ACOUSTIC CROSS-ENERGY MEASURES AND THEIR APPLICATIONS

Jonathan A. Hargreaves and Yiu W. Lam

Acoustics Research Centre, University of Salford, Salford, M5 4WT, UK
e-mail: j.a.hargreaves@salford.ac.uk

Acoustic energy density and acoustic intensity are quantities which are well established and widely applied. When these are applied to pressure and particle-velocity fields which are the linear superposition of more than one wave they naturally separate into multiple terms; some of these involve only one of the constituent waves and others involve combinations of two waves. When one-dimensional energy measures such as covariance or power-spectral density are applied to sums of signals in this way, the terms involving only one signal are designated the ‘auto’ terms and the terms involving the combination of two signals are designated the ‘cross’ terms. This paper proposes that this terminology be extended to the acoustic energy measures, hence acoustic ‘cross-energy density’ and acoustic ‘cross-intensity’, and their physical interpretations as measures of common acoustic energy flow are discussed. It is also well known that the divergence theorem may be applied to the acoustic energy density and acoustic intensity to produce the energy flux relation; this states that acoustic energy is conserved in a volume containing a lossless medium. Here it is shown that the same may be done for the cross-energy quantities, producing an equivalent energy-flux relationship for the new cross-energy measures and showing that conservation of energy applies to these also. Furthermore it is shown that the resulting integral is equivalent to the Kirchhoff-Helmholtz Boundary Integral Equation when one of the waves is a converging spherical wave. This leads to a new energy interpretation which is also informative for other choices of waves, such as plane-waves and higher-order spherical harmonic waves. Applications of these ideas include the near-field to far-field transformation used in Finite Difference Time Domain modelling, and near-field compensated Ambisonics, where it suggests new ways to couple computer simulation algorithms to auralisation systems and new microphone array designs.

1. Introduction

When attempting to measure the transfer function of a linear time-invariant system, such as the acoustic response of a room, in the presence of noise, a standard approach is to the cross-power spectral density method. In this, we repeatedly compute the cross-power between the excitation and measured signals. Upon averaging, this picks out the component of the measured signal which has a fixed phase relationship with the excitation, being the excitation multiplied by the transfer function, and ignores the background noise, since it is incoherent with the excitation signal. The result of this is normalised by the auto-power spectral density of the excitation to obtain the information desired.

In this paper we are primarily interested in devising a similar approach but for matching waves, which are functions of three dimensions plus time, in place of signals, which are functions only of time. The most obvious application here is microphone arrays designed for spatial audio recording[1]. These sample an acoustic pressure field (and/or possibly its gradient) and attempt to represent it as near-field compensated Ambisonic components (i.e. as a weighted sum of the spherical basis functions of the Helmholtz equation). Another possibility would be beam-forming, in which case the desired sound is usually a plane wave arriving from a particular angle to the absence of all else. Both of these approaches fundamentally try to measure the amount of one particular wave present in a sound-field which comprises many different waves superimposed. Our strategy toward achieving this is to look for an equivalent metric which quantifies the similarity of two waves, akin to how the cross-power approach works for signals. Given a sound field to measure and the definition of the wave term we want to extract (the 'testing' wave), we would compute this metric between the two (and then if necessary normalise by the metric for the testing wave with itself). As with cross-power spectral density measurement, the result would be a scalar stating the amount of the testing wave present in the measured sound-field. This has the potential to be a flexible framework since any wave can in principle be chosen as the testing wave. We refer to this approach as 'wave-matching'. It was recently published applied to the decoding of time-harmonic signals into near-field compensated Ambisonic components[2] and this paper aims to extend those ideas, not least to include the time domain.

Our starting point to establish a wave-matching (or wave similarity) metric is acoustic energy[3]. When acousticians talk about acoustic energy or sound power there is usually the implication that phase information has been discarded; consider for example energy methods such as ray-tracing in room acoustic modelling or the addition of sound power in environmental acoustics. However here we will start from exact statements for instantaneous acoustic energy density (and intensity) and focus specifically on the parts which are discarded when phase-free sound-power addition is performed. We will see that these parts are equivalent to the cross-power terms in signal processing, and they will become our metric. It will be seen that the corresponding energy and intensity quantities are related by an integral statement equivalent to Green's second theorem applied to acoustic waves, thereby bringing this a physical interpretation and a time-domain equivalent.

