$$

The Contribution over the Surface S_R as $R \to \infty$

The contribution over the surface S_R, for very large R, is

$$\int\int_{S_R} \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS. \tag{6.60}$$

which is identical to the steady Oseen case, since the oscillatory parts of the governing differential equation which include the term $\rho i\omega u_j$ have cancelled. The modulus of the far field integral over S_R for the oscillatory oseenlets is bounded by the far field integral for the steady oseenlets. This result, that the oscillatory oseenlets are bounded by the steady oseenlets in the far field, is shown as follows

$$|\phi^{(m)}(\mathbf{z})| \leq |-\frac{e^{-i\omega z_1/U}}{4\pi\rho U} \fint_{-\infty}^{z_1} e^{i\omega z_1'/U} \frac{\partial}{\partial z_m'} \left(\frac{1}{R'}\right) dz_1'|. \tag{6.61}$$

Away from the body $\omega \to 0$, then

$$|\phi^{(m)}(\mathbf{z})| \leq |\frac{1}{4\pi\rho U} \fint_{-\infty}^{z_1} \frac{\partial}{\partial z_m'} \left(\frac{-1}{R'}\right) dz_1'|. \tag{6.62}$$

Since $-\frac{1}{R} = \frac{\partial}{\partial z_1} \ln(R - z_1)$, then

$$|\phi^{(m)}(\mathbf{z})| \leq |\frac{1}{4\pi\rho U} \frac{\partial}{\partial z_m} \ln(R - z_1)|, \tag{6.63}$$

the right hand-side is the steady oseenlet solution for $\phi^{(m)}$ [13]. Similarly

$$\begin{aligned} |\chi^{(m)}(\mathbf{z})| &\leq |\frac{e^{-i\omega z_1/U}}{4\pi\rho U} \int_{-\infty}^{z_1} e^{i\omega z_1/U} \frac{\partial}{\partial z'_m} \left(\frac{e^{-k^* R + k z_1}}{R'} \right) dz'_1| \\ &\leq |\frac{1}{4\pi\rho U} \int_{-\infty}^{z_1} \frac{\partial}{\partial z'_m} \left(\frac{e^{-k^* R + k z_1}}{R'} \right) dz'_1| \end{aligned}$$

$$\begin{aligned} |\chi^{(m)}(\mathbf{z})| &\leq |\frac{-1}{4\pi\rho U} \int_{-\infty}^{z_1} \frac{\partial}{\partial z'_m} \left(e^{-k(R' - z'_1)} \frac{\partial}{\partial z_1} \ln(R' - z'_1) \right) dz'_1| \\ &\leq |\frac{-1}{4\pi\rho U} \int_{-\infty}^{z_1} \frac{\partial}{\partial z'_1} \left(e^{-k(R' - z'_1)} \frac{\partial}{\partial z_m} \ln(R' - z'_1) \right) dz'_1| \end{aligned} \tag{6.64}$$

since $R \longrightarrow \infty$ leads to $\omega \longrightarrow 0$ and $k^* = k$. Now we have

$$|\chi^{(m)}(\mathbf{z})| \leq |\frac{-1}{4\pi\rho U} e^{-k(R - z_1)} \frac{\partial}{\partial z_m} \ln(R - z_1)| \tag{6.65}$$

which is the steady oseenlet solution for $\chi^{(m)}$ [13]. Finally χ^* bounds by

$$\begin{aligned} |\chi^*(\mathbf{z})| &\leq |\frac{2k}{4\pi\rho U} \frac{e^{-k^* R + k z_1}}{R} \delta_{jm}| \\ &\leq |\frac{2k}{4\pi\rho U} \frac{e^{-k(R - z_1)}}{R} \delta_{jm}|, \end{aligned} \tag{6.66}$$

which is the steady oseenlet solution for χ^*. Therefore, the far field integral (6.60) is bounded by the far field integral for steady oseenlets, which has been shown to be zero in [32]. So

$$\int\int_{S_R} \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu\, u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu\, u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS = 0. \tag{6.67}$$

From (6.58), (6.59) and (6.67) we find the Green's integral represenation of the oscillatory Oseen velocity to be

$$u_m(\mathbf{x}) = \int\int_{S_B} \{\rho U u_j(\mathbf{y}) u_j^{(m)}(\mathbf{z}) n_1 + u_j^{(m)}(\mathbf{z}) p(\mathbf{y}) n_j + u_j(\mathbf{y}) p^{(m)}(\mathbf{z}) n_j$$
$$-\mu u_j^{(m)}(\mathbf{z}) \frac{\partial u_j(\mathbf{y})}{\partial y_l} n_l + \mu u_j(\mathbf{y}) \frac{\partial u_j^{(m)}(\mathbf{z})}{\partial y_l} n_l\} dS, \tag{6.68}$$

which is identical to the integral representation of steady Oseen velocity (5.75).

