NEW AND MODIFIED METHODS FOR ASSESSING RELIABILITY EQUIVALENCE

by

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SUBMITTED IN PARTIAL FULFILMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
AT
UNIVERSITY OF SALFORD
MANCHESTER M5 4WT UK
JUNE 2015

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

I dedicate this thesis to my wonderful and supportive parents

Mohammed & Ghawadah

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Acknowledgement

I have so much to be thankful for, first and foremost, for Allah the Lord of the universe for the countless blessings he has bestowed on me, both in general and particularly during my work on this thesis.

My main appreciation and special thanks go to my supervisor, Prof. David Percy, for his unlimited support, expert advice and guidance. Without his willingness and invaluable suggestions this PhD thesis could not have been completed. I appreciate his high standards, his ability, his knowledge of research methodology. One simply could not wish for a better and more friendly supervisor, which I appreciate from my heart. It is really hard to find the words to express my gratitude and appreciation for him.

I am extremely grateful to my parents who gave me everything a son could wish for, including support throughout the writing of this thesis, especially when I needed encouragement.

I want to acknowledge my closest supporters, my beloved wife Azzah, for her confidence and constant encouragement, and my kids Raghad, Mohammed, Rose and Omar, who lighten my life with their smiles and ever present love.

I am also grateful to my brothers and sisters who want the best and success for me, and have been a great source of motivation and support through all these years.

Finally, I wish to thank my country, Saudi Arabia for granting me the scholarship and providing me with this great opportunity for completing my studies in the UK and so enabling me to fulfil my ambition.

Abstract

This thesis investigates methods for assessing reliability equivalence factors for several common systems that comprise independent components or subsystems. We consider improving the reliability of the systems by (a) a reduction method and (b) several duplication methods: (i) hot duplication; (ii) cold duplication with perfect switching; (iii) cold duplication with imperfect switching. Two measures for comparing system improvements are considered in this study, survival reliability equivalence factors and mean reliability equivalence factors.

We apply our study to: (1) some simple systems including parallel-series and series-parallel systems, with flexible lifetime distributions including generalized quadratic failure rate and exponentiated Weibull lifetime distributions; (2) networks and complex systems with multiple types of components. We choose the exponentiated Weibull and generalized quadratic failure rate distributions because they are flexible and enable comparisons with other reliability equivalence studies.

We use the concept of survival signature to derive the reliability equivalence factors for any coherent system with any structure and with different lifetime distributions. In order to implement this approach, we use the `ReliabilityTheory` R package to derive survival reliability equivalence factors and mean reliability equivalence factors for networks and complex systems with multiple types of components.

Numerical examples for simple and complex systems are presented, to illustrate how to apply the theoretical results and demonstrate the relative benefits of various system improvements. We explain and discuss the results obtained by presenting summary tables and figures, before presenting conclusions and recommendations that

arise from this study. In particular, we deduce that considerable advances in reliability equivalence testing are made possible by specifying and analysing the survival signature, especially for the increasingly common context and practice of modelling networks and complex systems.

Abbreviations

The following abbreviations are used throughout the report.

ED	Exponential distribution
RD	Rayleigh distribution
WD	Weibull distribution
GED	Generalized exponential distribution
GRD	Generalized Rayleigh distribution
LFRD	Linear failure rate distribution
GLFRD	Generalized linear failure rate distribution
QFRD	Quadratic failure rate distribution
GQFRD	Generalized quadratic failure rate distribution
EWD	Exponentiated Weibull lifetimes
SREF	Survival reliability equivalence factor
MREF	Mean reliability equivalence factor
MTTF	Mean time to failure
RF	Reliability function
$i.i.d.$	Independent and identically distributed
$F(x)$	Cumulative distribution function (failure function)
$f(x)$	Probability density function
$R(x)$	Survival function (reliability function)
$h(x)$	Failure rate (hazard function)

Chapter 1

Introduction

System design involves a blend of efficient design techniques with optimal performance. In the reliability research domain, systems design can be improved by duplication (redundancy) methods. The duplication methods are plagued by space limitation and higher development costs. On the other hand, the reduction methods involve reducing the system components' failure rates which makes it more appealing. For example, the battery life of the smart phone is a serious problem facing users so many people carry an extra power source. In the meantime manufacturing companies are working hard to produce new types of batteries with expected lifetimes which are at least double that of the current battery.

This thesis investigates equivalence between the reduction and duplication methods, using carefully selected reliability equivalence factors. We study reliability equivalence factors for simple systems with flexible lifetime distributions including generalized quadratic failure rate and exponentiated Weibull lifetime distributions. We choose the exponentiated Weibull and generalized quadratic failure rate distributions because of their flexibility and because they generalize most of the studies in this field. In addition we introduce a new methodology to study the reliability equivalence factors for networks and complex systems with multiple types of components using the survival signature. This chapter gives an introduction to the fundamental concepts involved in our study including:

- generalized quadratic failure rate distribution;
- exponentiated Weibull distribution;
- reliability equivalence factors;
- signature and survival signature;
- `ReliabilityTheory` R package.

1.1 Flexible lifetime distributions

In the first part of our research we apply our study on systems with flexible lifetime distributions including generalized quadratic failure rate and exponentiated Weibull lifetime distributions. We apply our study at the beginning on a system of components with generalized quadratic failure rate distribution because this generalizes seven well known lifetime distributions and so generalizes several existing published studies on reliability equivalence factors. The Weibull distribution is a very popular lifetime distribution, unfortunately it is not a special case of the generalized quadratic failure rate distribution. Consequently, we also determine reliability equivalence factors for a system with exponentiated Weibull lifetime distributions, which include the Weibull distribution as a special case. Both generalized quadratic failure rate and exponentiated Weibull lifetime distributions are flexible and they have nice statistical properties as a result of generalizing many useful lifetime distributions. We present the key properties of those distributions by means of the following three points:

- derivation and definition of probability distributions;
- formulation of hazard functions and probability density functions;
- fitting the distributions to data.

1.1.1 Generalized quadratic failure rate distribution

This is a recently proposed lifetime distribution studied by Alghamdi (2008) and published by Sarhan and Alghamdi (2009) and Sarhan (2009a). The generalized quadratic failure rate distribution generalizes many useful lifetime distributions, including the generalized linear failure rate distribution, generalized exponential distribution, generalized Rayleigh distribution and quadratic failure rate distribution.

Researchers in statistics and life testing are interested in looking for suitable distributions with nice properties that enable them to describe the lifetimes of many industrial devices. Among those distributions are distributions with constant failure rate, distributions with increasing failure rate, distributions with decreasing failure rate, distributions with bath-tub shaped failure rate and distributions with upside down bath-tub failure rate. The generalized quadratic failure rate distribution has all of the aforementioned properties. This distribution can be used to describe the lifetime of an item (component) for which the failure rate may be constant, increasing, decreasing, the bath-tub shape or upside-down bath-tub shape.

We notate the generalized quadratic failure rate distribution by GQFRD$(\alpha, \beta, \gamma, \theta)$

and we say that the random variable T has a generalized quadratic failure rate distribution if its failure (cumulative distribution) function takes the form

$$F(t) = \left(1 - e^{-\alpha t - \frac{\beta}{2}t^2 - \frac{\gamma}{3}t^3}\right)^{\theta}, \qquad t \geq 0 \tag{1.1.1}$$

where $\alpha > 0$, $\gamma > 0$, $\theta > 0$ and $\beta \geq -2\sqrt{\alpha\gamma}$. This restriction on the parameter space is made to ensure that a hazard function with the following form is positive, as identified by Bain (1974) for the simpler, quadratic failure rate distribution:

$$\alpha + \beta t + \gamma t^2 \geq 0, \qquad t \geq 0.$$

If $T \sim \text{GQFRD}(\alpha, \beta, \gamma, \theta)$, then the reliability (survival) function of T is given by

$$R(t) = 1 - F(t) = 1 - \left[1 - e^{-\left(\alpha t + \frac{\beta}{2}x^2 + \frac{\gamma}{3}t^3\right)}\right]^{\theta}, \qquad t \geq 0. \tag{1.1.2}$$

where $\alpha > 0$, $\gamma > 0$, $\theta > 0$ and $\beta \geq -2\sqrt{\alpha\gamma}$. If T has a cumulative distribution function given by (1.1.1), the corresponding probability density function is given by

$$f(t) = F'(t) = \theta(\alpha + \beta t + \gamma t^2)\left[1 - e^{-\left(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3\right)}\right]^{\theta - 1} e^{-\left(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3\right)}, \qquad t \geq 0, \tag{1.1.3}$$

where $\alpha > 0$, $\gamma > 0$, $\theta > 0$ and $\beta \geq -2\sqrt{\alpha\gamma}$.

Figure 1.1 illustrates the probability density function of $\text{GQFRD}(\alpha, \beta, \gamma, \theta)$ for different parameter values. From this figure, it is apparent that the density can be decreasing or unimodal.

The failure rate (hazard) function of $\text{GQFRD}(\alpha, \beta, \gamma, \theta)$ takes the form

$$h(t) = \frac{f(t)}{R(t)} = \frac{\theta(\alpha + \beta t + \gamma t^2)\left[1 - e^{-\left(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3\right)}\right]^{\theta - 1} e^{-\left(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3\right)}}{1 - \left[1 - e^{-\left(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3\right)}\right]^{\theta}}, \qquad t \geq 0. \tag{1.1.4}$$

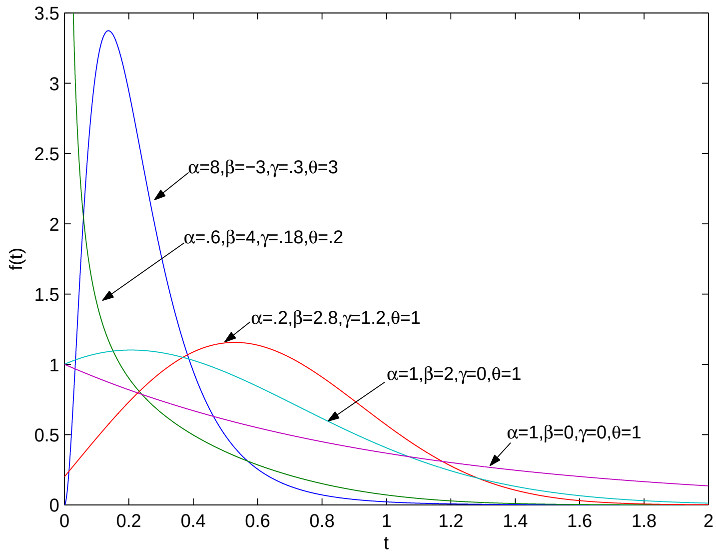


Figure 1.1: Probability density function of the GQFRD($\alpha, \beta, \gamma, \theta$).

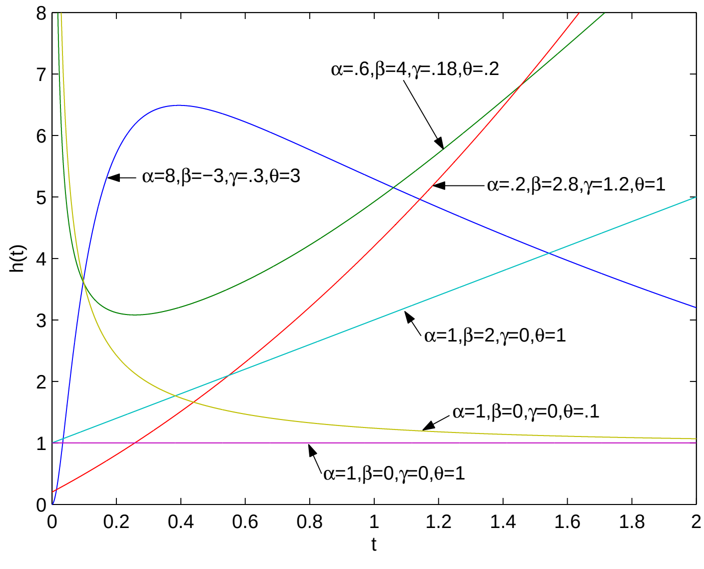


Figure 1.2: Failure rate (hazard function) of the GQFRD($\alpha, \beta, \gamma, \theta$).

Figure 1.2 presents the failure rate (hazard) function of GQFRD($\alpha, \beta, \gamma, \theta$) for different parameter values. From this figure, one can see that the hazard function can be increasing, decreasing, linear, constant, bath-tub shaped or upside-down bath-tub shaped. Further, one can easily verify from Identity (1.1.4) that:

- if $\theta = 1$, the hazard function is either increasing (if $\beta > 0$) or constant (if $\beta = 0$ and $\alpha > 0$);
- if $\theta > 1$, the hazard function is either increasing (if $\beta > 0$) or upside-down bath-tub shaped (if $\beta < 0$); and
- if $\theta < 1$, then the hazard function is either decreasing (if $\beta = 0$) or bath-tub shaped (if $\beta \neq 0$)

1.1.2 Exponentiated Weibull distribution

Mudholkar and Srivastava (1993) modified the standard two-parameter Weibull distribution through the introduction of an additional parameter. This distribution has been studied deeply in Mudholkar and Hutson (1996), Jiang and Murthy (1999) and Nassar and Eissa (2003). The exponentiated Weibull distribution hazard function resembles the hazard function of the generalized quadratic failure rate distribution and may be constant, increasing, decreasing, bath-tub shape or upside-down bath-tub shape.

We notate the exponentiated Weibull distribution by EWD(α, β, θ) and we say that the random variable T has an exponentiated Weibull distribution if its failure (cumulative distribution) function takes the form

$$F(t) = \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}, \qquad \alpha, \beta, \theta > 0, \quad t \geq 0. \tag{1.1.5}$$

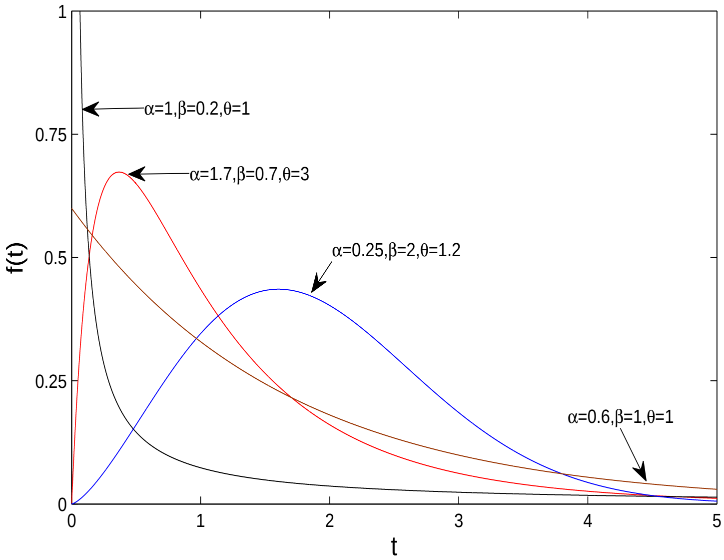


Figure 1.3: Probability density function of the exponentiated Weibull distribution.

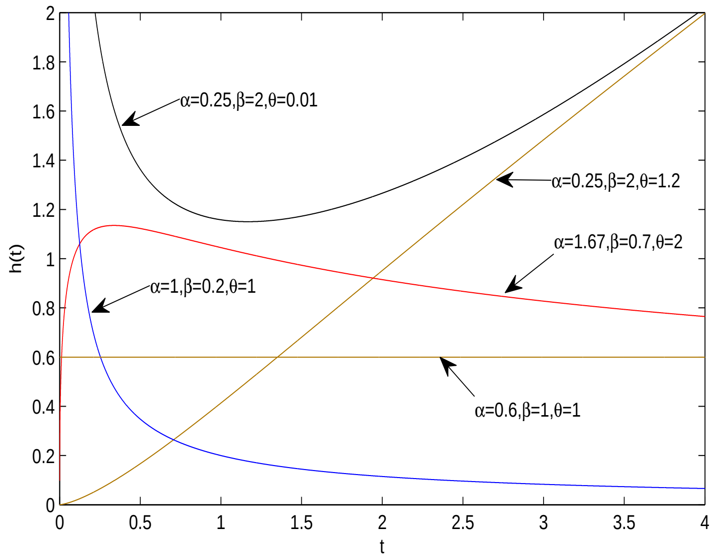


Figure 1.4: Failure rate (hazard function) of the exponentiated Weibull distribution.