2. Acoustic energy density

2.1 Instantaneous acoustic energy density for transient waves

Consider a wave which exists within a homogeneous isotropic medium with wavespeed c and density ρ. It is denoted by its velocity potential $\varphi(\mathbf{y},t)$, where t is time and vector $\mathbf{y}$ is a point in 3D Cartesian space, and satisfies the wave equation $\nabla^2\varphi = \ddot{\varphi}/c^2$ (where dots over a quantity indicate temporal differentiation) within a volume V. Pressure p and particle velocity $\mathbf{u}$ for this wave may respectively be found from its velocity potential by $p = -\rho\dot{\varphi}$ and $\mathbf{u} = \nabla\varphi$. The instantaneous acoustic energy density $E_\varphi(\mathbf{y},t)$ of φ is[Error! Reference source not found.]:

(1) $${}_{\varphi}(\mathbf{y},t) = \tfrac{1}{2}\rho\left[|\mathbf{u}(\mathbf{y},t)|^2 + \frac{p(\mathbf{y},t)^2}{\rho^2 \mathrm{c}^2}\right] = \tfrac{1}{2}\rho\left[\nabla\varphi(\mathbf{y},t)\cdot\nabla\varphi(\mathbf{y},t) + \frac{\dot{\varphi}(\mathbf{y},t)}{\mathrm{c}}\frac{\dot{\varphi}(\mathbf{y},t)}{\mathrm{c}}\right].$$

Here the statement on the left is the well-known definition, which combines the kinetic and potential energy present in φ. In the statement on the right, the two squared terms have been expanded out, replacing the square of the magnitude of $\mathbf{u}$ with a dot product, and then all quantities have been re-written in terms of the velocity potential φ.

Consider now the presence of a second acoustic wave ψ which also satisfies the wave equation in volume V. Its pressure and particle velocity will be denoted by q and $\mathbf{v}$ respectively, so $q = -\rho\dot{\psi}$ and $\mathbf{v} = \nabla\psi$. The acoustic energy density of the sum of these two waves is:

(2) $$ {}_{(\varphi+\psi)} = \tfrac{1}{2}\rho \left[[\nabla\varphi + \nabla\psi] \cdot [\nabla\varphi + \nabla\psi] + \frac{[\dot{\varphi} + \dot{\psi}]}{\mathrm{c}} \frac{[\dot{\varphi} + \dot{\psi}]}{\mathrm{c}} \right]. $$

Note that the dependence of $E_{\varphi+\psi}$, φ and ψ on $(\mathbf{y}, t)$ has been omitted here to save space. Expanding out the various combinations of φ and ψ which occur we find that terms of the following form naturally arise, which we denote $E_{\varphi\psi}$:

(3) $$ \begin{aligned} {}_{\varphi\psi}(\mathbf{y},t) &= \tfrac{1}{2}\rho \left[\nabla\varphi(\mathbf{y},t) \cdot \nabla\psi(\mathbf{y},t) + \frac{\dot{\varphi}(\mathbf{y},t)}{\mathrm{c}} \frac{\dot{\psi}(\mathbf{y},t)}{\mathrm{c}} \right] \\ &= \tfrac{1}{2}\rho \left[\mathbf{u}(\mathbf{y},t) \cdot \mathbf{v}(\mathbf{y},t) + \frac{p(\mathbf{y},t)}{\rho\mathrm{c}} \frac{q(\mathbf{y},t)}{\rho\mathrm{c}} \right]. \end{aligned} $$

Exploiting the subscript notation we can write that $E_{(\varphi+\psi)} = {}_{\varphi\varphi} + {}_{\varphi\psi} + {}_{\psi\varphi} + {}_{\psi\psi}$. It seems appropriate to call this new quantity ${}_{\varphi\psi}$ the 'acoustic cross-energy density', akin to the meaning of cross-covariance in signal processing. Appropriately, it is symmetric ($E_{\varphi\psi} = {}_{\psi\varphi}$) and reduces to the standard definition for acoustic energy density in Eq. 1 in the "auto" case (i.e. ${}_{\varphi\varphi} = {}_{\varphi}$).