6.6 Integral representation of the force

Denote the surface of the oscillating body by S_t. The force on the body due to the action of the fluid is then

$$F_j = \int\int_{S_t} \tau_{jl}^{\dagger} n_l dS \tag{6.69}$$

where for an incompressible fluid

$$\tau_{jl}^{\dagger} = -p^{\dagger}\delta_{jl} + \mu \left(\frac{\partial u_j^{\dagger}}{\partial x_l} + \frac{\partial u_l^{\dagger}}{\partial x_j} \right)$$

is the symmetric Navier-Stokes stress tensor.

Denoting the velocity on the body surface to be $u_j^\dagger = u_j^B$ then the total force F_j is

$$
\begin{aligned}
F_j &= \int\int_{S_t} \{\tau_{jl}^\dagger - \rho u_j^\dagger u_l^\dagger + \rho u_j^B u_l^B\} n_l dS \\
&= \int_{S_R} \{\tau_{jl}^\dagger - \rho u_j^\dagger u_l^\dagger\} n_l dS - \int\int\int_V \frac{\partial}{\partial x_l}(\tau_{jl}^\dagger - \rho u_j^\dagger u_l^\dagger) dV \\
&\quad + \int\int_{S_t} \rho u_j^B u_l^B n_l dS
\end{aligned}
\tag{6.70}
$$

The Navier-Stokes equation can be re-written as

$$
\rho \frac{\partial u_j^\dagger}{\partial t} = \frac{\partial}{\partial x_l}\left(\tau_{jl}^\dagger - \rho u_j^\dagger u_l^\dagger\right), \tag{6.71}
$$

since the continuity equation $\frac{\partial u_l^\dagger}{\partial x_l} = 0$ holds. So

$$
\begin{aligned}
F_j &= \int\int_{S_R} \{\tau_{jl}^\dagger - \rho u_j^\dagger u_l^\dagger\} n_l dS \\
&\quad - \int\int\int_{V_t} \rho \frac{\partial u_j^\dagger}{\partial t} dV + \int\int_{S_t} \rho u_j^B u_l^B n_l dS
\end{aligned}
\tag{6.72}
$$

where S_R is an enclosing surface a sufficiently large distance away from the body, and V_t is the volume of fluid exterior to S_t.

On S_R, assume that the surface is sufficiently far from the disturbance that the Oseen approximation

$$
u_j^\dagger = U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}, \; |u_j^n| \ll U, \tag{6.73}
$$