If $T \sim \text{EWD}(\alpha,\beta,\theta)$, then the reliability (survival) function of T is given by

$$R(t) = 1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}, \qquad \alpha,\beta,\theta > 0, \quad t \geq 0. \tag{1.1.6}$$

If T has a cumulative distribution function given by (1.1.5), the corresponding probability density function is given by

$$f(t) = \alpha\beta\theta t^{\beta-1} e^{-\alpha t^{\beta}} (1 - e^{-\alpha t^{\beta}})^{\theta-1}, \qquad \alpha,\beta,\theta > 0, \quad t \geq 0. \tag{1.1.7}$$

Figure 1.3 illustrates the probability density function of $\text{EWD}(\alpha,\beta,\theta)$ for different parameter values. From this figure, it is apparent that this density can also be decreasing or unimodal.

The failure rate (hazard) function of $\text{EWD}(\alpha,\beta,\theta)$ takes the form

$$h(t) = \frac{\alpha\beta\theta t^{\beta-1} e^{-\alpha t^{\beta}} (1 - e^{-\alpha t^{\beta}})^{\theta-1}}{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}}, \qquad \alpha,\beta,\theta > 0, \quad t \geq 0. \tag{1.1.8}$$

Figure 1.4 presents the failure rate (hazard) function of $\text{EWD}(\alpha,\beta,\theta)$ for different parameter values and can be increasing, decreasing, linear, constant, bath-tub shaped or upside-down bath-tub shaped. The shape of the hazard function does not depend on α but it depends on β and θ as follows:

- if $\beta \leq 1$ and $\beta\theta \leq 1$, then the hazard function is decreasing;
- if $\beta \geq 1$ and $\beta\theta \geq 1$, then the hazard function is increasing;
- if $\beta < 1$ and $\beta\theta > 1$, then the hazard function takes upside-down bathtub shape;
- if $\beta > 1$ and $\beta\theta < 1$, then the hazard function takes the bathtub shape.

1.1.3 Derivation and definition of probability distributions

We now explain (1) how the GQFRD$(\alpha, \beta, \gamma, \theta)$ generalizes the following distributions: exponential, generalized exponential, linear failure rate, generalized linear failure rate, quadratic failure rate, Rayleigh and generalized Rayleigh, and (2) how the EWD(α, β, θ) generalizes the exponential, generalized exponential, Rayleigh, generalized Rayleigh and Weibull distribution.

Firstly, by using the failure function (1.1.1), the following are seen to be special cases of the generalized quadratic failure rate distribution:

1. The generalized linear failure rate distribution GLFRD(α, β, θ), see Sarhan et al. (2008b), by setting $\gamma = 0$.
$$F(t) = \left(1 - e^{-\alpha t - \frac{\beta}{2}t^2}\right)^{\theta}, \quad \alpha, \beta, \theta > 0, \quad t \geq 0. \tag{1.1.9}$$

2. The quadratic failure rate distribution QFRD(α, β, γ), see Bain (1974), by setting $\theta = 1$.
$$F(t) = 1 - e^{-\alpha t - \frac{\beta}{2}t^2 - \frac{\gamma}{3}t^3}, \quad \alpha, \gamma > 0, \quad \beta \geq -2\sqrt{\alpha\gamma}, \quad t \geq 0. \tag{1.1.10}$$

3. The linear failure rate distribution LFRD(α, β), see Lee (2003), by setting $\theta = 1, \gamma = 0$.
$$F(t) = 1 - e^{-\alpha t - \frac{\beta}{2}t^2}, \quad \alpha, \beta > 0, \quad t \geq 0. \tag{1.1.11}$$

4. The generalized Rayleigh distribution GRD(β, θ), see Surles and Padgett (1998), by setting $\alpha = 0, \gamma = 0$.
$$F(t) = \left(1 - e^{-\frac{\beta}{2}t^2}\right)^{\theta}, \quad \beta, \theta > 0, \quad t \geq 0. \tag{1.1.12}$$

5. The Rayleigh distribution RD(β), see Krishnamoorth (2006), by setting $\alpha = 0, \gamma = 0$, and $\theta = 1$.

$$F(t) = 1 - e^{-\frac{\beta}{2}t^2}, \quad \beta > 0, \quad t \geq 0. \tag{1.1.13}$$

6. The generalized exponential distribution GED(α, θ), see Gupta and Kundu (1999), by setting $\beta = 0$,$\gamma = 0$.

$$F(t) = \left(1 - e^{-\alpha t}\right)^{\theta}, \quad \alpha, \theta > 0, \quad t \geq 0. \tag{1.1.14}$$

7. The exponential distribution ED(α), see Krishnamoorth (2006), by setting $\beta = 0, \gamma = 0$, and $\theta = 1$.

$$F(t) = 1 - e^{-\alpha t}, \quad \alpha > 0, \quad t \geq 0. \tag{1.1.15}$$

Secondly, by using the failure function (1.1.5), the following are special cases of the exponentiated Weibull distribution:

1. The generalized Rayleigh distribution GRD(α, θ), by setting $\beta = 2$.

$$F(t) = \left(1 - e^{-\alpha t^2}\right)^{\theta}, \quad \alpha, \theta > 0, \quad t \geq 0. \tag{1.1.16}$$

2. The Rayleigh distribution RD(α), by setting $\theta = 1$.

$$F(t) = 1 - e^{-\alpha t^2}, \quad \alpha > 0, \quad t \geq 0. \tag{1.1.17}$$

3. The generalized exponential distribution GED(α, θ), by setting $\beta = 1$.

$$F(t) = \left(1 - e^{-\alpha t}\right)^{\theta}, \quad \alpha, \theta > 0, \quad t \geq 0. \tag{1.1.18}$$

4. The exponential distribution ED(α), by setting $\beta = 1$ and $\theta = 1$.

$$F(t) = 1 - e^{-\alpha t}, \quad \alpha > 0, \quad t \geq 0. \tag{1.1.19}$$

5. The Weibull distribution WD(α, β), see Weibull (1951), by setting $\theta = 1$.

$$F(t) = 1 - e^{-\alpha t^{\beta}}, \quad \alpha, \beta > 0, \quad t \geq 0. \tag{1.1.20}$$

1.1.4 Formulation of hazard functions and probability density functions

The shape of the failure rate of a distribution plays an important role in deciding whether this distribution can be used to fit a given data set. It is known that some lifetime distributions may have a constant failure rate, corresponding to the exponential distribution, and some distributions may have increasing failure rates, such as the Weibull distribution when the shape parameter exceeds 1 and the increasing linear failure rate distribution. Some others may have decreasing failure rates, such as the Weibull distribution when the shape parameter does not exceed 1 and the decreasing linear failure rate distribution. Yet other distributions may have all of these types of failure rates over different periods of time, such as those distributions having failure rate of the bath-tub curve shape. There are other distributions that have upside-down bath-tub shape failure rate. All these shapes are of practical value, so a family of distributions that includes these forms would be a useful modelling tool. The generalized quadratic failure rate distribution and exponentiated Weibull distribution have all of the aforementioned curve shapes. The curve of the failure rate function of those distributions may be constant, increasing, decreasing, bath-tub shape or upside-down bath-tub shape, as we can see in Figures 1.2 and 1.4. This property suggests that the generalized quadratic failure rate distribution and exponentiated Weibull distribution are flexible models that could be used to fit real data in many different fields of application.

1.1.5 Fitting the distributions to data

Two recent publications demonstrate applications of the generalized quadratic failure rate distribution to three sets of real data to examine how the GQFRD($\alpha, \beta, \gamma, \theta$) works in practice: Alghamdi (2008); Sarhan and Alghamdi (2009). Two different data sets were used, simple data taken from Aarset (1987) and censored data taken from McCool (1974). They found that, based on the likelihood ratio test statistic and Kolmogorov-Smirnov test, the generalized quadratic failure rate distribution fits different real data sets better than do other very well known and commonly used distributions, including generalized exponential, generalized Rayleigh, generalized linear failure rate, generalized Weibull and quadratic failure rate distributions.

Mudholkar and Srivastava (1993) applied the exponentiated Weibull distribution to the data of Aarset (1987). Mudholkar et al. (1995) used the exponentiated Weibull distribution to analyse bus failure data. Furthermore, the exponentiated Weibull distribution was used to analyse flood data by Mudholkar and Hutson (1996).

1.2 Reliability equivalence factors

There are two main methods for improving system reliability. The first is a duplication method and the second is a reduction method. In a duplication method, system reliability can be improved by adding extra components in parallel to some of the system components. In a reduction method, system reliability can be improved by reducing the failure rate for all or some components in the system, see Figures 1.5. To illustrate reduction and duplication methods for system improvement, consider the possible arrangements for reinforcing the layout of aeroplane wheels. Improving the aeroplane wheels according to duplication methods means extra wheels are added in parallel to the main wheels in the case of hot duplication, see Figure 1.6. Cold duplication assumes that extra wheels are added as standby, so that the pilot switches to these extra wheels when the main wheels fail to do the required job. Reduction assumes that reducing the failure rate by replacing standard wheels with better wheels can improve aeroplane wheel system reliability. A simple definition of the reliability equivalence factors can be introduced as the factors by which the failure rates of some system components should be reduced to reach a reliability similar to that of a system improved using a duplication method.

Råde introduced the concept of reliability equivalence factors in 1993 and applied it to a simple system with two independent and identically distributed components connected in parallel and in series Råde (1993a,b). He assumed an exponential lifetime distribution for each component. Sarhan performed many extensions based on this concept: Sarhan (2000, 2002, 2004, 2005); Sarhan and Mustafa (2006); Sarhan et al. (2008a). He suggested two methods to derive the reliability equivalence factors, which are the survival reliability equivalence factors and mean reliability equivalence factors. He applied these approaches to a large system of components including parallel-series and series-parallel with exponential lifetime distribution with identical and non-identical parameters.

Xia and Zhang (2007) applied this concept for a parallel system with independent and identically distributed components assuming a gamma distribution for the

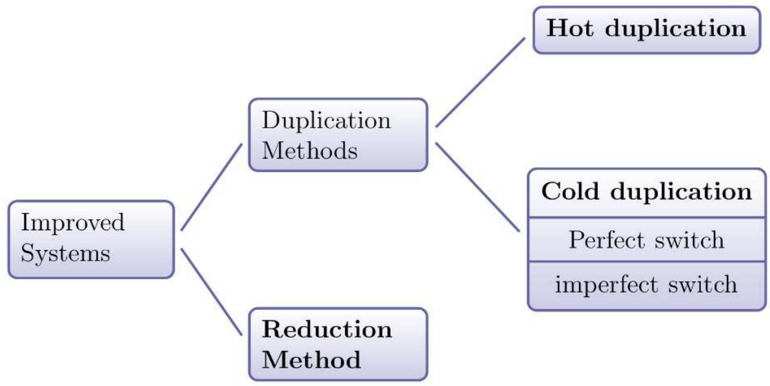


Figure 1.5: Reduction and duplication methods for improving systems.

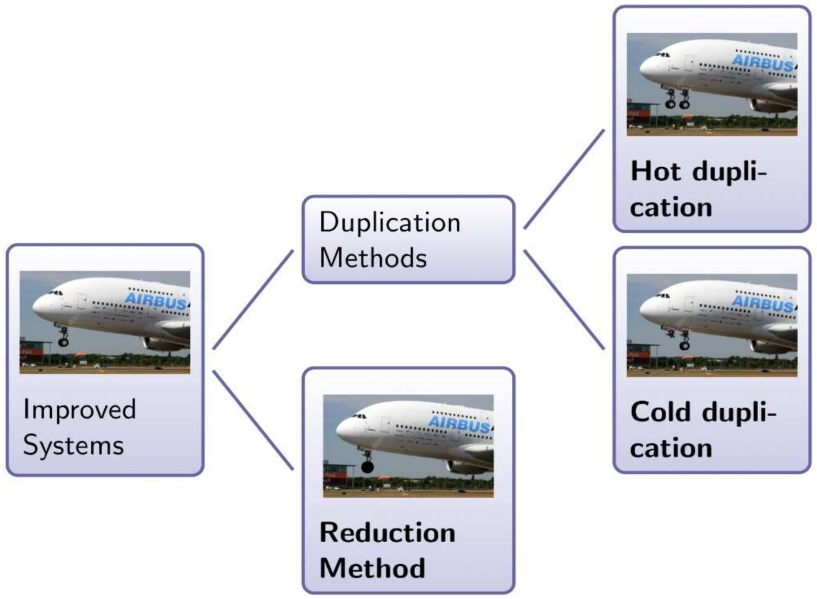


Figure 1.6: Illustrative example of reduction and duplication methods for improving systems.

lifetimes. El-Damcese (2009) assumed a series-parallel system with independent and identical components but with the Weibull distribution. Abdelkader et al. (2013) applied this concept on a system using the exponentiated exponential distribution and recently Migdadi and Al-Batah (2014) assumed a system with Burr type X distribution.

1.3 Signature

Samaniego (2007) introduced the concept of signature and provided a very good overview of this novel method for describing a system, while Coolen and Coolen-Maturi (2012) proposed several extensions relating to signature. Aslett (2012) developed a computer module based on the statistical programming language R to calculate the system signature, which is especially useful in systems with large numbers of components.

In order to present a definition of signature, there are some concepts that should be defined first, including system structure function, coherent system, minimal paths, minimal cuts and the reliability of a coherent system.

1.3.1 System structure function

For a system with m components, let x_i be the state of the ith component for $i = 1, 2, ..., m$ where $x_i = 1$ if it is working and $x_i = 0$ if it is not working. The vector $\underline{x} = (x_1, x_2, ..., x_m) \in (0, 1)^m$ is called the state vector. The system structure function $\phi(\underline{x})$ can be written as

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system is working} \\ 0 & \text{if the system is not working} \end{cases} \tag{1.3.1}$$

and the most common examples to illustrate the system structure function are series and parallel systems. The series system works only if every component is working, in which case the structure function of a series system can be written as

$$\phi(\underline{x}) = \min(x_1, x_2, ..., x_m) = \prod_{i=1}^{m} x_i. \tag{1.3.2}$$

Conversely, the parallel system works as long as at least one component is working, in which case the structure function for a parallel system can be written as

$$\phi(\underline{x}) = \max(x_1, x_2, ..., x_m) = 1 - \prod_{i=1}^{m} (1 - x_i). \tag{1.3.3}$$

1.3.2 Coherent system

A system is coherent if and only if every component is relevant and the structure function representing the system is monotone, Samaniego (2007).

The first condition refers to a system of order m components with a state vector $(x_1, ..., x_{i-1}, a, x_{i+1} ..., x_m)$ where $a \in \{0, 1\}$. The ith component is said to be irrelevant if:

$$\phi(x_1, ..., x_{i-1}, 1, x_{i+1} ..., x_m) = \phi(x_1, ..., x_{i-1}, 0, x_{i+1} ..., x_m)$$

for all possible state vectors. If a component is not irrelevant, then it is defined to be a relevant component.

The second condition is the monotone structure function. The structure function $\phi(\underline{.})$ of an order m system is said to be monotone if

$$\underline{x} \leq \underline{y} \Rightarrow \phi(\underline{x}) \leq \phi(\underline{y})$$

where $\underline{x}, \underline{y} \in \{0, 1\}^m$ and the inequality on the left is taken element-wise. In particular, this means that any improvement for any component cannot make the system worse.