2.2 Physical interpretation of cross-energy

Having introduced the concept of acoustic cross-energy and defined the new quantity $E_{\varphi\psi}$, it is appropriate to consider its physical interpretation. To this end we examine further the parallel with the use of covariance in signal processing. For two signals $x(t)$ and $y(t)$, both with zero mean, the cross-covariance is defined as:

(4) $$ s_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t+\tau) dt. $$

Equation 4 is typically understood as being a measure of the similarity of two signals, the second signal y being subjected to a 'lag' τ. It also has an established interpretation as a measure of the energy two signals have in common, and its Fourier transform is the cross-power spectral density of the two signals. Here we are interested primarily in the case where $\tau = 0$, for which the cross-covariance can also be written using inner product notation as $s_{xy}(0) = \langle x, y \rangle$. This also has an interpretation as an energy norm; indeed $s_{xx}(0)$ gives the mean-square value of signal x. It should be noted however that these quantities can only be regarded as proportional to the energy in the signal, since the characteristic impedance of the transmission medium is unknown; the mean-square value of signal x is proportional to the energy it would dissipate into a hypothetical resistive load.

In contrast, the instantaneous acoustic energy density in Eq. 1 is a precise statement for the acoustic energy density present in a wave. This is possible because it includes both pressure and particle velocity and takes into account the density ρ and wave-speed c in the medium. This means it gives information on the energy present without time integration. It is however a spatial density and must be integrated over a volume in order to compute Joules instead of Joules per metre cubed.

Returning to the new cross-energy density quantity in Eq. 3 and the relation that $E_{(\varphi+\psi)} = {}_{\varphi\varphi} + {}_{\varphi\psi} + {}_{\psi\varphi} + {}_{\psi\psi}$, it is clear that $E_{\varphi\varphi} + E_{\psi\psi}$ is the sum of energy density present in the two waves φ and ψ individually. This term is always positive and predicts an increase of +3dB if two waves of equal energy are present; it is equivalent to the sound power addition which occurs when signals that are incoherent in time are summed. In contrast to this, it is known that energy density would quadruple (+6dB) if the waves φ and ψ are identical, of vanish if $\varphi = -\psi$ (i.e. destructive interference is occurring); incoherent energy addition does not give the complete picture.

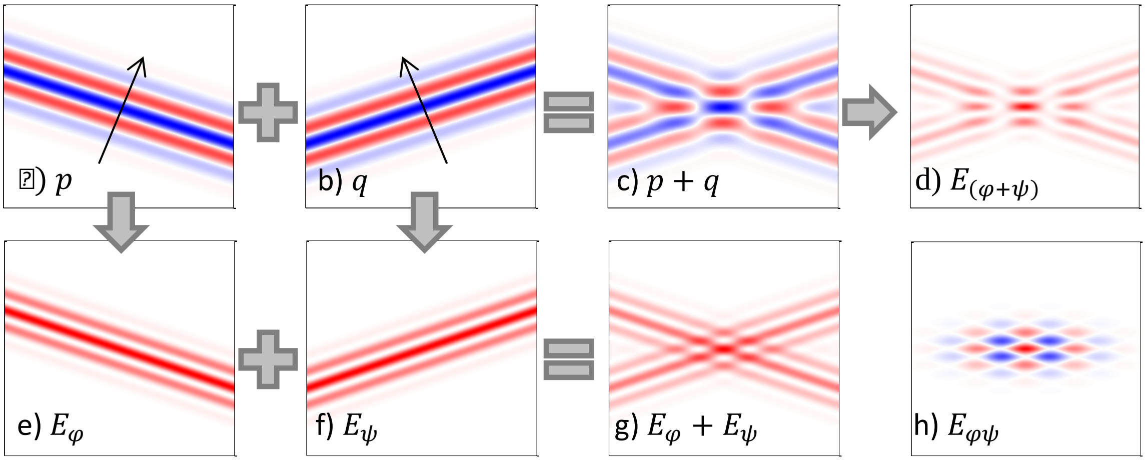

Figure 1: Two plane waves φ and ψ and their energy densities (see text for definition of symbols in subplots captions). Both are modulated Gaussian functions with identical pulse length and modulation parameters, but with propagation directions (indicated by arrows in a and b) which differ by 40°. Note that colour-scales differ between sub-plots, though red indicates positive values and blue indicates negative values in all cases.