holds, and

$$
p^\dagger = \sum_{n=-\infty}^{\infty} p^n e^{i\omega_n t}, \tag{6.74}
$$

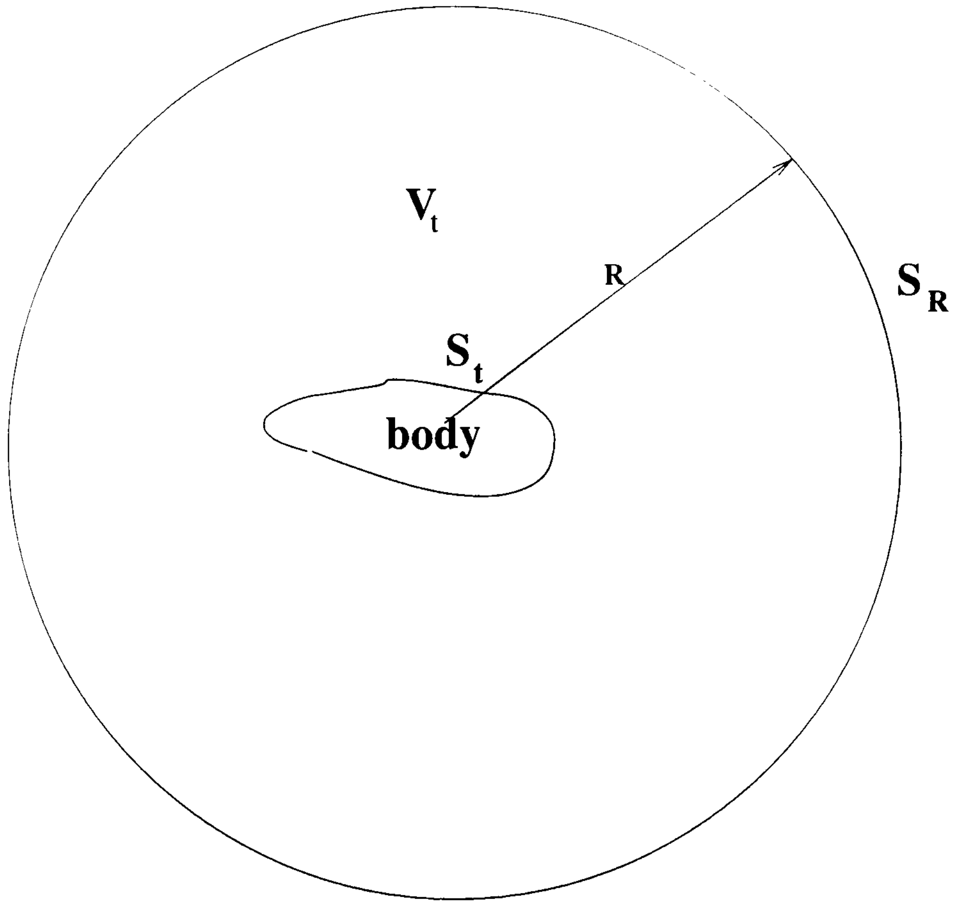

Figure 6.3: the surfaces S_R and S_t

where p^n is the pressure associated with the velocity field u_j^n. Then

$$\tau_{jl}^{\dagger} = \sum_{n=-\infty}^{\infty} \tau_{jl}^{n} e^{i\omega_n t}, \tag{6.75}$$

where we define $\tau_{jl}^{n} = -p^n\delta_{jl} + \mu\left(\frac{\partial u_j^n}{\partial x_l} + \frac{\partial u_l^n}{\partial x_j}\right)$, and

$$\begin{aligned}
u_j^{\dagger}u_l^{\dagger} &= \{U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}\}\{U\delta_{l1} + \sum_{m=-\infty}^{\infty} u_l^m e^{i\omega_m t}\} \\
&= U\delta_{j1}U\delta_{l1} + U\delta_{l1}\sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t} + U\delta_{j1}\sum_{m=-\infty}^{\infty} u_l^m e^{i\omega_n t} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}\sum_{m=-\infty}^{\infty} u_l^m e^{i\omega_m t} \\
&= U\delta_{j1}U\delta_{l1} + \sum_{n=-\infty}^{\infty}\{U\delta_{l1}u_j^n + U\delta_{j1}u_l^n + u_j^n\sum_{m=-\infty}^{\infty} u_l^m e^{i\omega_m t}\}e^{i\omega_n t} \\
&= U\delta_{j1}U\delta_{l1} + \sum_{n=-\infty}^{\infty}\{U\delta_{l1}u_j^n + U\delta_{j1}u_l^n + \sum_{m=-\infty}^{\infty} u_j^n u_l^m e^{i\omega_m t}\}e^{i\omega_n t} \\
&= U\delta_{j1}U\delta_{l1} + \sum_{n=-\infty}^{\infty}\{U\delta_{l1}u_j^n + U\delta_{j1}u_l^n\}e^{i\omega_n t}
\end{aligned} \tag{6.76}$$

because from Oseen approximation $u_j^n u_l^m = O(\epsilon^2)$.

$$\begin{aligned}\frac{\partial u_j^\dagger}{\partial t} &= \frac{\partial}{\partial t}(U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}) \\ &= \sum_{n=-\infty}^{\infty} i\omega u_j^n e^{i\omega_n t}.\end{aligned} \tag{6.77}$$

The force then becomes

$$\begin{aligned}F_j &= -\int\int_{S_R} \rho U^2 \delta_{j1}\delta_{j1} n_l ds + \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \{\tau_{jl}^n n_l - \rho U\delta_{j1} u_l^n n_l - \rho U\delta_{l1} u_j^n n_l\} dS \right. \\ &\left. - \int\int\int_V \rho i\omega_n u_j^n dV \right] e^{i\omega_n t} + \int\int_{S_t} \rho u_j^B u_l^B n_l dS.\end{aligned} \tag{6.78}$$

Since $\int\int n_l \; ds = 0$, the term $-\int\int_{S_R} U^2 \delta_{j1}\delta_{l1} n_l \; ds = 0$, so

$$\begin{aligned}F_j &= \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \{\tau_{jl}^n n_l - \rho U\delta_{j1} u_l^n n_l - \rho U\delta_{l1} u_j^n n_l\} dS \right. \\ &\left. - \int\int\int_V \rho i\omega_n u_j^n dV \right] e^{i\omega_n t} + \int\int_{S_t} \rho u_j^B u_l^B n_l dS.\end{aligned} \tag{6.79}$$