1.3.3 Minimal paths and minimal cut sets

For a coherent system, a set of components P is said to be a path set if the system works whenever all the components in the set P work. If no proper subset of P is a path set, then P is said to be a minimal path set. The algebraic union of all minimal path sets is the set of all the system's components.

However, a set of components C is said to be a cut set if the system fails whenever all the components in the set C fail. If no proper subset of C is a cut set, then C is said to be a minimal cut set. The algebraic union of all minimal cut sets is the set of all the system's components.

1.3.4 System signature definition

Consider a coherent system with m independent and identically distributed components. Let $T_s > 0$ be the random failure time of the system and $T_{i:m}$ the ith order statistic of the m random component failure times for $i = 1, 2, ..., m$, where $T_{1:m} \leq T_{2:m} \leq ... \leq T_{m:m}$. The signature of the system is the m-dimensional probability vector $S = (s_1, s_2, ..., s_m)$ with elements

$$s_i = P(T_s = T_{i:m}) \tag{1.3.4}$$

so the signature is the probability that the system failure occurs at the moment of the ith component failure: Samaniego (2007); Coolen and Coolen-Maturi (2012). Since T_s resides in the set $\{T_{1:m}, T_{2:m}, ..., T_{m:m}\}$ with probability one, it follows that $s_i \geq 0$ for all i and $\sum_{i=1}^{m} s_i = 1$.

Computing the signature is dependent on the number of components in the system and the system structure. For example, a series system fails when the first component fails, so the signature vector for a series system can be written as $S = (1, 0, ..., 0)$, while

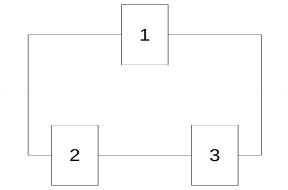


Figure 1.7: System with 3 independent and identically distributed components.

a parallel system fails whenever all the system components fail, so the signature vector for a parallel system can be written as $S = (0, 0, ..., 1)$. For other system signatures, let us consider an example of a system with three independent and identically distributed components as pictured in Figure 1.7. The failure times of these three components can be ordered in $3! = 6$ arrangements.

Table 1.1: Ordered component failure times for a system with 3 *i.i.d.* components.

ordered component failure times	order statistic equal to system failure time T_s
$T_1 < T_2 < T_3$	$T_{2:3}$
$T_1 < T_3 < T_2$	$T_{2:3}$
$T_2 < T_1 < T_3$	$T_{2:3}$
$T_2 < T_3 < T_1$	$T_{3:3}$
$T_3 < T_1 < T_2$	$T_{2:3}$
$T_3 < T_2 < T_1$	$T_{3:3}$

For this system, we can note that there are only two minimal cut sets, $\{1, 2\}$ and $\{1, 3\}$. The smallest minimal cut set has two members, which means that the system will not fail when the first component fails for all system components. The system will fail when the second component fails if the ordered component failure time takes any one of the minimal cut sets $\{1, 2\}$ and $\{1, 3\}$ (note we do not have any minimal cut set as a subset of those sets). Then the system signature is

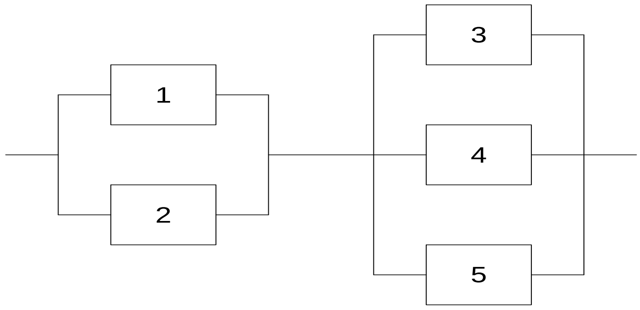


Figure 1.8: System with 5 independent and identically distributed components.

$S = (\frac{0}{6}, \frac{4}{6}, \frac{2}{6})$=$(0, 0.66, 0.33)$. For another example, let us consider a system with five independent and identically distributed components as pictured in Figure 1.8. The failure times of this system's components can be ordered in 5!=120 ways, and it has only two minimal cut sets, $\{1, 2\}$ and $\{3, 4, 5\}$. The system signature vector is $S = (\frac{0}{120}, \frac{12}{120}, \frac{36}{120}, \frac{72}{120}, \frac{0}{120})$= $(0, 0.1, 0.3, 0.6, 0)$.

1.3.5 Signature and system reliability

Samaniego (2007) introduced a very useful theorem to compute the reliability function for any coherent system with m independent and identically distributed components with a continuous lifetime distribution.

Theorem 1.3.1. *(Samaniego, 2007) Let $T_1, T_1, ..., T_m$ be the i.i.d. component lifetimes of an order m component coherent system with signature S. Let T_s be the system lifetime. Then*

$$P(T_s > t) = \sum_{i=1}^{m} s_i \sum_{j=0}^{i-1} \binom{m}{j} [F(t)]^j [R(t)]^{m-j} \tag{1.3.5}$$

where $F(t)$ and $R(t)$ are the failure function and reliability function of system components.

This theorem can also be written:

$$P(T_s > t) = \sum_{i=1}^{m} s_i \sum_{j=m-i+1}^{m} \binom{m}{j} [F(t)]^{m-j}[R(t)]^j; \tag{1.3.6}$$

see Coolen and Coolen-Maturi (2012) and Aslett et al. (2014).

1.4 Survival signature

Coolen and Coolen-Maturi (2012) recently introduced a new and useful method in this field, which they refer to as survival signature. Survival signature can be defined as the probability that a system functions given that a specified number of its components function.

For a coherent system with m independent and identically distributed components with a continuous lifetime distribution, let $\Phi(l)$ for $l = 0, 1, ..., m$ be the probability that the system functions given that precisely l of its components function. The system will not function when all system components fail, which means $\Phi(0) = 0$ and the system should function when all system components function, which means $\Phi(m) = 1$. There are $\binom{m}{l}$ state vectors $\underline{x}$ in which precisely l components function (l components with state $x_i = 1$), so $\sum_{i=1}^{m} x_i = l$; we will denote the set of these vectors by X_l. The system survival signature $\Phi(l)$ can be written as

$$\Phi(l) = \binom{m}{l}^{-1} \sum_{\underline{x} \in X_l} \phi(\underline{x}) \tag{1.4.1}$$

where $\phi(\underline{x})$ is the system structure function for each state vector in the set X_l. The survival signature can be derived from the signature thus:

$$\Phi(l) = \sum_{i=m-l+1}^{m} s_i \tag{1.4.2}$$

Let $C_t \in \{0, 1, 2, ..., m\}$ be the number of components in the system that function at time $t > 0$. If the components of the system have a continuous lifetime distribution with failure function $F(t)$ and reliability function $R(t)$ then, for $l \in \{0, 1, 2, ..., m\}$,

$$P(C_t = l) = \binom{m}{l}[F(t)]^{m-l}[R(t)]^l. \tag{1.4.3}$$

By applying Equations (1.4.1) and (1.4.3) in Theorem (1.3.1), the system reliability can then be written:

$$P(T_s > t) = \sum_{l=0}^{m} \Phi(l)\binom{m}{l}[F(t)]^{m-l}[R(t)]^l \tag{1.4.4}$$

We are going to use Relation (1.4.4) to compute the reliability functions and mean times to failure for different systems, which are improved according to different methods, in order to compare the efficiencies of these methods.

It is easy to compute the survival signature for a system with a small number of independent and identically distributed components. For example, the survival signature for the system in Figure 1.7 is shown in Table 1.2.

Table 1.2: Survival signature for the system in Figure 1.7.

$m = 3$				
l	0	1	2	3
$\Phi(l)$	0	$\frac{1}{3}$	1	1

To interpret Table 1.2, the probability that the system functions if none of its components function is zero, and the system definitely functions if all of its components function. The probability that the system functions if precisely one of its components functions is $\frac{1}{3}$ because $\binom{m}{l}=\binom{3}{1} = 3$, which means 3 state vectors have precisely one component functioning. These state vectors are $(1, 0, 0), (0, 1, 0)$ and

$(0,0,1)$. The structure function of the first state vector is $\phi(\underline{x}) = 1$; however, the structure function for the other two is $\phi(\underline{x}) = 0$. By using the definition of survival signature in Equation (1.4.1), we determine the value $\frac{1}{3}$ for the probability that the system functions if exactly one component functions. Using a similar method, we find that the probability that the system functions if precisely two of its components function is 1.

Using the same techniques, the survival signature for the system given in Figure 1.8 is shown in Table 1.3.

Table 1.3: Survival signature for the system in Figure 1.8.

$m = 5$						
l	0	1	2	3	4	5
$\Phi(l)$	0	0	$\frac{6}{10}$	$\frac{9}{10}$	1	1

Note that the survival signatures presented in Tables 1.2 and 1.3 are easily derived from the signature, as displayed in Equation (1.4.2).

1.4.1 Survival signature for systems with multiple types of component

Coolen and Coolen-Maturi (2012) also studied the survival signature for a system with multiple types of components. Studying systems with multiple types of component is more relevant to real applications. These authors consider a coherent system with m independent components classified into $n \geq 2$ types of components where type i has m_i identical components for $i = 1, 2, .., n$. Let $\Phi(l_1, l_2, ..., l_n)$, for $l_i = 0, 1, ..., m_i$, be the probability that a system functions given that precisely l_i of its components of type i function, for $i = 1, 2, .., n$. There are $\binom{m_i}{l_i}$ state vectors $\underline{x}^i$ where precisely l_i components of type i function (l_i of the m_i components have state $x^i_j = 1$ and the

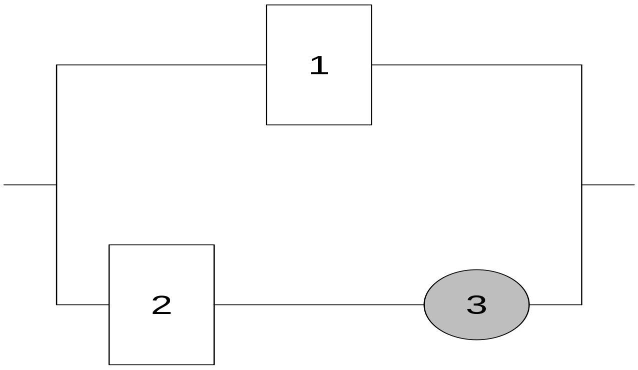


Figure 1.9: System with two types of component where the first type has 2 components and the second type has 1.

other $m_i - l_i$ have state $x_j^i = 0$), so $\sum_{j=1}^{m_i} x_j^i = l_i$. Let $X_{l_1,...,l_n}$ be the set of all state vectors for the whole system for which $\sum_{j=1}^{m_i} x_j^i = l_i$, $i = 1, 2, ..., n$. Then the survival signature of such a system is

$$\Phi(l_1, l_2, ..., l_n) = \left[\prod_{i=1}^{n} \binom{m_i}{l_i}^{-1}\right] \times \sum_{\underline{x} \in X_{l_1,...,l_n}} \phi(\underline{x}) \tag{1.4.5}$$

To illustrate the concept of the survival signature for a system with multiple types of components, we adapt Figures 1.7 and 1.8 in order to construct systems with more than one type of component. First consider a system with two types of component ($n = 2$), as pictured in Figure 1.9. The first type has two components, $m_1 = 2$, and the second has only one component, $m_2 = 1$. The whole number of system components is $\sum_{i=1}^{n} m_i = 3$.

Tables 1.4 and 1.5 present the technique to derive the survival signature for the system in Figure 1.9. Firstly, we derive all the state factors for the whole system. Secondly, we find the value of the system structure function for each state factor.

Table 1.4: Component state and system state for the system in Figure 1.9.

Number	Component state Type 1 x_1	x_2	Type 2 x_1	System state $\phi(\underline{x})$
1	0	0	0	0
2	0	0	1	0
3	0	1	0	0
4	0	1	1	1
5	1	0	0	1
6	1	0	1	1
7	1	1	0	1
8	1	1	1	1

Table 1.5: Survival signature for the system in Figure 1.9.

l_1	l_2	$\binom{m_1}{l_1}$	$\binom{m_2}{l_2}$	Analogous state vector in Table 1.4	$\sum \phi(\underline{x})$	$\Phi(l_1, l_2)$
0	0	1	1	1	0	0
0	1	1	1	2	0	0
1	0	2	1	3,5	1	$\frac{1}{2}$
1	1	2	1	4,6	2	1
2	0	1	1	7	1	1
2	1	1	1	8	1	1

Thirdly, we compute all possible combinations of the numbers of components that function for each type. Finally, we apply the aforementioned results in Relation (1.4.5) to derive the survival signature for this system.

We now provide a second example further to illustrate this methodology. Consider a system with two types of component, where the first has two components and the second has three components, as shown in Figure 1.10. Using the same technique which we used in the first example, the survival signature for the system in Figure 1.10 is shown in Table 1.6.

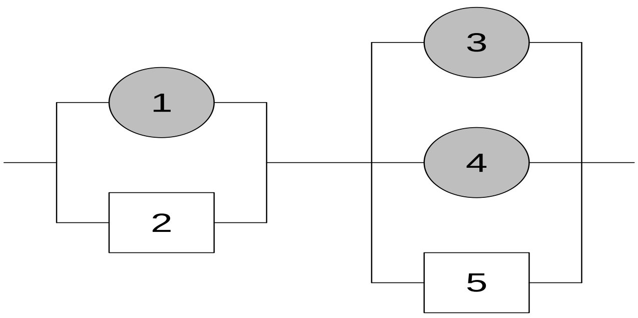


Figure 1.10: System with two types of component where the first type has 2 components and the second type has 3.

Table 1.6: Survival signature for the system in Figure 1.10.

l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_2	$\Phi(l_1, l_2)$
0	0	0	1	2	$\frac{5}{6}$
0	1	0	1	3	1
0	2	$\frac{2}{3}$	2	0	1
0	3	1	2	1	1
1	0	0	2	2	1
1	1	$\frac{1}{2}$	2	3	1

1.4.2 Reliability function for systems with multiple types of component using survival signature

Theorem 1.3.1 and Relation (1.4.1) can be used to derive the reliability function for a system with multiple types of components, as the following

$$P(T_s > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_n=0}^{m_n} \left[\Phi(l_1, ..., l_n) \prod_{i=1}^{n} \left\{ \binom{m_i}{l_i} [1 - R_i(t)]^{m_i - l_i} [R_i(t)]^{l_i} \right\} \right]. \quad (1.4.6)$$

Using this relation to derive reliability equivalence factors is very useful for several reasons. The first reason is that it matches real applications well and we can avoid assuming systems with identically distributed components and specific structures as do most studies in the reliability equivalence factors field. The second reason is that there are computer packages that can be used to compute the signature and survival signature for systems. Such a computer package helps to compute two of the reliability equivalence factors measures, which are the survival reliability equivalence factors and the mean reliability equivalence factors. Thirdly, we present a new technique to find the reliability equivalence factors.

1.5 `ReliabilityTheory`: R package

`ReliabilityTheory` is a software R package presented by Aslett (2012). This package includes very useful functions to compute signature (`computeSystemSignature`) and survival signature (`computeSystemSurvivalSignature`) for coherent systems. These functions are helpful especially for complex systems. The `graph.formula` function is used for representation of the system or network whose signature or survival signature is to be computed.