The cross-energy terms ${}_{\varphi\psi} + {}_{\psi\varphi}$ are the difference between the exact result and the one given by simple power addition. It appears therefore that these terms must in some sense measure the extent to which the waves φ and ψ are 'similar'; this is consistent with the understanding of cross-covariance as a measure of the similarity of signals. In particular, if φ and ψ are identical then it is easy to show that $E_{\varphi\psi} = E_{\psi\varphi} = E_{\varphi\varphi} = E_{\psi\psi}$, so $E_{(\varphi+\psi)} = 4E_{\varphi\varphi}$ (+6dB increase), or if $\varphi = -\psi$ then ${}_{\varphi\psi} = {}_{\psi\varphi} = -E_{\varphi\varphi} = -E_{\psi\psi}$, so $E_{(\varphi+\psi)} = 0$. Based on these two extremes (and the form of Eq. 3) we would expect the cross-energy density to obey a Cauchy-Schwarz inequality of the form $|E_{\varphi\psi}| \leq \sqrt{E_{\varphi\varphi}E_{\psi\psi}}$. Other axioms such as linearity and positive-definiteness are also easily proven.

Other more subtle combinations produce patterns in which $E_{\varphi\psi}$ differs in sign in different locations. Figure 1 illustrates an example when φ and ψ are plane waves which are identical except for their propagation direction. Here it can be seen that simply adding the energy density of the two waves (Fig. 1g) fails to capture the interference effects which occur between in the two waves and are visible in both Fig. 1c and Fig. 1d. The cross-energy density, plotted in Fig. 1h, is (half) the difference between $E_{(\varphi+\psi)}$ in Fig. 1d and $E_\varphi + E_\psi$ in Fig. 1g. This shows what the error in due to incoherent energy addition is.

2.3 Time-averaged acoustic energy density for time-harmonic waves

Care has to be taken when using complex time-harmonic notation with energy measures, since there is potential for confusion between instantaneous and time-averaged quantities, and for differing amplitude definitions when mixing real and complex-valued waves. In particular the instantaneous acoustic energy density of a real-valued time harmonic wave comprises a time-invariant part plus a part which oscillates at twice the frequency of the field quantities.

The time-averaged acoustic energy density for a time-harmonic wave with angular frequency ω is:

(5) $$\bar{}_{\varphi}(\mathbf{y},\omega) = \tfrac{1}{2}\rho[\nabla\Phi(\mathbf{y},\omega)\cdot\nabla\Phi^*(\mathbf{y},\omega) + k^2\Phi(\mathbf{y},\omega)\Phi^*(\mathbf{y},\omega)].$$

Here we have added a bar over ${}_{\varphi}$ to indicate the presence of temporal averaging. Φ denotes the complex spatial amplitude of φ, i.e. $\varphi(\mathbf{y},t) = \mathrm{Real}\big(\Phi(\mathbf{y},\omega)e^{-\mathrm{i}\omega t}\big)$, and conjugates (indicated by an asterisk) have been introduces on the second quantity in each product to be consistent with the original definition of acoustic energy density, which is stated in terms of magnitudes squared. In addition, the time derivatives have been evaluated in the second term; these amount to multiplication by $-\mathrm{i}\omega$ and $\mathrm{i}\omega$ for each term respectively (due to the presence of the conjugate), which after division by c^2 produces the term k^2, where $k = \omega/c$ is the wavenumber in radians per meter.

Equation 5 differs from that used by Pierce in section 1-11 of Ref. 3 in that he has a factor of a ¼ outside the bracket. This arises due to the aforementioned differences in notation convention, and is discussed in more detail by Morse and Ingard[4] (see section 6.2). We have opted to use a ½ in Eq. 5 since this then matches with the standard definition of time-averaged acoustic intensity used in the next section. Morse and Ingard point out that for this to be exact the complex amplitudes, such as Φ, must be regarded as RMS quantities, that is the peak amplitude normalized by a factor of $\sqrt{2}$.