However,

$$\begin{aligned}\int\int\int_V u_j^n dV &= \int\int\int_V \frac{\partial}{\partial y_k}(y_j u_k^n) dV \\ &= \int\int_{S_R} y_j u_k^n n_k dS - \int\int_{S_t} y_j u_k^n n_k dS.\end{aligned} \tag{6.80}$$

The volume integral term becomes

$$
\begin{aligned}
-\int\int\int_V \rho i\omega_n u_j^n \, dV &= -i\rho\omega_n[\int\int_{S_R} y_j u_k^n n_k ds - \int\int_{S_t} y_j u_k^n n_k ds] \\
&= -\int\int_{S_R} i\rho\omega_n y_j u_k^n n_k ds + \int\int_{S_t} i\rho\omega_n y_j u_k^n n_k ds
\end{aligned}
\tag{6.81}
$$

Also, consider I such that

$$
\begin{aligned}
I &= \sum_{n=-\infty}^{\infty} \left(\int\int_{S_t} \rho i\omega_n u_k^n y_j n_k dS \right) e^{i\omega_n t} \\
&= \int\int_{S_t} \sum_{n=-\infty}^{\infty} \rho y_j (i\omega_n u_k^n e^{i\omega_n t}) n_k dS \\
&= \int\int_{S_t} \sum_{n=-\infty}^{\infty} \rho y_j \frac{\partial}{\partial t}(u_k^n e^{i\omega_n t}) n_k \; dS \\
&= \int\int_{S_t} \rho y_j \frac{\partial}{\partial t}(\sum_{n=-\infty}^{\infty} u_j k^n e^{i\omega_n t}) n_k \; dS.
\end{aligned}
\tag{6.82}
$$

Since $u_j^\dagger = U\delta_{j1} + \sum_{n=-\infty}^{\infty} u_j^n e^{i\omega_n t}$, then $\sum_{n=-\infty}^{\infty} u_k^n e^{i\omega_n t} = u_k^\dagger - U\delta_{k1} = u_k^B - U\delta_{k1}$ on the body surface, then

$$
\begin{aligned}
I &= \int\int_{S_t} \rho y_j \frac{\partial}{\partial t}(u_k^B - U\delta_{k1}) \; n_k \; dS. \\
&= \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k \; dS.
\end{aligned}
\tag{6.83}
$$

Therefore

$$
\begin{aligned}
F_j &= \sum_{n=-\infty}^{\infty} \left[\int\int_{S_R} \{\tau_{jl}^n n_l - \rho U \delta_{j1} u_l^n n_l - \rho U \delta_{l1} u_j^n n_l\} dS \right. \\
&\left. - \int\int_{S_R} i\rho\omega_n y_j u_k^n n_k ds \right] e^{i\omega_n t} + \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k dS + \int\int_{S_t} \rho u_j^B u_l^B n_l \ dS.
\end{aligned}
\tag{6.84}
$$

The continuity equation yields not fluid outflow, which means $\int\int_{S_R} u_l n_l dS = 0$. Thus, the term

$$
\int\int_{S_R} U \delta_{j1} u_l n_l \ dS = 0.
$$

Let us therefore write

$$
F_j = \sum_{n=-\infty}^{\infty} f_j^n e^{i\omega_n t} + f_j^B \tag{6.85}
$$

where

$$
f_j^n = \int\int_{S_R} \{\tau_{jl}^n n_l - \rho U \delta_{l1} u_j^n n_l - \rho i \omega_n y_j u_k^n n_k\} \ dS \tag{6.86}
$$

and

$$
f_j^B = \int\int_{S_t} \{\rho u_j^B u_l^B n_l + \rho y_j \frac{\partial u_k^B}{\partial t} n_k\} \ dS. \tag{6.87}
$$

Let us consider the force associated with an oscillatory oseenlet of frequency ω be f_j where

$$
\begin{aligned}
f_j &= \int\int_{S_R} \{\tau_{jl} n_l - \rho U \delta_{l1} u_j n_l - \rho i \omega y_j u_k n_k\} \ dS \\
&= \int\int_{S_R} \{(-p\delta_{jl} + \mu \left(\frac{\partial u_l}{\partial y_j} + \frac{\partial u_j}{\partial y_l} \right)) n_l - \rho U \delta_{l1} u_j n_l - \rho i \omega y_j u_k n_k\} \ dS. \\
&= \int\int_{S_R} \{(-p n_j + \mu \frac{\partial u_j}{\partial y_l} n_l - \rho U \delta_{l1} u_j n_l - \rho i \omega y_j u_k n_k\} \ dS.
\end{aligned}
\tag{6.88}
$$