1.5.1 Package input

We can use the `ReliabilityTheory` R package if we have a coherent system with m independent components classified into $n \geq 2$ types. We load the `ReliabilityTheory` R package at the beginning and we give each component in the system a separate number then we follow these steps. Firstly, we define the system structure whereby each end of the system is denoted by “s” and “t” and the double dashes $--$ indicate a series connection while the colon : indicates a parallel connection. Secondly, we specify the prevalent types of system component. Thirdly, we compute the survival signature using `computeSystemSurvivalSignature`. To illustrate these steps the survival signatures for the systems in Figures 1.9 and 1.10 are calculated as follows:

- Computing the survival signature of the system in Figure 1.9. ⋆

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
# First, define system structure
g <- graph.formula(s--1--t,s--2--3--t)
# Second, specify types where components 1 and 2 represent the
# first type and component 3 represents the second type.
V(g)$compType <- NA
V(g)$compType[match(c("1","2"), V(g)$name)] <- "T1"
V(g)$compType[match(c("3"), V(g)$name)] <- "T2"
# Third, compute the survival signature
computeSystemSurvivalSignature(g,frac=TRUE)
```

- Computing the survival signature of the system in Figure 1.10. ⋆⋆

```
library(ReliabilityTheory)
# First, define system structure
g2 <- graph.formula(s--2:1--3--t,s--2:1--4--t,s--2:1--5--t,
                      1:2--3:4:5)
# Second, specify types where components 2 and 5 represent the
# first type and components 1, 3 and 4 represent the second type.
V(g2)$compType <- NA
V(g2)$compType[match(c("2","5"), V(g2)$name)] <- "T1"
V(g2)$compType[match(c("1","3","4"), V(g2)$name)] <- "T2"
# Third, compute the survival signature
computeSystemSurvivalSignature(g2,frac=TRUE)
```

We cannot use the function `computeSystemSurvivalSignature` for a system with a single type of component ($n = 1$), so we turn to the function `computeSystemSignature` to compute the signature. Then using Equation (1.4.2) we can compute the survival signature. For example:

- Computing the signature of the system in Figure 1.7. ♦

```
library(ReliabilityTheory)
g <- graph.formula(s--1--t,s--2--3--t)
computeSystemSignature(g,frac=TRUE)
```

- Computing the signature of the system in Figure 1.8. ♦♦

```
library(ReliabilityTheory)
g2 <- graph.formula(s--2:1--3--t,s--2:1--4--t,s--2:1--5--t,
                      1:2--3:4:5)
computeSystemSignature(g2,frac=TRUE)
```

1.5.2 Package output

The output of the `computeSystemSurvivalSignature` function is a table with $n+1$ columns. The first n columns contain the numbers of each type of component which are functional and the last column contains the probabilities that the system functions. The output of that functions presented in the Input section take the following forms:

- The survival signature of the system in Figure 1.9. ⋆

```
   T1 T2      Probability
1  0  0            0
2  0  1            0
3  1  0           1/2
4  1  1            1
5  2  0            1
6  2  1            1
```

- The survival signature of the system in Figure 1.10. ⋆⋆

```
   T1 T2      Probability
1   0  0           0
2   0  1           0
3   0  2          2/3
4   0  3           1
5   1  0           0
6   1  1          1/2
7   1  2          5/6
8   1  3           1
9   2  0           1
10  2  1           1
11  2  2           1
12  2  3           1
```

The output of the `computeSystemSignature` function is a vector which is the system signature. The output of the function presented in the Input section for systems with single types is:

- The signature of the system in Figure 1.7. ♦

  ```
  s = ( 0/1, 2/3, 1/3  )
  ```

- The signature for the system in Figure 1.8. ♦♦