The time averaged acoustic cross-energy density is equivalently given by:

$$\overline{\ }_{\varphi\psi}(\mathbf{y},\omega) = \tfrac{1}{2}\rho[\nabla\Phi(\mathbf{y},\omega)\cdot\nabla\Psi^*(\mathbf{y},\omega) + k^2\Phi(\mathbf{y},\omega)\Psi^*(\mathbf{y},\omega)]. \tag{6}$$

Here $\Psi(\mathbf{y},\omega)$ is the complex spatial amplitude of wave ψ. Unlike $\overline{\ }_{\varphi}$ which is purely real, $\overline{\ }_{\varphi\psi}$ may be complex. However it is also conjugate symmetric, meaning $\overline{\ }_{\varphi\psi} + \overline{\ }_{\psi\varphi}$ is purely real and $\overline{\ }_{(\varphi+\psi)} = \overline{\ }_{\varphi\varphi} + \overline{\ }_{\varphi\psi} + \overline{\ }_{\psi\varphi} + \overline{\ }_{\psi\psi}$ as expected.

3. Acoustic Intensity

3.1 Instantaneous acoustic intensity for transient waves

An accompanying quantity is instantaneous acoustic power-flux density (i.e. the flow of acoustic energy density), better known as acoustic intensity. It is defined as:

$$\mathbf{I}_{\varphi}(\mathbf{y},t) = p(\mathbf{y},t)\mathbf{u}(\mathbf{y},t) = -\rho\dot{\varphi}(\mathbf{y},t)\nabla\varphi(\mathbf{y},t) \tag{7}$$

Using only the property that φ satisfies the wave equation it is straightforward to show that $\ _{\varphi}$ and $\mathbf{I}_{\varphi}$ are related by the well-known energy-flux relation $\dot{\ }_{\varphi} = -\nabla\cdot\mathbf{I}_{\varphi}$. The divergence theorem may be applied to this over a connected volume V bounded by a surface S:

$$\iiint_V \dot{E}_{\varphi}(\mathbf{y},t)dV = \iint_S \hat{\mathbf{n}}_{\mathbf{y}}\cdot\mathbf{I}_{\varphi}(\mathbf{y},t)\,dS \tag{8}$$

Here $\hat{\mathbf{n}}_{\mathbf{y}}$ is a unit vector normal to surface S at point $\mathbf{y}$; we have followed the convention used in the Boundary Element Method (BEM) and defined $\hat{\mathbf{n}}_{\mathbf{y}}$ to point <u>into</u> the volume V enclosed by S, hence there is no minus sign in Eq. 8. This statement is sometimes referred to as the "acoustic energy conservation law" (see section 1.11 of Ref. 4, where this is in turn cited to Kirchhoff). It has the physical interpretation that *within a lossless medium, energy is not created or destroyed and any change in total energy (versus time) is due to power flow through the surface bounding the volume under consideration*. Note that if the medium includes sources these must be excluded from V in order for Eq. 8 to hold (as is done in the derivation of the Kirchhoff-Helmholtz Boundary Integral Equation).

We propose a corresponding quantity "cross-intensity" $\mathbf{I}_{\varphi\psi}$, defined as:

$$\begin{aligned}\mathbf{I}_{\varphi\psi}(\mathbf{y},t) &= \tfrac{1}{2}[p(\mathbf{y},t)\mathbf{v}(\mathbf{y},t) + q(\mathbf{y},t)\mathbf{u}(\mathbf{y},t)] \\ &= -\tfrac{1}{2}\rho\left[\dot{\varphi}(\mathbf{y},t)\nabla\psi(\mathbf{y},t) + \dot{\psi}(\mathbf{y},t)\nabla\varphi(\mathbf{y},t)\right]\end{aligned} \tag{9}$$