because $\int\int_{S_R} \frac{\partial u_l}{\partial y_j} n_l \, dS = \int\int\int_V \frac{\partial^2 u_l}{\partial y_l \partial y_j} = 0$. Comparing with

$$A_j = -\int\int_{S_B} \left\{ \rho U u_j n_1 + p n_j - \mu \frac{\partial u_j}{\partial y_l} n_l \right\} \, dS \tag{6.89}$$

as it is defined in [30], we see that all the terms are the same except for an additional term $\int\int_{S_R} \rho i\omega y_j u_k n_k \, n_k \, dS$ in f_j. However, in the limit as $\omega \to 0$, $\int\int_{S_R} \rho i\omega y_j u_k n_k \, n_k \, dS = 0$, and so for the steady case $A_j = f_j$ as expected from [30]. Also at the limit $U \to 0$ the force generated by the oscillatory stokeslets (4.82) recovers from (6.88).

6.7 Force generated by Oscillatory oseenlet

In order to obtain the force generated by the oscillatory oseenlet we consider an oseenlet $u_j^{(m)}$ in the body. A sphere S_δ of radius δ, is central at the force point $u_j^{(m)}$ and the volume V_t is bounded by the body surface S_t and S_δ. Considering the body as a fluid we can write the force in similar way to what we done in last section taking in consideration that $\int\int\int_{V_t} = \int\int_{S_t} - \int\int_{S_\delta}$ as the normal n_l points out of the volume V_t. Hence

$$\begin{aligned} F_j &= \sum_{n=-\infty}^{\infty} \left[\int\int_{S_\delta} \{\tau_{jl}^n n_l - \rho U \delta_{l1} u_j^n n_l\} dS \right. \\ &\quad \left. - \int\int_{S_\delta} i\rho\omega_n y_j u_k^n \, n_k \, ds \right] e^{i\omega_n t} + \int\int_{S_t} \rho y_j \frac{\partial u_k^B}{\partial t} n_k dS + \int\int_{S_t} \rho u_j^B u_l^B n_l \, dS. \end{aligned} \tag{6.90}$$

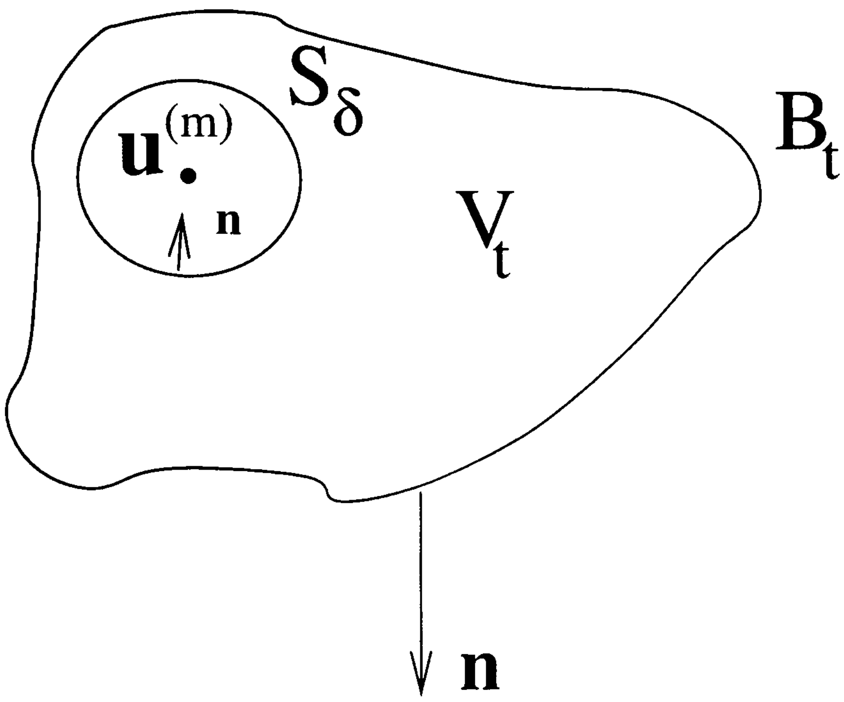

Figure 6.4: oseenlet inside the oscillatory body

Substituting an oscillatory oseenlet $u_j^{(m)}$ into the force, gives

$$F_j^{(m)} = \sum_{n=-\infty}^{\infty}\Big[\int\int_{S_\delta}\{\tau_{jl}^{n(m)}n_l - \rho U\delta_{l1}u_j^{n(m)}n_l\}dS + \int\int_{S_\delta} i\rho\omega_n z_j u_k^{n(m)}\ n_k\ ds\Big]e^{i\omega_n t} + \int\int_{S_t}\rho y_j\frac{\partial u_k^B}{\partial t}n_k dS + \int\int_{S_t}\rho u_j^B u_l^B n_l\ dS, \tag{6.91}$$