  ```
  s = ( 0/1, 1/10, 3/10, 3/5, 0/1 )
  ```

1.6 Outline of thesis

The motivation of this thesis is to present a generalization to apply the concept of the reliability equivalence factors on real application systems better than before. This thesis is divided into six chapters classified into three parts. The first part includes applying reliability equivalence on a system of components with generalized quadratic failure rate and exponentiated Weibull distributions. The second part introduces a new methodology to derive reliability equivalence using the concept of survival signature. The last part presents some illustrative real application examples, a discussion and a conclusion. The main chapters of this thesis (2, 3 and 5) are presented in a format suitable for submission for publication in a peer reviewed journal which inevitably lead to some duplication for some sections. In addition to this chapter the remainder of the thesis is organized as follows:

In Chapter 2 we apply the reliability equivalence factors on a parallel-series system of components with generalized quadratic failure rate distribution. Part of this chapter was presented to the Mathematical Methods in Reliability Joint Research Group at the University of Salford on 22 March 2013. We recently submitted a paper based on the content of this chapter to a well regarded journal and it is still under review.

In Chapter 3 we derive the reliability equivalence factors for a series-parallel system of components with exponentiated Weibull lifetimes. This chapter was presented as a proceedings paper at the 8th IMA international conference on modelling in industrial maintenance and reliability (MIMAR) at University of Oxford from 10-12 July 2014, see Alghamdi and Percy (2014). An extended version containing much of the material in this chapter was subsequently published in the IMA Journal of Management Mathematics, Alghamdi and Percy (2015).

Chapter 4 presents the various steps for using survival signature to derive the reliability equivalence factors. In this chapter we use survival signature to compute reliability equivalence factors for simple systems including series-parallel and parallel-series systems.

In Chapter 5 survival signature is used to derive the reliability equivalence factors for complex systems and networks. To our knowledge, this is the first attempt to use survival signature to compute the reliability equivalence factors for different systems and for systems with multiple types of components. We are preparing to submit this chapter to an established peer reviewed journal.

In Chapter 6 we present real application examples for our study and we give conclusions for our thesis. Finally, we present some further research challenges which can be considered for future work.

Part I

Reliability equivalence for systems with flexible lifetime distributions

Chapter 2

Reliability equivalence factors for a parallel-series system assuming a generalized quadratic failure rate distribution

The aim of this study is to apply reliability equivalence techniques to a parallel-series system comprising several parallel subsystems connected in series. The lifetimes of all system components are assumed to be independent and identically distributed, according to a generalized quadratic failure rate distribution. Four different methods are used to improve any such system: (a) reduction; (b) hot duplication; (c) cold duplication with perfect switch; (d) cold duplication with imperfect switch. Two measures for comparing system improvements are considered in this study, survival reliability equivalence factors and mean reliability equivalence factors. Numerical examples are presented for a specific parallel-series formulation, to illustrate how to apply the theoretical results and demonstrate the relative benefits of various system improvements.

2.1 Introduction

The concept of reliability equivalence factors was introduced by Råde (1993a,b). He applied this concept to simple systems that consist of one component or two components connected in series or parallel. Later, Sarhan (2000, 2005) and Sarhan et al. (2008a) applied this concept to more general systems. Most of the designs considered have components with exponential lifetime distributions although some studies applied this concept to other lifetime distributions, such as the Weibull distribution, El-Damcese (2009), gamma distribution, Xia and Zhang (2007), exponentiated exponential distribution, Abdelkader et al. (2013) and recently Burr type X distribution, Migdadi and Al-Batah (2014).

There are two main methods for improving a system's design. The first method is reduction, which involves improving the reliability of the system by reducing the failure rate by a factor ρ for some of the system components, where $\rho \in (0, 1)$. This can be achieved by replacing standard components with more expensive, higher quality components. The second method for improving a system's design is redundancy duplication, which involves adding extra components in parallel to existing system components. There are three ways to add extra components to the system: hot duplication; cold duplication with perfect switch; cold duplication with imperfect switch. Sometimes, and for many different reasons such as high cost and space limitation, it is impossible to improve the reliability of the system by the redundancy duplication method. Reliability equivalence factors refer to the factors by which the failure rates (hazard functions) of some of the system's components must be reduced in order to reach equality of the system reliability with that of a better system.

In this study, we consider a parallel-series system in a broader context by assuming

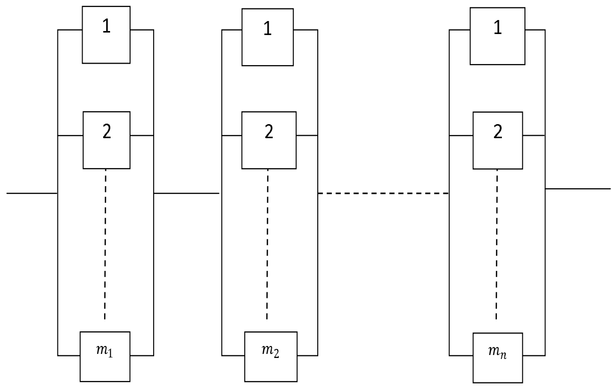


Figure 2.1: Parallel-series system.

that all the system's components are independent and follow the generalized quadratic failure rate distribution proposed by Sarhan and Alghamdi (2009) with identical parameters. First, we compute the reliability function (RF) and the mean time to failure (MTTF) of the original system. Second, we compute the RFs and MTTFs of the systems following improvement according to reduction, hot duplication and cold duplication (perfect and imperfect) methods. Third, we equate the RF and MTTF of the system improved according to the reduction method with the RF and MTTF of the system improved according to each of the duplication methods to determine the reliability equivalence factors. Finally, we illustrate the results obtained with an application example by presenting summary tables and figures.

2.2 Parallel-series system

The system we consider here is shown in Figure 2.1 and consists of n subsystems connected in series, where subsystem i consists of m_i components that are connected in parallel for $i = 1, 2, ..., n$. Such a system is usually referred to as a parallel-series system Sarhan et al. (2008a), though some authors refer to it as a series-parallel system.

We assume that the lifetimes of all the system's components are independent and follow the generalized quadratic failure rate distribution with identical parameters, GQFRD$(\alpha, \beta, \gamma, \theta)$ in the notation of Sarhan and Alghamdi (2009). As explained by those authors, this distribution offers much flexibility in the form of hazard function and includes several familiar models as special cases, including generalized exponential, Rayleigh and linear failure rate distributions. Let $r_{ij}(t)$ be the reliability function of component j ($j = 1, 2, ..., m_i$) in subsystem i ($i = 1, 2, ..., n$) and let $R_i(t)$ be the reliability function of subsystem i. The above assumption implies that $r_{ij}(t) = r(t)$ where

$$r(t) = 1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{\theta} \tag{2.2.1}$$

for $t \geq 0$, as the lifetimes of components are unaffected by failures of other components. The reliability function of subsystem i then takes the form

$$\begin{aligned} R_i(t) &= 1 - \prod_{j=1}^{m_i} \{1 - r_{ij}(t)\} \\ &= 1 - \{1 - r(t)\}^{m_i} \\ &= 1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{m_i \theta} \end{aligned} \tag{2.2.2}$$

for $t \geq 0$, so the reliability function of the parallel-series system is

$$\begin{aligned} R(t) &= \prod_{i=1}^{n} R_i(t) \\ &= \prod_{i=1}^{n} \left[1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{m_i \theta}\right] \end{aligned} \tag{2.2.3}$$

for $t \geq 0$, and the mean time to failure of the parallel-series system is given by

$$\begin{aligned} MTTF &= \int_0^\infty R(t)dt \\ &= \int_0^\infty \left(\prod_{i=1}^{n} \left[1 - \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{m_i \theta} \right] \right) dt. \end{aligned} \tag{2.2.4}$$

2.3 Designs of improved systems

The two main approaches for improving a system are reduction methods and standby redundancy (duplication) methods. The latter comprise two variations, hot duplication and cold duplication. Furthermore, cold duplication can be performed with perfect switch or imperfect switch. In this section, we derive the reliability function and the mean time to failure for parallel-series systems improved according to the methods identified above.

2.3.1 Reduction method

As mentioned in the introduction, the reliability of a system can be improved by scaling the hazard function for some of the system's components by a factor $\rho \in (0, 1)$. For the generalized quadratic failure rate distribution GQFRD$(\alpha, \beta, \gamma, \theta)$, reducing one or more of the parameters α, β and γ can reduce the failure rate. Here, we consider reducing all three parameters α, β and γ of a set A of the system's components by a factor $\rho \in (0, 1)$, in order to reduce the failure rate for the whole system. This is a logical procedure for the GQFRD, as the corresponding hazard function varies with time only through linear combinations of these parameters, as evident from the reliability function in Equation (2.2.1).

Define a_i $(i = 1, 2, ..., n)$ to be the number of components in subsystem i whose failure rate is reduced, so $a_i \in \{0, 1, \ldots, m_i\}$ and the cardinality of the set of improved components is $|A| = \sum_{i=1}^{n} a_i$.

By comparison with Equation (2.2.2), we see that the reliability function $R_i^{(A)}(t)$ of subsystem i is then given by

$$R_i^{(A)}(t) = 1 - \left\{1 - e^{-\rho(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{a_i\theta} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(m_i - a_i)\theta} \tag{2.3.1}$$

for $t \geq 0$ from Equation (2.2.1), since the components are connected in parallel. Then the reliability function of the system takes the form

$$\begin{aligned} R^{(A)}(t) &= \prod_{i=1}^{n} R_i^{(A)}(t) \\ &= \prod_{i=1}^{n} \left[1 - \left\{1 - e^{-\rho(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{a_i\theta} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(m_i - a_i)\theta}\right] \end{aligned} \tag{2.3.2}$$

for $t \geq 0$, since the subsystems are connected in series. We can then compute the mean time to failure of this parallel-series system as

$$\begin{aligned} MTTF^{(A)} &= \int_0^{\infty} R^{(A)}(t)dt \\ &= \int_0^{\infty} \left(\prod_{i=1}^{n} \left[1 - \left\{1 - e^{-\rho(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{a_i\theta} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(m_i - a_i)\theta}\right]\right) dt. \end{aligned} \tag{2.3.3}$$

2.3.2 Duplication methods

Now we obtain the corresponding reliability measures of the system when it is improved by duplication. We derive the reliability function and the mean time to failure of the parallel-series system improved according to the hot duplication method and the cold duplication methods with perfect and imperfect switches.

2.3.2.1 Hot duplication

This means that some of the system components are duplicated in parallel by similar components. We assume that in the hot duplication method each component of the set B is augmented by introducing a new but identical component in the same subsystem.

Let b_i $(i = 1, 2, ..., n)$ be the number of components in subsystem i whose reliability is improved according to the hot duplication method, so $b_i \in \{0, 1, \ldots, m_i\}$ and $|B| = \sum_{i=1}^{n} b_i$. By comparison with Equation (2.2.2), we see that the reliability function $R_i^{(B)}(t)$ of subsystem i is given by

$$\begin{aligned} R_i^{(B)}(t) &= 1 - \prod_{i=1}^{b_i+m_i} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{\theta} \\ &= 1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(b_i+m_i)\theta} \end{aligned} \tag{2.3.4}$$

for $t \geq 0$ from Equation (2.2.1), since the components are connected in parallel. Then the reliability function of the system takes the form

$$\begin{aligned} R^{(B)}(t) &= \prod_{i=1}^{n} R_i^{(B)}(t) \\ &= \prod_{i=1}^{n} \left[1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(b_i+m_i)\theta}\right] \end{aligned} \tag{2.3.5}$$

for $t \geq 0$, since the subsystems are connected in series. We can then compute the mean time to failure of this parallel-series system as

$$\begin{aligned} MTTF^{(B)} &= \int_0^{\infty} R^{(B)}(t)dt \\ &= \int_0^{\infty} \left(\prod_{i=1}^{n} \left[1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(b_i+m_i)\theta}\right]\right) dt. \end{aligned} \tag{2.3.6}$$

2.3.2.2 Cold duplication with perfect switch

This approach to improving system reliability means that a similar component is connected with an original component in such a way that it is activated immediately upon failure of the original component. For this aspect of our analysis, the cold duplication method assumes that each component of a set C is improved by introducing a new but identical component with a perfect switch.

Let c_i $(i = 1, 2, ..., n)$ be the number of components in subsystem i, whose reliability is improved according to the cold duplication method with perfect switch, so $c_i \in \{0, 1, \ldots, m_i\}$ and $|C| = \sum_{i=1}^{n} c_i$.

Let $s_1(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with perfect switch. Regarding a definition of cold duplication with perfect switch, we can describe this improvement as a renewal process with only one renewal. Using the convolution technique, the reliability function of each component whose reliability is improved according to cold duplication with perfect switch can be derived as:

$$s_1(t) = 1 - \int_0^t \frac{-dr(x)}{dx}[1 - r(t - x)]dx \tag{2.3.7}$$

where $r()$ is the reliability function for the generalized quadratic failure rate lifetime distribution presented in Equation (2.2.1).

To prove Equation (2.3.7), assume a standby duplication mode as present in Figure 2.2 where:

A	Original component
B	Standby component
S	Switch
T_1	Failure time of the original component
T_2	Failure time of the standby component
T	Failure time of the whole system
$N(t)$	Number of failures (renewal process) in the interval $(0, t]$.

According to the definition of the cold duplication method with perfect switch in this study, we obtained the following:

- A, B are independent and identically distributed with a generalized quadratic failure rate distribution;

- S is 100% reliable (perfect switch);

- Component B does not fail when in the standby position. It can only fail given that the original component A has already failed;