Importantly, this definition satisfies $\dot{E}_{\varphi\psi} = -\nabla\cdot\mathbf{I}_{\varphi\psi}$ so $\mathbf{I}_{\varphi\psi}$ naturally has an interpretation as the flux of $\dot{E}_{\varphi\psi}$. It could therefore equivalently be called the instantaneous acoustic cross-power-flux density. Like $E_{\varphi\psi}$ it is symmetric and reduces to the standard definition of acoustic intensity in the "auto" case i.e. $\mathbf{I}_{\varphi\varphi} = \mathbf{I}_{\varphi}$. Exploiting the subscript notation we can write a similar addition rule:

$\mathbf{I}_{(\varphi+\psi)} = \mathbf{I}_\varphi + \mathbf{I}_\psi + \mathbf{I}_{\varphi\psi} + \mathbf{I}_{\psi\varphi}$. Following the reasoning in section 2.2, since $\mathbf{I}_\varphi + \mathbf{I}_\psi$ represents the incoherent power addition of the energy flow due to the two waves φ and ψ, $\mathbf{I}_{\varphi\psi}$ must in some sense represent the power flow which these two waves have in common.

In addition, we can apply the divergence theorem to the relation $\dot{}_{\varphi\psi} = -\nabla \cdot \mathbf{I}_{\varphi\psi}$ to obtain the surprising and remarkable statement:

(10) $$\iiint_V \dot{E}_{\varphi\psi}(\mathbf{y},t)dV = \iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot \mathbf{I}_{\varphi\psi}(\mathbf{y},t)\, dS.$$

Equation 10 shows that the acoustic energy conservation law also applies to cross-energy, and can be interpreted equivalently i.e. *within a lossless medium cross-energy is not created or destroyed and any change in total cross-energy (versus time) is due to cross-power flow through the surface bounding the volume under consideration*.

In addition, integrating Eq. 10 in time gives:

(11) $$\iiint_V \ _{\varphi\psi}(\mathbf{y},t_1)dV = \iiint_V E_{\varphi\psi}(\mathbf{y},t_0)dV + \int_{t_0}^{t_1} \iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot \mathbf{I}_{\varphi\psi}(\mathbf{y},t)\, dS\, dt$$

Equation 11 states that the difference between the total cross-energy of wave φ in volume V at time t_0 and at time t_1 is the time integral of the cross-intensity passing through the boundary of the volume between those two times. If we assume that φ and ψ are transient and t_0 is sufficiently far in the past that they had not entered V yet, then it follows that $E_{\varphi\psi}(\mathbf{y},t_0)$ is zero in all of V and the volume integral on the right hand side vanishes. This means that anything integrating $\ _{\varphi\psi}$ over V can tell us about the similarity of φ and ψ at the present time, can also be found by integrating the time history of $\hat{\mathbf{n}}_\mathbf{y} \cdot \mathbf{I}_{\varphi\psi}$**.** Thus the latter can also be used as a wave similarity measure.

3.2 Time-averaged acoustic intensity

Time-averaged acoustic intensity is given by:

(12) $$\begin{aligned} \bar{\mathbf{I}}_\varphi(\mathbf{y},\omega) &= \mathrm{Real}\big(P(\mathbf{y},\omega)\boldsymbol{U}^*(\mathbf{y},\omega)\big) \\ &= \tfrac{1}{2}[P(\mathbf{y},\omega)\boldsymbol{U}^*(\mathbf{y},\omega) + P^*(\mathbf{y},\omega)\boldsymbol{U}(\mathbf{y},\omega)] \\ &= \tfrac{1}{2}i\omega\rho[\Phi(\mathbf{y},\omega)\nabla\Phi^*(\mathbf{y},\omega) - \Phi^*(\mathbf{y},\omega)\nabla\Phi(\mathbf{y},\omega)] \end{aligned}$$

Here the first statement is the standard definition, the second statement has the Real operator expanded as half the sum of the argument and its conjugate, and the third statement has been written in terms of velocity potential Φ. Note that the minus sign between the two terms arise because $\dot{\Phi}(\mathbf{y}) = -\mathrm{i}\omega\Phi(\mathbf{y})$ whereas $\dot{\Phi}^*(\mathbf{y},\omega) = +\mathrm{i}\omega\Phi^*(\mathbf{y},\omega)$. Note also that this Eq. 12 differs from Eq. 20 in Ref. 2 because the Φ represented pressure there and here it represents velocity potential.