where $F_j^{(m)}$ is the force generated by the oscillatory oseenlet, $\tau_{jl}^{n(m)}$ is the oscillatory Oseen stress tensor. When $\delta \to 0$, $\tau_{jl}^{(m)} \to \tau_{jl}^{s(m)}$ the steady Stokes tress tensor, and we have shown that $\int\int_{S_\delta}\tau_{jl}^{s(m)}\ n_l\ dS = \delta_{jm}$. Hence

$$\int\int_{S_\delta}\tau_{jl}^{n(m)}n_l\ dS = \delta_{jm}. \tag{6.92}$$

Also, around zero the oscillatory oseenlet can be approximated to the steady stokeslet;

$$u_j^{(m)} \approx \frac{-1}{8\pi\mu}\{\frac{\delta_{jm}}{R} + \frac{z_j z_m}{R^3}\}, \tag{6.93}$$

so the second and third integrals in the force representation (6.91) can be approximated using the approximation series of the oscillatory oseenlet, as follows. The second integral is,

$$\begin{aligned}
\rho U\delta_{l1}\int\int_{S_\delta} u_j^{n(m)}\ n_l\ dS &\approx \frac{-\rho U\delta_{l1}}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}}{R}+\frac{z_jz_m}{R^3}\}\ n_l\ dS \\
&\approx \frac{-\rho U\delta_{l1}}{8\pi\mu}\int\int_{S_\delta}\{\frac{\delta_{jm}z_l}{R^2}+\frac{z_jz_mz_l}{R^4}\}\ dS \\
&\approx O(R)\rightarrow 0
\end{aligned} \tag{6.94}$$

as $R(=\delta)\rightarrow 0$. And the third integral is

$$\begin{aligned}
i\rho\omega_n\int\int_{S_\delta} z_ju_k^{n(m)}\ n_k\ dS &\approx \frac{-i\rho\omega_n}{8\pi\mu}\int\int_{S_\delta} z_j\{\frac{\delta_{kj}}{R}+\frac{z_kz_m}{R^3}\}\ n_k\ dS \\
&\approx \frac{-i\rho\omega_n}{8\pi\mu}\int\int_{S_\delta}\{\frac{z_kz_j}{R^2}+\frac{z_k^2z_jz_m}{R^4}\}\ dS \\
&\approx \frac{-i\rho\omega_n}{4\pi\mu}\int\int_{S_\delta}\frac{z_mz_j}{R^2}\ dS \\
&\approx \frac{-i\rho\omega_n}{3\mu}\delta_{jm}R^2 = O(R^2)\rightarrow 0
\end{aligned} \tag{6.95}$$

when $\delta\rightarrow 0$, using$z_m = z_k\delta_{km}$, $n_k = \frac{z_k}{R}$, $z_j^2 = R^2$ and $\int\int_{S_\delta} z_mz_k = \frac{4\pi}{3}\delta_{jk}R^3$. Therefore, from (6.92) , (6.94) and (6.95), the force generated by an oscillatory oseenlet is

$$F_j^{(m)} = \sum_{n=-\infty}^{\infty}\delta_{jm}e^{i\omega_nt} + \int\int_{S_t}\{\rho y_j\frac{\partial u_k^B}{\partial t}n_k + \rho u_j^Bu_l^Bn_l\}\ dS \tag{6.96}$$

As the limit $\omega\rightarrow 0$, $u_j^B = 0$ hence $F_j^{(m)} = \delta_{jm}$ which is the force generated by a steady oseenlet (5.91). Also, the equation (6.96) shows that the force is oscillatory.

6.8 Chapter Conclusion

The main new result is presented, which is the description of the oscillatory oseenlet, obtained using the approach of potential decomposition. The new solutions are checked against known solutions; which are steady Stokes, oscillatory Stokes, and steady Oseen flows. The oscillatory oseenlet tends to the steady stokeslet in the near-field limit, and this is a similar result satisfied by existing oseenlet solutions in the literature. The force generated by the oseenlet is calculated and we show that the forces are oscillating themselves.