- We can describe this system as a renewal process with only one renewal, Gamiz et al. (2011). After the original component A fails the standby component B takes over for the remainder of the mission and therefore the system does not fail. If the standby component B fails the system also fails;

- Such a process is called a renewal process or perfect maintenance, which means that after a failure, the system behaviour is exactly as good as new.

- The switch immediately transfers load to the standby component B when the original component A fails, which means the repair time is negligible.

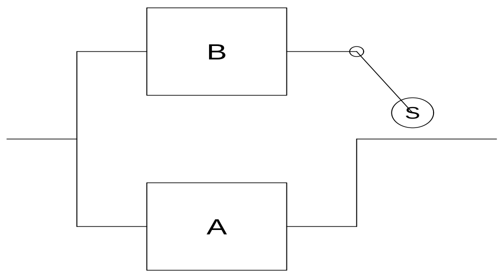


Figure 2.2: Standby duplication modes.

- The system fails when A and B fail which means $T = T_1 + T_2$. Thus, the whole system reliability can be derived using the number of failures or number of renewals $N(t)$, Ross (2006).

$$P(T \leq t) = P(N(t) \geq 2)$$

$$\Leftrightarrow P(T > t) = P(N(t) < 2)$$

Thus, the reliability function of each component whose reliability is improved according to cold duplication with perfect switch is a convolution of two generalized quadratic failure rate distributions as presented in Equation (2.3.7). $\square$

By comparison of Equation (2.3.7) with Equation (2.2.2), we see that the reliability function $R_i^{(C)}(t)$ of subsystem i is given by

$$R_i^{(C)}(t) = 1 - \{1 - s_1(t)\}^{c_i} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(m_i - c_i)\theta} \tag{2.3.8}$$

for $t \geq 0$ from Equation (2.2.1), since the components are connected in parallel. Then the reliability function of the system takes the form

$$
\begin{aligned}
R^{(C)}(t) &= \prod_{i=1}^{n} R_i^{(C)}(t) \\
&= \prod_{i=1}^{n} \left[1 - \{1 - s_1(t)\}^{c_i} \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{(m_i - c_i)\theta} \right] \qquad (2.3.9)
\end{aligned}
$$

for $t \geq 0$, and $s_1(t)$ as defined in Equation (2.3.7), since the subsystems are connected in series. We can then compute the mean time to failure of this parallel-series system as

$$
\begin{aligned}
MTTF^{(C)} &= \int_0^\infty R^{(C)}(t)dt \\
&= \int_0^\infty \left(\prod_{i=1}^{n} \left[1 - \{1 - s_1(t)\}^{c_i} \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{(m_i - c_i)\theta} \right] \right) dt. \qquad (2.3.10)
\end{aligned}
$$

2.3.2.3 Cold duplication with imperfect switch

This approach to improving system reliability means that a similar component is connected with an original component by a cold standby via a random switch having a constant failure rate. For this aspect of our analysis, the cold duplication method assumes that each component of a set D is improved by introducing a new but identical component with an imperfect switch.

Let d_i ($i = 1, 2, ..., n$) be the number of components in subsystem i, whose reliability is improved according to cold duplication with imperfect switch, so $d_i \in \{0, 1, \ldots, m_i\}$ and $|D| = \sum_{i=1}^{n} d_i$.

Let $s_2(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with imperfect switch. Following the same technique that we used for cold duplication with perfect switch but with the extra condition that the switch is not 100% reliable, Billinton and Allan (1992), we have

$$s_2(t) = 1 - \int_0^t \frac{-dr(x)}{dx}[1 - r(t-x)s_3(x)]dx \tag{2.3.11}$$

where $r()$ was defined previously for cold duplication with perfect switch, and $s_3()$ is the reliability function for the imperfect switch. The imperfect switch is chosen to have a constant failure rate λ, which means it has an exponential lifetime distribution with parameter λ

$$s_3(t) = e^{-\lambda t}. \tag{2.3.12}$$

To prove Equation (2.3.11), as we did in the cold perfect switch case, the only difference is that the reliability of the switch will affect the reliability of the standby component B. An imperfect switch makes with the standby component a series system with two components (component B and imperfect switch). The imperfect switch is chosen to have a constant failure rate prior to use, which means the reliability of the switch corresponds to an exponential distribution. This is the most common form of imperfect switch investigated in the literature relating to reliability equivalence and is appropriate for many practical purposes.

The reliability function $R_i^{(D)}(t)$ of subsystem i is given by

$$R_i^{(D)}(t) = 1 - \{1 - s_2(t)\}^{d_i} \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{(m_i - d_i)\theta} \tag{2.3.13}$$

for $t \geq 0$ from Equation (2.2.1), since the components are connected in parallel. Then the reliability function of the system takes the form

$$
\begin{aligned}
R^{(D)}(t) &= \prod_{i=1}^{n} R_i^{(D)}(t) \\
&= \prod_{i=1}^{n} \left[1 - \{1 - s_2(t)\}^{d_i} \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{(m_i - d_i)\theta} \right]
\end{aligned} \tag{2.3.14}
$$

for $t \geq 0$, and $s_2(t)$ as defined in Equation (2.3.11), since the subsystems are connected in series. We can then compute the mean time to failure of this parallel-series system as

$$
\begin{aligned}
MTTF^{(D)} &= \int_0^{\infty} R^{(D)}(t) dt \\
&= \int_0^{\infty} \left(\prod_{i=1}^{n} \left[1 - \{1 - s_2(t)\}^{d_i} \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{(m_i - d_i)\theta} \right] \right) dt.
\end{aligned} \tag{2.3.15}
$$

2.4 Reliability equivalence factors

According to El-Damcese (2009), a reliability equivalence factor is a factor by which a characteristic of components of a system design has to be multiplied in order to reach equality of a characteristic of this design and a different design regarded as a standard.

We compute two measures of reliability equivalence. The first involves survival reliability equivalence factors (SREFs) and these are determined from the reliability or survival function. The second involves mean reliability equivalence factors (MREFs) and these are determined from the mean time to failure.

2.4.1 Survival reliability equivalence factors

The idea of SREFs is to assess what degrees of intervention are required to establish equivalence between the reliability functions of a system whose reliability is improved according to one of the duplication methods and a system whose reliability is improved according to the reduction method.

That is, to derive the SREFs, we have to solve the following set of equations

$$R^{(A)}(t) = R^{(H)}(t) = \omega, \quad H = B, C, D \tag{2.4.1}$$

for the appropriate reduction factor ρ and time fractile t corresponding to a specified reliability requirement ω. The system of equations in (2.4.1) has no closed form solutions and can be solved using a mathematical package such as Matlab.

2.4.2 Mean reliability equivalence factors

The idea of MREFs is to assess what degrees of intervention are required to establish equivalence between the mean times to failure of a system whose reliability is improved according to one of the duplication methods and a system whose reliability is improved according to the reduction method.

That is, to derive the MREFs, we have to solve the following set of equations

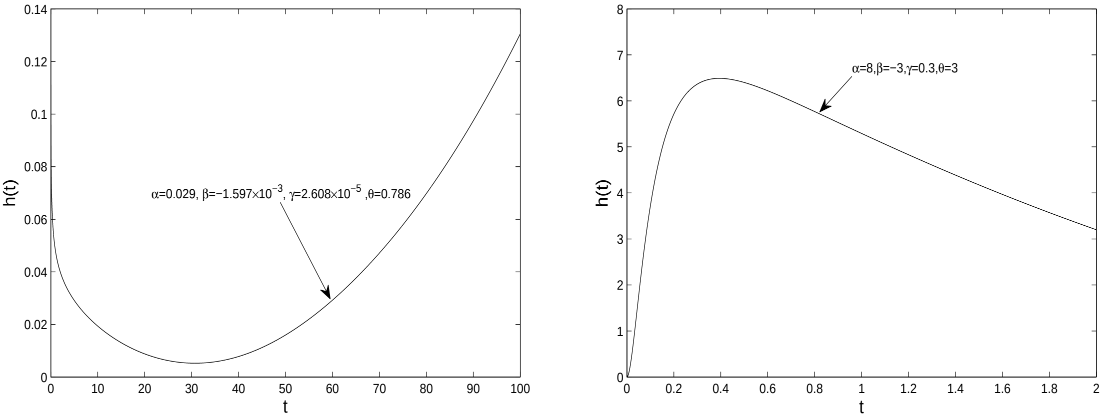
$$MTTF^{(A)} = MTTF^{(H)}, \quad H = B, C, D \tag{2.4.2}$$

for the appropriate reduction factor ρ. The system of equations in (2.4.2) also has no closed form solutions and can be solved using a mathematical package such as Matlab.

2.5 Numerical results and analysis

2.5.1 Example 1

Suppose that we have a parallel-series system consisting of two subsystems connected in series. The first subsystem has two components connected in parallel and the



(a) Bath-tub shape. (b) Upside down bath-tub shape.

Figure 2.3: Hazard function of the GQFRD$(\alpha, \beta, \gamma, \theta)$ for different parameter values.

second subsystem has three components connected in parallel. This means that $n = 2, m_1 = 2, m_2 = 3$ and the total number of components is $m = 5$. All of the system's components are assumed to be independent and identically distributed, with lifetimes that behave according to a generalized quadratic failure rate distribution with parameters $\alpha = 0.029, \beta = -1.597 \times 10^{-3}, \gamma = 2.608 \times 10^{-5}$ and $\theta = 0.786$. The values of these parameters derive from real data as described in Aarset (1987) and Sarhan and Alghamdi (2009). The hazard function for each component in the system takes bath-tub shape, see Figure 2.3a. We define:

1. $A_k^{(i,j)}$, $i = 0, 1, 2$, $j = 0, 1, 2, 3$ and $k = i+j$, to represent a reduction method that requires us to reduce the failure rate of i components from the first subsystem and j from the second subsystem.

2. $B_k^{(i,j)}$, $i = 0, 1, 2$, $j = 0, 1, 2, 3$ and $k = i + j$, to represent hot duplication methods when i components are added to the first subsystem and j to the second subsystem.

3. $C_k^{(i,j)}$, $i = 0, 1, 2$, $j = 0, 1, 2, 3$ and $k = i + j$, to represent cold duplication methods with perfect switch when i components are added to the first subsystem and j components are added to the second subsystem.

4. $D_k^{(i,j)}$, $i = 0, 1, 2$, $j = 0, 1, 2, 3$ and $k = i + j$, to represent cold duplication methods with imperfect switch when i components are added to the first subsystem and j components are added to the second subsystem.

Table 2.1: Hot survival reliability equivalence factors.

	ω	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
	0.1	0.6680	0.5083	0.4109	0.5514	0.3799	0.2846	0.2224	0.3799	0.2549	0.1824	0.1341
$A_1^{(0,1)}$	0.5	0.4590	0.2541	0.1526	0.1961	0.0178	-	-	0.0178	-	-	-
	0.9	0.1995	0.0458	0.0107	-	-	-	-	-	-	-	-
	0.1	0.8022	0.6803	0.5953	0.7151	0.5663	0.4703	0.4011	0.5663	0.4380	0.3529	0.2901
$A_2^{(0,2)}$	0.5	0.6621	0.4764	0.3581	0.4120	0.1115	-	-	0.1115	-	-	-
	0.9	0.4379	0.2056	0.0946	-	-	-	-	-	-	-	-
	0.1	0.8587	0.7618	0.6895	0.7903	0.6639	0.5756	0.5086	0.6639	0.5447	0.4601	0.3946
$A_3^{(0,3)}$	0.5	0.7545	0.5985	0.4882	0.5397	0.2130	0.0259	0.0160	0.2130	0.0183	-	-
	0.9	0.5732	0.3430	0.0298	0.0846	0.0527	0.0337	0.0237	0.0527	0.0040	-	-
	0.1	0.7704	0.6403	0.5544	0.6767	0.5259	0.4346	0.3720	0.5259	0.4051	0.3301	0.2777
$A_1^{(1,0)}$	0.5	0.6866	0.5325	0.4431	0.4828	0.2933	0.1962	0.1380	0.2933	0.1483	0.0751	0.0330
	0.9	0.7320	0.6503	0.6239	0.2516	0.0903	0.0355	0.0142	0.0903	-	-	-
	0.1	0.8538	0.7685	0.7116	0.7925	0.6927	0.6322	0.5908	0.6927	0.6127	0.5633	0.5291
$A_2^{(1,1)}$	0.5	0.7870	0.6802	0.6178	0.6456	0.5120	0.4424	0.4000	0.5120	0.4076	0.3525	0.3185
	0.9	0.7890	0.7251	0.7044	0.4146	0.2863	0.2389	0.2180	0.2863	0.1830	0.1442	0.1256
	0.1	0.8897	0.8210	0.7732	0.8406	0.7571	0.7040	0.6666	0.7571	0.6865	0.6412	0.6090
$A_3^{(1,2)}$	0.5	0.8335	0.7449	0.6912	0.7152	0.5965	0.5312	0.4901	0.5965	0.4976	0.4428	0.4079
	0.9	0.8219	0.7663	0.7481	0.4816	0.3539	0.3045	0.2821	0.3539	0.2438	0.2000	0.1783
	0.1	0.9107	0.8522	0.8103	0.8692	0.7958	0.7474	0.7126	0.7958	0.7312	0.6884	0.6575
$A_4^{(1,3)}$	0.5	0.8621	0.7847	0.7363	0.7582	0.6483	0.5854	0.5448	0.6483	0.5522	0.4971	0.4612
	0.9	0.8443	0.7939	0.7773	0.5216	0.3905	0.3381	0.3141	0.3905	0.2726	0.2244	0.2002
	0.1	0.8702	0.7815	0.7158	0.8076	0.6927	0.6135	0.5544	0.6927	0.5862	0.5124	0.4570
$A_2^{(2,0)}$	0.5	0.8231	0.7170	0.6473	0.6791	0.5120	0.4060	0.3315	0.5120	0.3455	0.2341	0.1482
	0.9	0.8544	0.8044	0.7876	0.4911	0.2863	0.1756	0.1092	0.2863	-	-	-
	0.1	0.9029	0.8392	0.7936	0.8576	0.7779	0.7255	0.6879	0.7779	0.7079	0.6619	0.6288
$A_3^{(2,1)}$	0.5	0.8618	0.7832	0.7337	0.7561	0.6427	0.5771	0.5345	0.6427	0.5423	0.4843	0.4465
	0.9	0.8739	0.8319	0.8179	0.5896	0.4603	0.4052	0.3792	0.4603	0.3329	0.2769	0.2479
	0.1	0.9209	0.8686	0.8307	0.8838	0.8175	0.7733	0.7411	0.8175	0.7583	0.7187	0.6897
$A_4^{(2,2)}$	0.5	0.8845	0.8186	0.7768	0.7957	0.6996	0.6433	0.6064	0.6996	0.6132	0.5622	0.5284
	0.9	0.8876	0.8503	0.8378	0.6368	0.5234	0.4749	0.4518	0.5234	0.4106	0.3598	0.3331
	0.1	0.9329	0.8875	0.8543	0.9008	0.8427	0.8032	0.7741	0.8427	0.7897	0.7536	0.7268
$A_5^{(2,3)}$	0.5	0.9000	0.8420	0.8049	0.8217	0.7355	0.6840	0.6499	0.7355	0.6561	0.6085	0.5765
	0.9	0.8979	0.8638	0.8525	0.6669	0.5603	0.5141	0.4920	0.5603	0.4521	0.4026	0.3761

Table 2.2: Cold survival reliability equivalence factors with perfect switch.

	ω	$C_1^{(0,1)}$	$C_2^{(0,2)}$	$C_3^{(0,3)}$	$C_1^{(1,0)}$	$C_2^{(1,1)}$	$C_3^{(1,2)}$	$C_4^{(1,3)}$	$C_2^{(2,0)}$	$C_3^{(2,1)}$	$C_4^{(2,2)}$	$C_5^{(2,3)}$
$A_1^{(0,1)}$	0.1	0.2577	0.1064	0.0503	0.1105	-	-	-	0.0005	-	-	-
	0.5	0.2776	0.0966	0.0361	0.0084	-	-	-	-	-	-	-
	0.9	0.1391	0.0236	0.0042	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.1	0.4411	0.2508	0.1588	0.2567	-	-	-	0.0136	-	-	-
	0.5	0.5006	0.2777	0.1625	0.0754	-	-	-	-	-	-	-
	0.9	0.3638	0.1466	0.0627	-	-	-	-	-	-	-	-
$A_3^{(0,3)}$	0.1	0.5477	0.3521	0.2472	0.3587	-	-	-	0.0428	-	-	-
	0.5	0.6198	0.4071	0.2776	0.1615	-	-	-	0.0064	-	-	-
	0.9	0.5054	0.2726	0.1557	0.0784	-	-	-	-	-	-	-
$A_1^{(1,0)}$	0.1	0.4079	0.2465	0.1795	0.2511	-	-	-	0.1056	-	-	-
	0.5	0.5516	0.3877	0.3185	0.2783	-	-	-	0.1014	-	-	-
	0.9	0.7031	0.6346	0.6174	0.2077	0.0544	0.0017	-	0.0707	-	-	-
$A_2^{(1,1)}$	0.1	0.6145	0.5089	0.4658	0.5119	0.1832	0.1180	0.0983	0.4186	0.1180	0.0851	0.0743
	0.5	0.6935	0.5788	0.5300	0.5014	0.2751	0.1924	0.1559	0.3726	0.1748	0.1062	0.0773
	0.9	0.7664	0.7127	0.6993	0.3802	0.2558	0.2027	0.1793	0.2699	0.1405	0.0898	0.0686
$A_3^{(1,2)}$	0.1	0.6881	0.5896	0.5474	0.5925	0.2336	0.1512	0.1260	0.4998	0.1512	0.1091	0.0952
	0.5	0.7562	0.6569	0.6129	0.5867	0.3620	0.2691	0.2252	0.4630	0.2482	0.1612	0.1209
	0.9	0.8023	0.7555	0.7436	0.4482	0.3223	0.2656	0.2397	0.3370	0.1957	0.1348	0.1075
$A_4^{(1,3)}$	0.1	0.7327	0.6386	0.5968	0.6414	0.2629	0.1704	0.1419	0.5487	0.1703	0.1229	0.1073
	0.5	0.7947	0.7049	0.6638	0.6390	0.4131	0.3125	0.2634	0.5176	0.2892	0.1903	0.1433
	0.9	0.8267	0.7841	0.7732	0.4879	0.3571	0.2962	0.2680	0.3726	0.2196	0.1515	0.1208
$A_2^{(2,0)}$	0.1	0.5888	0.4220	0.3419	0.4273	-	-	-	0.2422	-	-	-
	0.5	0.7310	0.6004	0.5368	0.4969	-	-	-	0.2777	-	-	-
	0.9	0.8371	0.7944	0.7833	0.4440	0.2194	0.0376	-	0.2517	-	-	-
$A_3^{(2,1)}$	0.1	0.7096	0.6087	0.5645	0.6117	0.2343	0.1512	0.1260	0.5141	0.1512	0.1091	0.0952
	0.5	0.7934	0.7013	0.6588	0.6329	0.3958	0.2907	0.2404	0.5059	0.2667	0.1677	0.1234
	0.9	0.8593	0.8236	0.8144	0.5574	0.4254	0.3595	0.3278	0.4417	0.2713	0.1871	0.1474
$A_4^{(2,2)}$	0.1	0.7598	0.6719	0.6322	0.6746	0.2911	0.1892	0.1577	0.5857	0.1892	0.1366	0.1192
	0.5	0.8271	0.7494	0.7134	0.6913	0.4822	0.3814	0.3295	0.5813	0.3571	0.2474	0.1908
	0.9	0.8746	0.8429	0.8347	0.6086	0.4927	0.4343	0.4059	0.5071	0.3547	0.2751	0.2350
$A_5^{(2,3)}$	0.1	0.7910	0.7102	0.6728	0.7127	0.3237	0.2110	0.1759	0.6283	0.2110	0.1523	0.1329
	0.5	0.8496	0.7804	0.7479	0.7279	0.5321	0.4323	0.3789	0.6265	0.4074	0.2912	0.2278
	0.9	0.8860	0.8571	0.8496	0.6405	0.5311	0.4751	0.4476	0.5448	0.3975	0.3179	0.2768

For this scenario, in Tables 2.1, 2.2 and 2.3 the SREFs for hot and cold (perfect and imperfect) duplication are calculated using Matlab according to the above formulae where ω is chosen to be $0.1, 0.5, 0.9$ and the imperfect switch has a constant failure rate $\lambda = 0.01$. For more discussions based on the results presented in the Tables 2.1, 2.2 and 2.3, it may be observed that:

- Reducing the failure rate of one component in the second subsystem (which we denote as $A_1^{(0,1)}$) by setting $\rho = 0.6680$ improves the reliability of the system like adding one component to the second subsystem (which we denote as $B_1^{(0,1)}$) according to a hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$, see Table 2.1.