The corresponding time-averaged acoustic cross-intensity is defined as:

(13) $$\bar{\mathbf{I}}_{\varphi\psi}(\mathbf{y},\omega) = \tfrac{1}{2}i\omega\rho[\Phi(\mathbf{y},\omega)\nabla\Psi^*(\mathbf{y},\omega) - \Psi^*(\mathbf{y},\omega)\nabla\Phi(\mathbf{y},\omega)]$$

Like was the case for $\bar{\ }_{\varphi\psi}$, it is possible that $\bar{\mathbf{I}}_{\varphi\psi}$ may be complex. However it is also conjugate symmetric, meaning $\bar{\mathbf{I}}_{\varphi\psi} + \bar{\mathbf{I}}_{\psi\varphi}$ is purely real and $\bar{\mathbf{I}}_{(\varphi+\psi)} = \bar{\mathbf{I}}_{\varphi\varphi} + \bar{\mathbf{I}}_{\varphi\psi} + \bar{\mathbf{I}}_{\psi\varphi} + \bar{\mathbf{I}}_{\psi\psi}$ as expected.

3.3 Equivalence with Green's 2nd Theorem

The time-averaged quantities obey the same divergence identities that the instantaneous quantities do, except that the time-averaged energy density is of course time-invariant so $\partial\bar{E}_\varphi/\partial t = 0$, leading to the well-known result for time-averaged intensity (see e.g. section 1-11 of Ref 3) that:

(14) $$\iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot \bar{\mathbf{I}}_\varphi(\mathbf{y},\omega)dS = 0.$$

This equation is extremely useful since it allows the sound power of a source to be measured in situ by summing a set of intensity measurements performed over an enclosing surface.

The same properties also apply to the cross quantities we have proposed, hence we also have:

(15) $$\iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot \bar{\mathbf{I}}_{\varphi\psi}(\mathbf{y},\omega)dS = 0.$$

This remarkable statement has the physical interpretation that time-averaged cross-energy entering and leaving a region must sum to zero for a time-harmonic problem; i.e. total acoustic cross-energy in the enclosed volume V is constant. It also has a measurement interpretation, which is discussed in more details in Ref. 2. As suggested in the introduction, this allows us to find the amount of a testing wave ψ present in a measured wave φ, simply by weighting measurements of $\nabla\Phi$ and Φ according to Eq. 13 and integrating over a surface.

Furthermore, examining the form of $\bar{\mathbf{I}}_{\varphi\psi}$ in Eq. 13 it is also apparent that Eq. 15 is equivalent to Green's second theorem applied to acoustic waves, once the fact that Φ and Ψ satisfy the Helmholtz equation has been used to eliminate the volume integral. The only key different is that a conjugate has been applied to Ψ in Eq. 13, but this is really just a minor change in the definition of Ψ; omitting it corresponds to time reversal (under which the wave equation is still satisfied).

This can be exemplified by considering the Kirchhoff-Helmholtz boundary integral equation:

(16) $$\Phi(\mathbf{x},\omega) = \iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot [\Phi(\mathbf{y},\omega)\nabla G(\mathbf{x},\mathbf{y},\omega) - G(\mathbf{x},\mathbf{y},\omega)\nabla\Phi(\mathbf{y},\omega)]dS.$$

Here $G(\mathbf{x},\mathbf{y},\omega) = e^{ik|\mathbf{x}-\mathbf{y}|}/4\pi|\mathbf{x}-\mathbf{y}|$ is the free-space Green's function. The standard physical interpretation of this is of boundary monopole and dipole sources, with amplitudes weighted by $\hat{\mathbf{n}}_\mathbf{y} \cdot \nabla\Phi$ and Φ respectively, emanating into the volume V and reconstructing Φ at an observer position $\mathbf{x}$. This interpretation is depicted in Fig. 2a and is the basis of Wave Field Synthesis.