Chapter 7

Conclusion and Future Work

7.1 Applications Discussion:

7.1.1 Modelling a miniaturized swimming robot

In developing a model for a miniaturized swimming robot, the most important quantities for assessing the manoeuvring characteristics are the force and moment calculations. The near-field is governed by Stokes flow, but in the far-field the Oseen approximation is valid. Similar to the steady case, this is important; In the steady case, for flow past a circular cylinder Proudman and Pearson [34] match a near-field Stokes flow to a far-field Oseen flow.In the representation presented here in section (6.4.1), we note that the near-field limit of the oscillatory oseenlet is the oscillatory stokeslet, and this matching is provided in this thesis. For this case of a swimming robot, a near-field stokeslet distribution has the additional challenge that the oscillating stokeslets are themselves oscillating rather than stationary. Also, the robot is designed to have a slender body tail with elastic response, see figure (7.1).

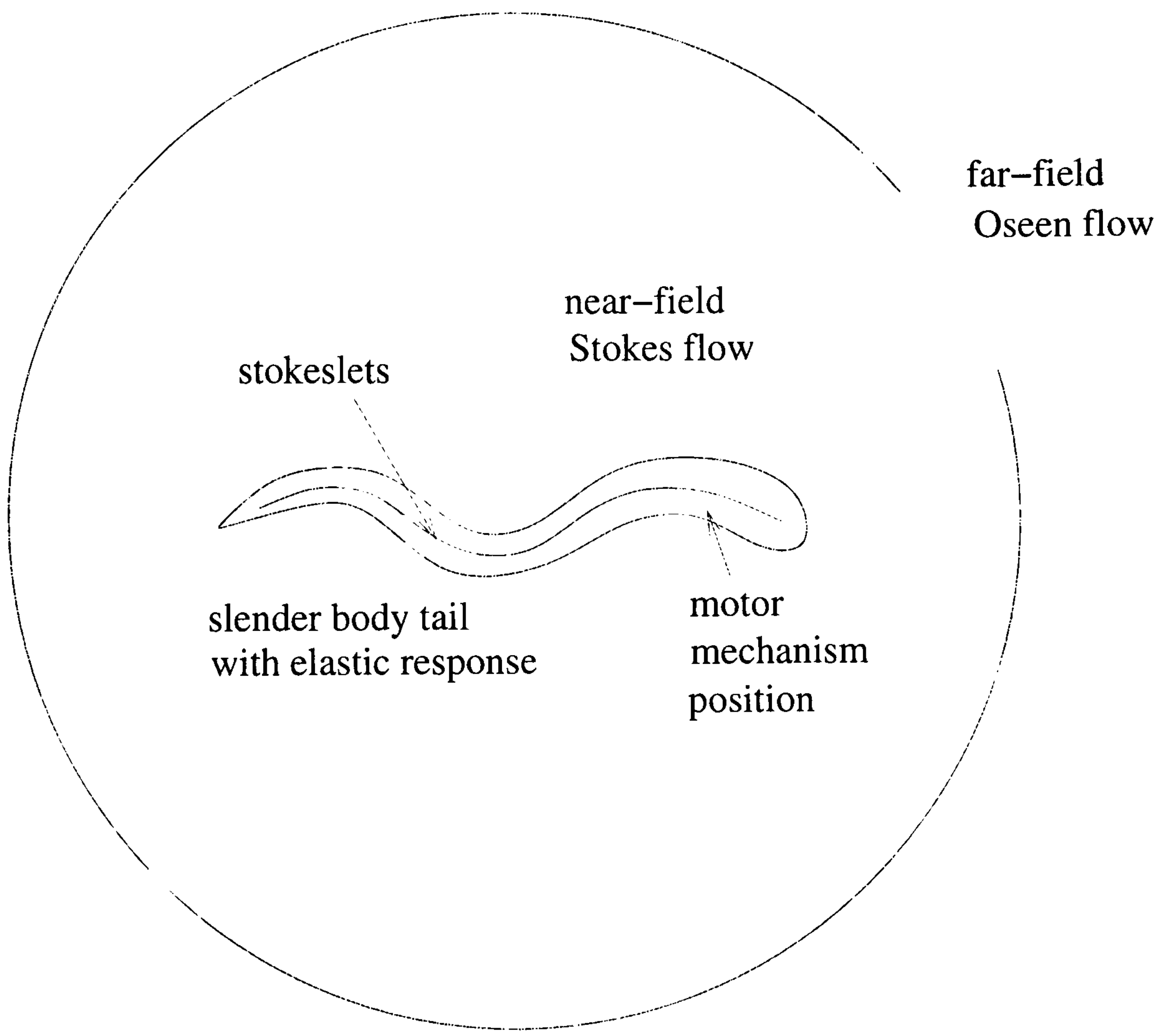

Figure 7.1: A design for the swimming robot

A slender-body theory has been presented by Chadwick [35] but only for the steady case. However, the work in this thesis provides the far-field representation for a formulation to model a miniaturized swimming robot motion which is left for future work.