- Reducing the failure rate of each component belonging to the set $A_5^{(2,3)}$ of the system components by setting $\rho = 0.8496$ improves the reliability of the system like adding a set $C_1^{(0,1)}$ of components to the system according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 2.2.

- Reducing the failure rate of each component belonging to the set $A_5^{(2,3)}$ of the system components by setting factor $\rho = 0.3661$ improves the reliability of the system like adding a set $D_5^{(2,3)}$ of components to the system according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 2.3.

- Missing values of the SREFs mean that it is not possible to reduce the failure rate for the set A of components in order to improve the system reliability to be equivalent with the system reliability that can be obtained by improving the sets B, C, D of components according to duplication methods.

- In the same manner, one can interpret the other results presented in Tables 2.1, 2.2 and 2.3.

Tables 2.4, 2.5 and 2.6 present the MREFs for hot and cold (perfect and imperfect) duplication. Based on the results presented in those tables, we see that:

- The modified system that can be obtained by improving the set $H_1^{(0,1)}$, where $H = B, C, D$ of the system components, according to hot and cold (perfect and imperfect) duplication has the same mean time to failure of that system which can be obtained be reducing the failure rate of each component belonging to the set $A_1^{(0,1)}$ by factors $\rho = 0.465, 0.257, 0.367$ respectively.

- Empty cells of MREFs mean that it is not possible to reduce the failure rate of the set A components in order to improve the mean time to failure of the system to be equivalent with the mean time to failure of the system that can be obtained by improving the sets B, C, D of components according to the duplication methods.

- In the same manner, one can interpret the other results presented in Tables 2.4, 2.5 and 2.6.

Table 2.7 presents the mean time to failure of the modified systems assuming hot and cold duplication methods, the latter with perfect and imperfect switch, assuming two constant failure rates $\lambda = 0.01$ and $\lambda = 0.05$. The mean time to failure of the original system is 53.063. From this table, one can conclude that

- If the failure rate of the imperfect switch is $\lambda = 0.01$, then

$$MTTF < MTTF^{(B)} < MTTF^{(D)} < MTTF^{(C)}$$

- If the failure rate of the imperfect switch is $\lambda = 0.05$, then

$$MTTF < MTTF^{(D)} < MTTF^{(B)} < MTTF^{(C)}$$

- This implies that the improvement due to hot duplication is better than using cold duplication with low reliability switch.

Figure 2.4 explains the improvement strategies to calculate the SREFs. Figure 2.5 presents reliability functions of the original and some modified systems. From this figure, one may observe that, for this scenario:

- Improving the reliability of all components according to cold duplication with perfect switch gives the best system.

- For the same number of components

$$R(t) < R^{(B)}(t) < R^{(D)}(t) < R^{(C)}(t)$$

  where $\lambda = 0.01$.

Figures 2.6 and 2.7 present the behaviour of MTTF against the appropriate reduction factor ρ. It seems from these two figures that:

- MTTFs are non-decreasing with decreasing ρ for all possible sets A.

- Reducing the failure rate of one or two components from the first subsystem gives a better system than that obtained by reducing the failure rate of one or two components in the second subsystem, see Figure 2.6. This means that improving a component from the subsystem with the smaller number of components is better than improving a component from the subsystem with the larger number of components.

- Reducing the failure rates of all components in the system gives the best system, see Figure 2.7.

- It is not possible to reduce the failure rate of the sets $A_1^{(0,1)}$ or $A_2^{(0,2)}$ of the system components to reach the mean time to failure which we can achieve by improving the sets $B_2^{(1,1)}$ or $C_3^{(1,2)}$ of the system components according to hot duplication and cold duplication with perfect switch respectively, see Figure 2.6.

- Improving a number of components selected from two subsystems, with equal numbers if they are even, gives a better system than that obtained by improving the number of components selected from the same subsystem or selected from the two subsystems with unequal numbers, see Figure 2.7.

2.5.2 Example 2

In order to generalise these results and conclusions for broader applicability, we now consider a contrasting analysis for the same system that we presented in Example 1 but with different parameter values. All of the system's components are assumed to be independent and identically distributed, with a generalized quadratic failure rate lifetime distribution with parameters $\alpha = 8, \beta = -3, \gamma = 0.3$ and $\theta = 3$. By using these parameter values the hazard function for each component in the system takes upside down bath-tub shape, see Figure 2.3b.

For this scenario, the hot survival reliability equivalence factors for this system with these parameter values are calculated according to the above formulae where ω is chosen to be $0.1, 0.5, 0.9$ and presented in Table 2.8. The hot mean equivalence factors for this system are presented in Table 2.9. We used version 2012a of Matlab software to derive both hot survival reliability equivalence factors and hot mean equivalence factors. All results presented in Tables 2.8 and 2.9 can be discussed in the same manner as for Tables 2.1 and 2.4 respectively. As expected, the numbers differ between the corresponding tables, although the patterns are similar for the bath-tub shape and upside down bath-tub shape hazard functions. Cold survival reliability equivalence factors and cold mean equivalence factors with perfect and imperfect switch can be derived in the same manner as for Example 1.

2.6 Conclusions

In this study, the system reliability function and system mean time to failure are used to study the reliability equivalence factors for a parallel-series system. All the system components are assumed to be independent and identically distributed, according to a generalized quadratic failure rate distribution. We discuss four different methods to improve such a system.

We derive analytical results for both survival and mean reliability equivalence factors of this system. Some numerical results are presented for a representative system in order to illustrate how one can apply the theoretical results obtained and to compare the various approaches in this context. Accordingly, detailed recommendations are discussed for improving the system considered in this study.

Several extensions of this study are identified, including analysis of other important parallel-series configurations, equivalent systems with non-identical components and simpler systems with dependent components. The methods described in this study adapt readily to deal with these other scenarios.

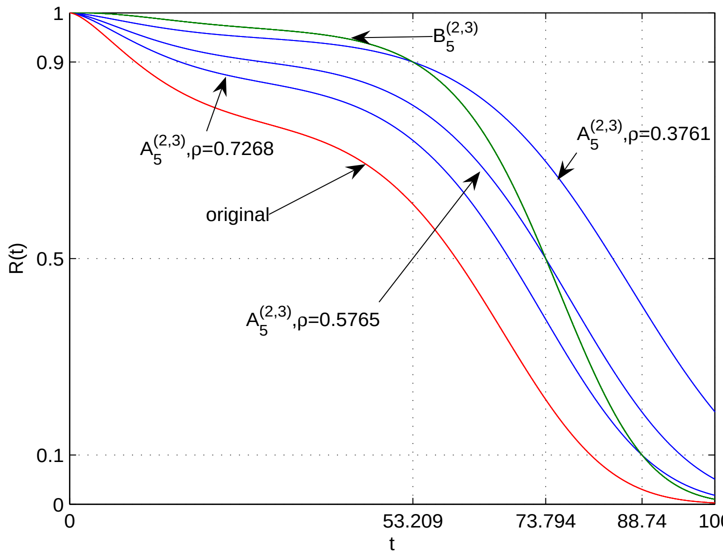


Figure 2.4: Use of survival reliability equivalence factors to recommend system improvement strategies.

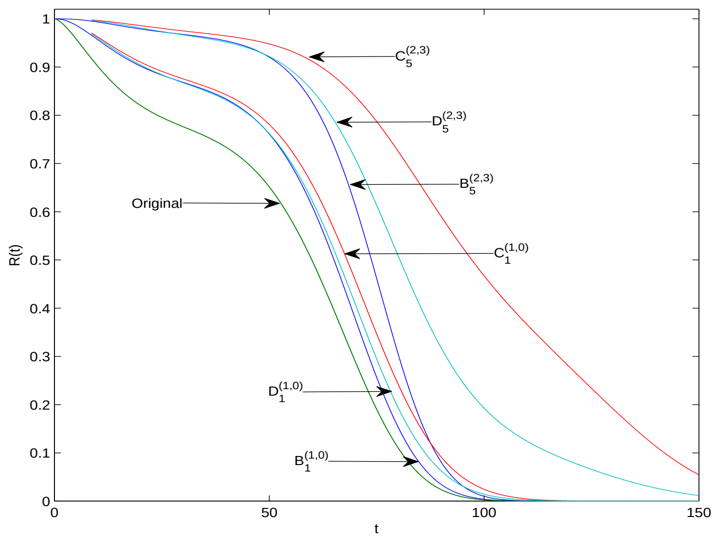


Figure 2.5: Reliability function of the original and some modified systems.

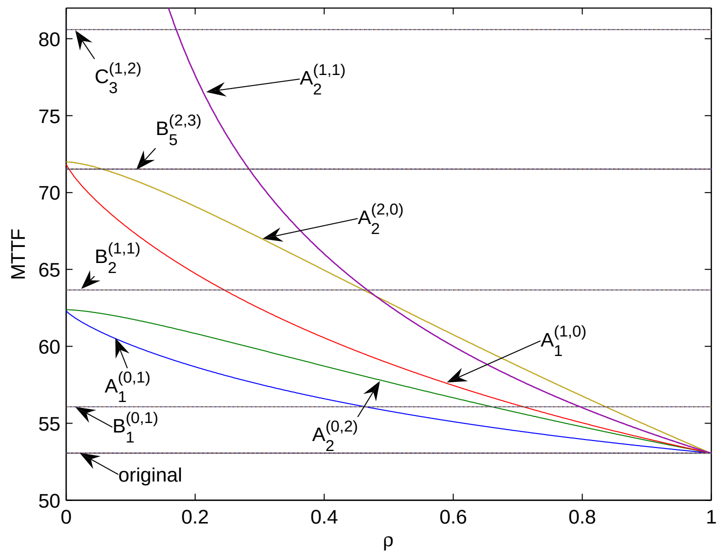


Figure 2.6: The behaviour of MTTF against ρ, when $|A| \leq 2$.

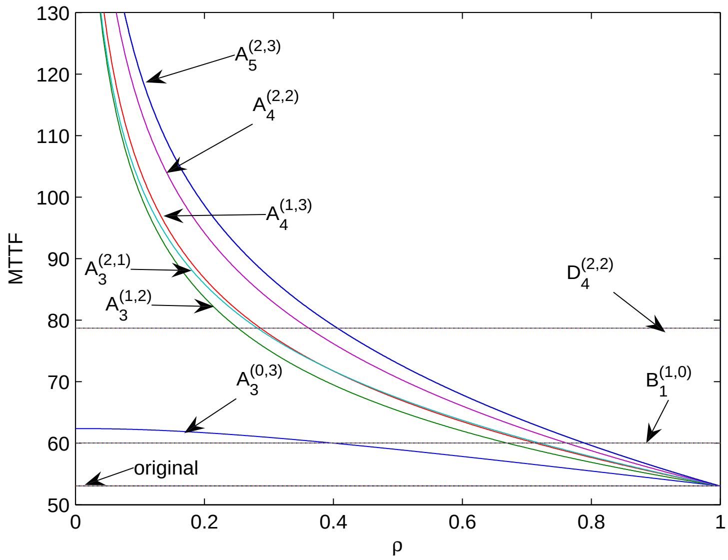


Figure 2.7: The behaviour of MTTF against ρ, when $|A| > 2$.

Table 2.3: Cold survival reliability equivalence factors with imperfect switch ($\lambda = 0.01$).

	ω	$D_1^{(0,1)}$	$D_2^{(0,2)}$	$D_3^{(0,3)}$	$D_1^{(1,0)}$	$D_2^{(1,1)}$	$D_3^{(1,2)}$	$D_4^{(1,3)}$	$D_2^{(2,0)}$	$D_3^{(2,1)}$	$D_4^{(2,2)}$	$D_5^{(2,3)}$
	0.1	0.4344	0.2477	0.1560	0.2806	-	-	-	0.1064	-	-	-
$A_1^{(0,1)}$	0.5	0.4021	0.1943	0.1006	0.1213	-	-	-	-	-	-	-
	0.9	0.1673	0.0329	0.0069	-	-	-	-	-	-	-	-
	0.1	0.6166	0.4299	0.3192	0.4661	-	-	-	0.2508	-	-	-
$A_2^{(0,2)}$	0.5	0.6154	0.4099	0.2840	0.3151	-	-	-	-	-	-	-
	0.9	0.4002	0.1737	0.0791	-	-	-	-	-	-	-	-
	0.1	0.7080	0.5369	0.4254	0.5716	0.0009	-	-	0.3521	-	-	-
$A_3^{(0,3)}$	0.5	0.7170	0.5377	0.4136	0.4455	0.0229	-	-	0.0268	-	-	-
	0.9	0.5391	0.3090	0.1806	0.0830	-	-	-	-	-	-	-
	0.1	0.5755	0.3978	0.3017	0.4308	0.0900	-	-	0.2465	-	-	-
$A_1^{(1,0)}$	0.5	0.6465	0.4812	0.3918	0.4129	0.1729	0.0614	0.0078	0.2107	0.0212	-	-
	0.9	0.7170	0.6415	0.6202	0.2395	0.0973	0.0442	0.0205	0.0945	-	-	-
	0.1	0.7257	0.6079	0.5447	0.6296	0.4086	0.3086	0.2531	0.5089	0.3002	0.2060	0.1589
$A_2^{(1,1)}$	0.5	0.7593	0.6445	0.5818	0.5965	0.4255	0.3418	0.2947	0.4529	0.3080	0.2370	0.1970
	0.9	0.7773	0.7181	0.7015	0.4051	0.2920	0.2468	0.2245	0.2897	0.1856	0.1409	0.1190
	0.1	0.7852	0.6821	0.6238	0.7017	0.4895	0.3819	0.3182	0.5896	0.3725	0.2617	0.2032
$A_3^{(1,2)}$	0.5	0.8109	0.7143	0.6594	0.6725	0.5150	0.4319	0.3830	0.5413	0.3969	0.3202	0.2745
	0.9	0.8117	0.7602	0.7456	0.4724	0.3598	0.3129	0.2891	0.3575	0.2467	0.1961	0.1704
	0.1	0.8209	0.7271	0.6718	0.7453	0.5381	0.4253	0.3564	0.6386	0.4152	0.2941	0.2289
$A_4^{(1,3)}$	0.5	0.8426	0.7573	0.7073	0.7193	0.5695	0.4860	0.4352	0.5952	0.4498	0.3684	0.3185
	0.9	0.8352	0.7884	0.7750	0.5124	0.3967	0.3470	0.3216	0.3942	0.2757	0.2201	0.1914
	0.1	0.7326	0.5793	0.4827	0.6100	0.2188	-	-	0.4220	-	-	-
$A_2^{(2,0)}$	0.5	0.7970	0.6779	0.6040	0.6221	0.3774	0.2091	0.0683	0.4232	0.1165	-	-
	0.9	0.8455	0.7988	0.7851	0.4785	0.2977	0.1969	0.1322	0.2932	-	-	-
	0.1	0.8051	0.7036	0.6441	0.7232	0.5031	0.3886	0.3216	0.6087	0.3787	0.2630	0.2035
$A_3^{(2,1)}$	0.5	0.8422	0.7552	0.7037	0.7161	0.5604	0.4726	0.4191	0.5873	0.4345	0.3489	0.2969
	0.9	0.8663	0.8273	0.8159	0.5809	0.4666	0.4147	0.3874	0.4641	0.3365	0.2718	0.2371
	0.1	0.8403	0.7546	0.7031	0.7714	0.5754	0.4628	0.3914	0.6719	0.4524	0.3250	0.2539
$A_4^{(2,2)}$	0.5	0.8680	0.7950	0.7515	0.7620	0.6289	0.5518	0.5036	0.6522	0.5176	0.4382	0.3876
	0.9	0.8808	0.8461	0.8361	0.6292	0.5290	0.4833	0.4591	0.5268	0.4138	0.3551	0.3230
	0.1	0.8628	0.7863	0.7392	0.8014	0.6183	0.5059	0.4317	0.7102	0.4953	0.3607	0.2828
$A_5^{(2,3)}$	0.5	0.8855	0.8211	0.7823	0.7917	0.6707	0.5987	0.5527	0.6922	0.5662	0.4890	0.4385
	0.9	0.8918	0.8601	0.8509	0.6598	0.5656	0.5221	0.4990	0.5635	0.4552	0.3979	0.3661

Table 2.4: Hot mean equivalence factors.

	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
$A_1^{(0,1)}$	0.465	0.280	0.193	0.104	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.660	0.493	0.396	0.276	-	-	-	-	-	-	-
$A_3^{(0,3)}$	0.752	0.609	0.519	0.397	-	-	-	-	-	-	-
$A_1^{(1,0)}$	0.709	0.580	0.510	0.430	0.245	0.162	0.117	0.245	0.096	0.033	0.005
$A_2^{(1,1)}$	0.797	0.705	0.656	0.599	0.467	0.407	0.375	0.467	0.359	0.310	0.284
$A_3^{(1,2)}$	0.839	0.762	0.719	0.669	0.547	0.488	0.455	0.547	0.440	0.389	0.360
$A_4^{(1,3)}$	0.865	0.798	0.759	0.713	0.596	0.537	0.504	0.596	0.488	0.435	0.404
$A_2^{(2,0)}$	0.837	0.750	0.699	0.635	0.461	0.363	0.302	0.462	0.270	0.150	0.056
$A_3^{(2,1)}$	0.870	0.803	0.765	0.719	0.602	0.543	0.508	0.602	0.492	0.437	0.405
$A_4^{(2,2)}$	0.890	0.833	0.801	0.761	0.660	0.607	0.577	0.660	0.562	0.512	0.482
$A_5^{(2,3)}$	0.903	0.854	0.824	0.789	0.696	0.647	0.618	0.696	0.604	0.556	0.527

Table 2.5: Cold mean equivalence factors with perfect switch.

	$C_1^{(0,1)}$	$C_2^{(0,2)}$	$C_3^{(0,3)}$	$C_1^{(1,0)}$	$C_2^{(1,1)}$	$C_3^{(1,2)}$	$C_4^{(1,3)}$	$C_2^{(2,0)}$	$C_3^{(2,1)}$	$C_4^{(2,2)}$	$C_5^{(2,3)}$
$A_1^{(0,1)}$	0.257	0.092	0.037	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.469	0.258	0.153	-	-	-	-	-	-	-	-
$A_3^{(0,3)}$	0.587	0.377	0.256	-	-	-	-	-	-	-	-
$A_1^{(1,0)}$	0.562	0.419	0.360	0.253	-	-	-	0.087	-	-	-
$A_2^{(1,1)}$	0.693	0.591	0.549	0.473	0.227	0.160	0.132	0.353	0.143	0.093	0.073
$A_3^{(1,2)}$	0.752	0.662	0.624	0.552	0.297	0.217	0.183	0.433	0.196	0.132	0.106
$A_4^{(1,3)}$	0.789	0.706	0.670	0.601	0.336	0.248	0.209	0.481	0.224	0.152	0.122
$A_2^{(2,0)}$	0.738	0.626	0.574	0.469	-	-	-	0.255	-	-	-
$A_3^{(2,1)}$	0.794	0.713	0.677	0.607	0.333	0.241	0.201	0.485	0.217	0.143	0.113
$A_4^{(2,2)}$	0.826	0.756	0.725	0.665	0.413	0.319	0.275	0.556	0.292	0.205	0.167
$A_5^{(2,3)}$	0.847	0.784	0.756	0.701	0.459	0.362	0.315	0.598	0.333	0.239	0.196

Table 2.6: Cold mean equivalence factors with imperfect switch ($\lambda = 0.01$).

	$D_1^{(0,1)}$	$D_2^{(0,2)}$	$D_3^{(0,3)}$	$D_1^{(1,0)}$	$D_2^{(1,1)}$	$D_3^{(1,2)}$	$D_4^{(1,3)}$	$D_2^{(2,0)}$	$D_3^{(2,1)}$	$D_4^{(2,2)}$	$D_5^{(2,3)}$
$A_1^{(0,1)}$	0.376	0.184	0.101	0.037	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.585	0.385	0.271	0.153	-	-	-	-	-	-	-
$A_3^{(0,3)}$	0.689	0.509	0.392	0.256	-	-	-	-	-	-	-
$A_1^{(1,0)}$	0.650	0.502	0.427	0.359	0.105	0.011	-	0.169	-	-	-
$A_2^{(1,1)}$	0.755	0.651	0.597	0.549	0.366	0.290	0.250	0.413	0.255	0.192	0.160
$A_3^{(1,2)}$	0.805	0.715	0.668	0.624	0.447	0.367	0.323	0.494	0.328	0.256	0.217
$A_4^{(1,3)}$	0.835	0.755	0.712	0.670	0.495	0.411	0.364	0.543	0.370	0.291	0.248
$A_2^{(2,0)}$	0.799	0.693	0.633	0.574	0.284	0.080	-	0.372	-	-	-
$A_3^{(2,1)}$	0.840	0.762	0.718	0.677	0.499	0.412	0.363	0.548	0.369	0.286	0.241
$A_4^{(2,2)}$	0.865	0.798	0.761	0.725	0.569	0.489	0.442	0.613	0.448	0.366	0.319
$A_5^{(2,3)}$	0.882	0.822	0.788	0.756	0.611	0.533	0.487	0.652	0.493	0.411	0.362

Table 2.7: Mean times to failure of the modified systems.

	$\{0_1, 1_2\}$	$\{0_1, 2_2\}$	$\{0_1, 3_2\}$	$\{1_1, 0_2\}$	$\{1_1, 1_2\}$	$\{1_1, 2_2\}$	$\{1_1, 3_2\}$	$\{2_1, 0_2\}$	$\{2_1, 1_2\}$	$\{2_1, 2_2\}$	$\{2_1, 3_2\}$
hot	56.068	57.744	58.764	60.045	63.672	65.746	67.038	63.672	67.697	70.042	71.530
cold perfect	58.005	60.250	61.312	63.516	75.421	82.003	85.862	68.017	84.259	93.639	99.362
cold imperfect ($\lambda = 0.01$)	56.816	58.887	60.102	61.316	67.425	71.216	73.728	65.572	73.401	78.488	82.000
cold imperfect ($\lambda = 0.05$)	55.098	56.473	57.453	57.925	60.235	61.815	62.956	61.004	63.52	65.261	66.532

Table 2.8: Hot survival reliability equivalence factors for system in Example 2.

	ω	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
	0.1	0.7847	0.6709	0.5962	0.7025	0.5713	0.4901	0.4322	0.5713	0.4631	0.3917	0.3384
$A_1^{(0,1)}$	0.5	0.6839	0.5254	0.4248	0.4708	0.1955	-	-	0.1955	-	-	-
	0.9	0.5390	0.3298	0.2114	-	-	-	-	-	-	-	-
	0.1	0.8735	0.8498	0.8103	0.8148	0.7093	0.6367	0.5817	0.7093	0.6215	0.5417	0.4870
$A_2^{(0,2)}$	0.5	0.8121	0.8523	0.6061	0.6466	0.3709	-	-	0.3709	-	-	-
	0.9	0.7182	0.5434	0.5139	-	-	-	-	-	-	-	-
	0.1	0.9101	0.8454	0.8327	0.8647	0.7772	0.7131	0.6626	0.7772	0.6901	0.6248	0.5721
$A_3^{(0,3)}$	0.5	0.8659	0.7710	0.6980	0.7328	0.4795	0.0021	-	0.4795	-	-	-
	0.9	0.7969	0.6538	0.5449	-	-	-	-	-	-	-	-
	0.1	0.8527	0.7635	0.7016	0.7890	0.6804	0.6100	0.5589	0.6804	0.5862	0.5232	0.4764
$A_1^{(1,0)}$	0.5	0.8266	0.7292	0.6673	0.6954	0.5499	0.4592	0.3949	0.5499	0.4071	0.3087	0.2273
	0.9	0.8786	0.8369	0.8228	0.5732	0.3921	0.2848	0.2121	0.3921	0.2345	0.2187	0.1876
	0.1	0.9069	0.8498	0.8103	0.8661	0.7970	0.7530	0.7219	0.7970	0.7385	0.7007	0.6737
$A_2^{(1,1)}$	0.5	0.8843	0.8207	0.7812	0.7990	0.7095	0.6581	0.6246	0.7095	0.6307	0.5849	0.5547
	0.9	0.9059	0.8746	0.8641	0.6930	0.5938	0.5505	0.5297	0.5938	0.4920	0.4448	0.4194
	0.1	0.9299	0.8844	0.8517	0.8976	0.8405	0.8027	0.7752	0.8405	0.7899	0.7561	0.7314
$A_3^{(1,2)}$	0.5	0.9103	0.8586	0.8257	0.8406	0.7642	0.7187	0.6884	0.7642	0.6940	0.6517	0.6232
	0.9	0.9212	0.8945	0.8856	0.7353	0.6447	0.6042	0.5845	0.6447	0.5485	0.5026	0.4776
	0.1	0.9434	0.9048	0.8763	0.9161	0.8663	0.8323	0.8070	0.8663	0.8206	0.7892	0.7658
$A_4^{(1,3)}$	0.5	0.9260	0.8815	0.8524	0.8657	0.7966	0.7540	0.7252	0.7966	0.7305	0.6896	0.6616
	0.9	0.9315	0.9077	0.8996	0.7592	0.6702	0.6295	0.6095	0.6702	0.5727	0.5253	0.4993