Alternatively, Eq. 15 gives the following if it assumed that $\Psi = G^*$, i.e. a contracting spherical wave which coalesces at $\mathbf{x}$. Note that a vanishingly small region around $\mathbf{x}$ must be excluded from V before the divergence theorem can be applied, and it is this (as in the derivation of Eq. 16) which produces the term $\Phi(\mathbf{x},\omega)$ on the left hand side:

(17) $$\begin{aligned}\Phi(\mathbf{x},\omega) &= \iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot \bar{\mathbf{I}}_{\varphi\psi}(\mathbf{y},\omega)dS \\ &= \iint_S \hat{\mathbf{n}}_\mathbf{y} \cdot [\Phi(\mathbf{y},\omega)\nabla\Psi^*(\mathbf{x},\mathbf{y},\omega) - \Psi^*(\mathbf{x},\mathbf{y},\omega)\nabla\Phi(\mathbf{y},\omega)]dS.\end{aligned}$$

It should be immediately obvious that these are equal given the definition of Ψ used. However given what has been discussed so far, Eq. 17 has another interpretation, of a microphone array designed to sense the wave Ψ as depicted in Fig. 2b. This essentially performs a spatial cross-correlation over S, collecting cross-intensity and mapping it across V such that the correct value of Φ at $\mathbf{x}$ is found (assuming S encloses $\mathbf{x}$).

Finally, we ask what the time domain equivalent of Eq. 15 might be for transient signals. For this we return to Eq. 11 and assume that $t_0 \to -\infty$, before any acoustic energy has arrived, and $t_1 \to +\infty$, at which point it has all left. Substituting in these two limits eliminates the volume integrals giving:

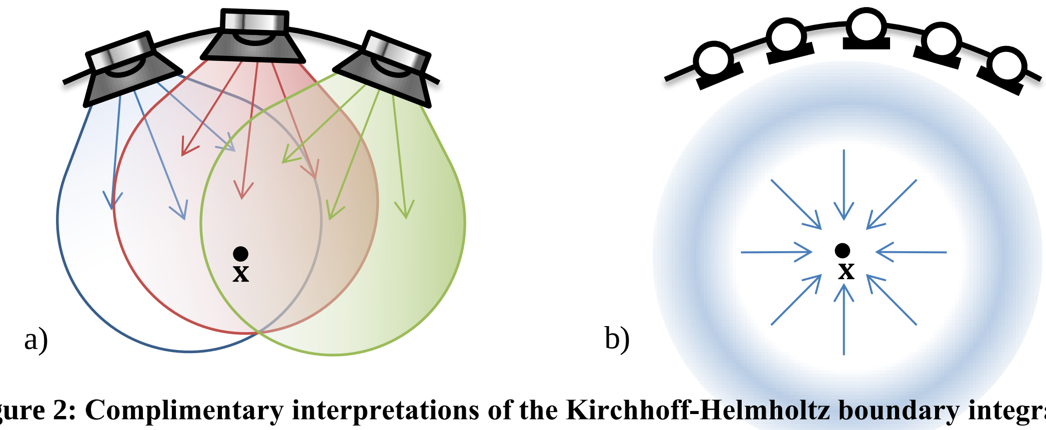

Figure 2: Complimentary interpretations of the Kirchhoff-Helmholtz boundary integral equation. a): the standard 'Wave Field Synthesis' interpretation of monopole and dipole boundary sources radiating waves which sum to form the wave at an observer point x. b): the new 'microphone array' interpretation, computing the cross-intensity between the measured wave and a 'testing wave' (blue), being a contracting spherical wave which coalesces at x.

(18) $$\int_{-\infty}^{\infty} \iint_S \hat{\mathbf{n}}_{\mathbf{y}} \cdot \mathbf{I}_{\varphi\psi}(\mathbf{y}, t)\, dS\, dt = 0$$

This states that for a transient problem *the net acoustic cross-power flux through a surface bounding a lossless medium is zero*, i.e. any acoustic cross-energy which enters a region will ultimately leave it again.

4. Conclusions and further work

That concludes the findings which it has been possible to report in these conference proceedings. Further application ideas pertaining to microphone array design, in particular relating this work to that of Hulsebos *et al*[5], and to coupling acoustic prediction models with Auralisation hardware, will be discussed in the lecture presentation.

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