7.1.2 Biological fluid dynamics:

Micro-organisms are very small in size, so that the Reynolds number of their motion, that based on a characteristic dimension of the body L and velocity of propulsion U, is very small, $Re = \rho UL/\mu$. Hence in the near field (when between $R = O(1)$ and $R = O(Re^{-1})$) the dominant forces on micro-organisms are viscous forces and the inertia forces are negligible, [36], and the flow which is generated by the movement of micro-organisms, is slow viscous flow (with as low as $Re = O(10^{-5})$ in water) that can be represented with same accuracy both by Stokes or Oseen approximation . Away from the micro-organism (when $R = O(Re^{-1})$) the inertial forces are dominant and can not be neglected, so Oseen flow is better to represent the flow.

Self-propulsion Micro-organisms

There are may different types of micro-organisms base on their swimming way, and more predominant ones are flagella and cilia. Flagella are consist of a head and one or more motile; such as bacteria, sperm cell and make wave-like motion to move. Cilia are hair-like organelles and they cover the outer surface of micro-organism. The cilia move back and forth to enable the micro-organism to swim, see [36] and [37]. The flow singularity solutions which describe point forces (stokeslets) cannot be considered in the far field as they represent exact solutions only at zero Reynolds number. In the literature, the far field of a swimming micro-organism is represented by a symmetric force dipole or stresslet, see [38]. However, the oscillatory oseenlets, that we introduce in chapter 6, replace the stokeslets in the far field and give needed representation for low non-zero Reynolds number.

Example: Swimming Flagella

Considering a flagellum swims with wave its centre-line is

$$y(x) = (-ut + x, a\cos(kx - \omega t))$$

where u is the velocity, k is the wavenumber, ω is angular frequency, and a is amplitude of the wave. The xy is an adopted coordinate system in which x moves along the position of the flagellum, see [39]. Using slender body theory and zero net force on flagellum body, implies

$$u = \frac{1}{2}a^2 k\omega$$

which is not small, but the Reynolds number $Re = \frac{\rho L u}{\mu}$ is because of the flagellum size. To study the effect of the flagellum motion on the far field, the oscillatory oseenlets are needed to give the most correct flow representation. Also, the slender theory is applicable as the flagellum has slender body.

7.1.3 Micro-Electro-Mechanical System (MEMS)

Clarke et. al [8] consider the problem of a MEMS device vibrating in a fluid at rest. The device is treated as a slender body and the Stokes approximation is used. The oscillatory stokeslet given by Pozrikidis [9] is used. A further development on Clarke's work would be to consider the effect of a uniform rather than stationary flow field, which would for example replicate blood flow. Within such a development, the oscillatory stokeslet is an inner near-field description to be matched to an outer far-field oscillatory oseenlet. In order to enable this, there is a requirement for the oscillatory oseenlet solution.

7.1.4 Acoustic Devices

Considering an acoustic device such as loud speaker under the water, the speaker surface vibrates and generates oscillatory wave in the water. To measure the loudness of the sound away from the speaker and to make improvement of quality of the sound away from the speaker, the far field representation is needed and the oscillatory oseenlet give better far field modeling.

7.2 Conclusion and Future Work

Oscillatory stokeslets are given using potentials shown to be equivalent to Pozrikidis' form [9]. The oscillatory oseenlet and corresponding Green's integral representation have been presented which enables time-periodic Oseen flow to be modelled. A far-field velocity expansion is given. Low Reynolds number limit oscillatory oseenlets reduce to oscillatory stokeslets. In the limit as $\omega \rightarrow 0$ it is shown that the far-field steady Oseen velocity expansion of Chadwick [30] is recovered. The force generated by the oscillatory oseenlet is shown itself to be oscillatory, and so any net propulsive force is related to the steady oseenlet only. This completes all the cases for oseenlets and stokeslets in steady, transient and oscillatory flows, the oscillatory oseenlet being the final case not yet present in the literature.

Its use for modelling a problem as a far-field formulation is discussed, that being the swimming motion of small robotic devices at low Reynolds number where it would be matched to a near-field Stokes flow. The matching of the far-field oseenlet to the near-field stokeslet required for this application is that presented in this thesis. However, there are further challenges left for future work, which are that the stokeslet distribution itself may be moving relative to the reference frame and also that the tail exhibits an elastic response from the action of the fluid.

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