	0.1	0.9175	0.8584	0.8132	0.8760	0.7970	0.7400	0.6959	0.7970	0.7197	0.6637	0.6199
$A_2^{(2,0)}$	0.5	0.9045	0.8424	0.7993	0.8192	0.7095	0.6321	0.5728	0.7095	0.5843	0.4868	0.3983
	0.9	0.9361	0.9126	0.9045	0.7411	0.5938	0.4910	0.4123	0.5938	0.0081	0.0013	0.0011
	0.1	0.9384	0.8963	0.8653	0.9087	0.8544	0.8174	0.7900	0.8544	0.8047	0.7707	0.7455
$A_3^{(2,1)}$	0.5	0.9259	0.8807	0.8509	0.8645	0.7931	0.7486	0.7183	0.7931	0.7239	0.6808	0.6512
	0.9	0.9449	0.9254	0.9188	0.7981	0.7159	0.6766	0.6569	0.7159	0.6198	0.5704	0.5425
	0.1	0.9499	0.9155	0.8898	0.9256	0.8808	0.8498	0.8266	0.8808	0.8391	0.8101	0.7885
$A_4^{(2,2)}$	0.5	0.9382	0.9007	0.8759	0.8872	0.8279	0.7907	0.7652	0.8279	0.7700	0.7335	0.7081
	0.9	0.9511	0.9339	0.9280	0.8239	0.7547	0.7219	0.7056	0.7547	0.6748	0.6341	0.6110
	0.1	0.9575	0.9278	0.9054	0.9366	0.8974	0.8699	0.8491	0.8974	0.8603	0.8341	0.8143
$A_5^{(2,3)}$	0.5	0.9466	0.9138	0.8920	0.9020	0.8493	0.8159	0.7928	0.8493	0.7971	0.7637	0.7403
	0.9	0.9557	0.9401	0.9348	0.8399	0.7763	0.7461	0.7309	0.7763	0.7024	0.6642	0.6425

Table 2.9: Hot mean equivalence factors for system in Example 2.

	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
$A_1^{(0,1)}$	0.732	0.604	0.526	0.542	0.363	0.247	0.206	0.363	0.153	-	-
$A_2^{(0,2)}$	0.840	0.744	0.678	0.692	0.523	0.397	0.284	0.523	0.283	-	-
$A_3^{(0,3)}$	0.885	0.809	0.753	0.765	0.611	0.486	0.349	0.611	0.366	-	-
$A_1^{(1,0)}$	0.846	0.761	0.705	0.717	0.587	0.510	0.391	0.587	0.460	0.378	0.319
$A_2^{(1,1)}$	0.897	0.841	0.805	0.812	0.731	0.685	0.326	0.731	0.657	0.615	0.587
$A_3^{(1,2)}$	0.920	0.874	0.845	0.851	0.781	0.740	0.520	0.781	0.715	0.592	0.649
$A_4^{(1,3)}$	0.934	0.895	0.869	0.874	0.811	0.773	0.537	0.811	0.748	0.725	0.685
$A_2^{(2,0)}$	0.915	0.860	0.822	0.830	0.732	0.666	0.472	0.732	0.620	0.541	0.480
$A_3^{(2,1)}$	0.933	0.893	0.867	0.872	0.807	0.767	0.533	0.807	0.742	0.702	0.675
$A_4^{(2,2)}$	0.944	0.911	0.889	0.893	0.839	0.805	0.554	0.839	0.784	0.749	0.726
$A_5^{(2,3)}$	0.952	0.923	0.903	0.907	0.858	0.828	0.567	0.858	0.808	0.777	0.755

Chapter 3

Reliability equivalence factors for a series-parallel system of components with exponentiated Weibull lifetimes

We now study reliability equivalence factors for a system of independent and identically distributed components with exponentiated Weibull lifetimes. The system we consider has n subsystems connected in parallel and subsystem i has m_i components connected in series, $i = 1, .., n$. We chose this series-parallel system structure to complement our parallel-series analysis in Chapter 2. As before, we consider improving the reliability of this system by (a) a reduction method and (b) several duplication methods: (i) hot duplication; (ii) cold duplication with perfect switching; (iii) cold duplication with imperfect switching. We again compute two types of reliability equivalence factors, survival equivalence factors and mean equivalence factors. Although our methods adapt to allow for general lifetime models, we use the exponentiated Weibull distribution because it is flexible and enables comparisons with other reliability equivalence studies. The example we present demonstrates the potential for applying these methods to address specific questions that arise when attempting to improve the reliability of simple systems or simple configurations of possibly complex sub-systems in many diverse applications.

3.1 Introduction

Series-parallel and parallel-series system configurations are the building blocks for more complicated systems, and an understanding of the analytical processes and optimal strategies involved for these systems enables and informs arbitrary generalisation to complex situations. However, only one of these is needed to illustrate the methodology and we choose the series-parallel system here.

In this study, we also assume that all the system's components are independent and follow the exponentiated Weibull distribution of Mudholkar and Srivastava (1993) with identical parameters. We choose this distribution because it complement the GQFRD and includes all common shapes of hazard function and because its hazard and reliability are elementary functions. In particular, it includes the monotone hazard function of the Weibull distribution but also permits bathtub and inverted bathtub hazard functions. Special cases of the exponentiated Weibull distribution include the Weibull, exponentiated exponential and Burr type X distributions mentioned above.

Firstly, we compute the reliability function and the mean time to failure (MTTF) of the original system. Secondly, we compute the reliability functions and MTTFs of the systems following improvement according to reduction, hot duplication and cold duplication (perfect and imperfect) methods. Thirdly, we equate the reliability function and the MTTF of the system improved according to the reduction method with the reliability function and the MTTF of the system improved according to each

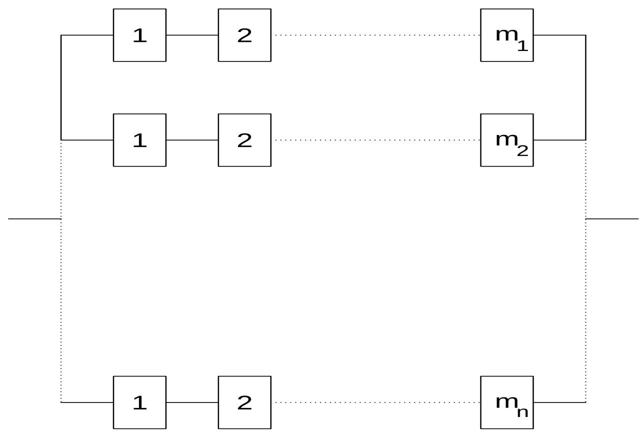


Figure 3.1: Series-parallel system.

of the duplication methods to determine the reliability equivalence factors.

Finally, we illustrate the results obtained with an application example by presenting summary tables and figures. This study expands considerably upon some preliminary ideas that Alghamdi and Percy (2014) presented, by investigating both survival and mean reliability equivalence factors for a series-parallel system, and both hot and cold duplication methods.

3.2 Series-parallel system

The system we consider here is shown in Figure 3.1 and consists of n subsystems connected in parallel, where subsystem i consists of m_i components that are connected in series for $i = 1, 2, ..., n$. Such a system is usually referred to as a series-parallel system, El-Damcese (2009).

We assume that the lifetimes of all the system's components are independent and follow the exponentiated Weibull distribution with identical parameters, see Mudholkar and Srivastava (1993) and Lai (2014). The exponentiated Weibull distribution generalizes well known lifetime distributions including exponential, Rayleigh and Weibull, and has the desirable properties of flexibility and tractability noted earlier. It provides a useful complement to the GQFRD family which does not include the Weibull distribution.

Under this assumption, the reliability function for each component j ($j = 1, 2, ..., m_i$) in subsystem i ($i = 1, 2, ..., n$) is given by

$$r(t) = 1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta} \tag{3.2.1}$$

for $t \geq 0$, as the lifetimes of components are unaffected by failures of other components. The reliability function of subsystem i then takes the form

$$\begin{aligned} R_i(t) &= \prod_{j=1}^{m_i} r_{ij}(t) \\ &= \prod_{j=1}^{m_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\} \\ &= \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i} \end{aligned} \tag{3.2.2}$$

for $t \geq 0$, so the reliability function of the series-parallel system is

$$\begin{aligned} R(t) &= 1 - \prod_{i=1}^{n} \{1 - R_i(t)\} \\ &= 1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i}\right] \end{aligned} \tag{3.2.3}$$

for $t \geq 0$, and the mean time to failure of the series-parallel system is given by

$$\begin{aligned} MTTF &= \int_0^\infty R(t)dt \\ &= \int_0^\infty \left(1 - \prod_{i=1}^n \left[1 - \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i}\right]\right) dt. \end{aligned} \tag{3.2.4}$$

3.3 Designs of improved systems

As explained in Chapter 2, the two main approaches for improving a system are reduction methods and standby redundancy (duplication) methods. In this section, we derive the reliability function and the mean time to failure, primarily for the series-parallel system, when improved according to the methods identified above.

3.3.1 Reduction method

For the exponentiated Weibull distribution, reducing only the scale parameter α reduces the failure rate. Here, we consider reducing α for a set A of the system's components by a factor $\rho \in (0, 1)$, in order to reduce the failure rate (hazard function) for the whole system. This is a logical procedure for the exponentiated Weibull distribution.

Define a_i $(i = 1, 2, ..., n)$ to be the number of components in subsystem i whose failure rate is reduced, so $a_i \in \{0, 1, \ldots, m_i\}$ and the cardinality of the set of improved components is $|A| = \sum_{i=1}^{n} a_i$. By comparison with Equation (3.2.2), we see that the reliability function $R_i^{(A)}(t)$ of subsystem i is then given by

$$R_i^{(A)}(t) = \prod_{j=1}^{a_i} \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\} \prod_{j=1}^{m_i - a_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}$$
$$= \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\}^{a_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i} \quad (3.3.1)$$

for $t \geq 0$ from Equation (3.2.1) and by comparison with Equation (3.2.3), since the components are connected in series. Then the reliability function of the system takes the form

$$R^{(A)}(t) = 1 - \prod_{i=1}^{n} \left\{1 - R_i^{(A)}(t)\right\}$$
$$= 1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\}^{a_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i}\right] \quad (3.3.2)$$

since the subsystems are connected in parallel. We can then compute the mean time to failure of this series-parallel system as

$$MTTF^{(A)} = \int_0^\infty R^{(A)}(t) dt$$
$$= \int_0^\infty \left(1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\rho\alpha t^\beta}\right)^\theta\right\}^{a_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - a_i}\right]\right) dt. \quad (3.3.3)$$

3.3.2 Duplication methods

We derive the reliability function and the mean time to failure for the series-parallel system, when improved according to the hot duplication method and the cold duplication methods with perfect and imperfect switches.

3.3.2.1 Hot duplication method

This means that some of the system components are duplicated in parallel by similar components. We assume that in the hot duplication method each component of the set B is augmented by introducing a new but identical component in the same subsystem.

Let b_i $(i = 1, 2, ..., n)$ be the number of components in subsystem i whose reliability is improved according to the hot duplication method, so $b_i \in \{0, 1, \ldots, m_i\}$ and $|B| = \sum_{i=1}^{n} b_i$. The reliability function $R_i^{(B)}(t)$ of subsystem i is given by

$$\begin{aligned} R_i^{(B)}(t) &= \prod_{j=1}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{2\theta}\right\} \prod_{j=1}^{m_i - b_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{\theta}\right\} \\ &= \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{\theta}\right\}^{m_i - b_i} \end{aligned} \tag{3.3.4}$$

for $t \geq 0$ from Equation (3.2.1), since the components are connected in series. Then the reliability function of the whole system takes the form

$$R^{(B)}(t) = 1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{\theta}\right\}^{m_i - b_i}\right] \tag{3.3.5}$$

for $t \geq 0$, and the mean time to failure of this series-parallel can then computed as

$$\begin{aligned} MTTF^{(B)} &= \int_0^\infty R^{(B)}(t)dt \\ &= \int_0^\infty \left(1 - \prod_{i=1}^{n} \left[1 - \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{2\theta}\right\}^{b_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^{\theta}\right\}^{m_i - b_i}\right]\right) dt. \end{aligned} \tag{3.3.6}$$

3.3.2.2 Cold duplication method with perfect switch

This approach to improving system reliability means that a similar component is connected with an original component in such a way that it is activated immediately upon failure of the original component. For this aspect of our analysis, the cold duplication method assumes that each component of a set C is improved by introducing a new but identical component with a perfect switch. The switch immediately transfers load to the standby component when the original component fails, which means the switch operation time is negligible.

Let c_i $(i = 1, 2, ..., n)$ be the number of components in subsystem i, whose reliability is improved according to the cold duplication method with perfect switch, so $c_i \in \{0, 1, \ldots, m_i\}$ and $|C| = \sum_{i=1}^{n} c_i$. Let $s_1(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with perfect switch. Regarding a definition of cold duplication with perfect switch, we can describe this improvement as a renewal process with only one renewal, Gamiz et al. (2011). Using the convolution technique, the reliability function of each component whose reliability is improved according to cold duplication with perfect switch can be derived as:

$$s_1(t) = 1 - \int_0^t \frac{-dr(x)}{dx}[1 - r(t - x)]dx \tag{3.3.7}$$

where $r()$ is the reliability function for the exponentiated Weibull lifetime distribution presented in Equation (3.2.1). By comparison with Equation (3.2.2), we see that the reliability function $R_i^{(C)}(t)$ of subsystem i is given by

$$\begin{aligned} R_i^{(C)}(t) &= \prod_{j=1}^{c_i} s_1(t) \prod_{j=1}^{m_i - c_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\} \\ &= \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - c_i} \end{aligned} \tag{3.3.8}$$

for $t \geq 0$, from Equation (3.2.1), since the components are connected in series. Then the reliability function of the system takes the form

$$R^{(C)}(t) = 1 - \prod_{i=1}^{n} \left\{1 - R_i^{(C)}(t)\right\}$$
$$= 1 - \prod_{i=1}^{n} \left[1 - \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - c_i}\right] \qquad (3.3.9)$$

for $t \geq 0$, and $s_1(t)$ as defined in Equation (3.3.7), since the subsystems are connected in parallel. We can then compute the mean time to failure of this series-parallel system as

$$MTTF^{(C)} = \int_0^{\infty} R^{(C)}(t)dt$$
$$= \int_0^{\infty} \left(1 - \prod_{i=1}^{n} \left[1 - \{s_1(t)\}^{c_i} \left\{1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta}\right\}^{m_i - c_i}\right]\right) dt. \qquad (3.3.10)$$

3.3.2.3 Cold duplication method with imperfect switch

This approach to improving system reliability means that a similar component is connected with an original component by a cold standby via a random switch having a constant failure rate. For this aspect of our analysis, the cold duplication method assumes that each component of a set D is improved by introducing a new but identical component with an imperfect switch.

Let d_i $(i = 1, 2, ..., n)$ be the number of components in subsystem i, whose reliability is improved according to cold duplication with imperfect switch, so $d_i \in \{0, 1, \ldots, m_i\}$ and $|D| = \sum_{i=1}^{n} d_i$. Let $s_2(t)$ be the reliability function of each component whose reliability is improved according to cold duplication with imperfect switch. Following the same technique that we used for cold duplication with perfect switch but with the extra condition that the switch is not 100% reliable, Billinton and Allan

(1992), we have

$$s_2(t) = 1 - \int_0^t \frac{-dr(x)}{dx}[1 - r(t-x)s_3(x)]dx \tag{3.3.11}$$

where $r()$ was defined in Equation (3.2.1), and $s_3()$ is the reliability function for the imperfect switch. The imperfect switch is chosen to have a constant failure rate λ, which means that it has an exponential lifetime distribution with parameter λ and so

$$s_3(t) = e^{-\lambda t}. \tag{3.3.12}$$

The reliability function $R_i^{(D)}(t)$ of subsystem i is given by

$$R_i^{(D)}(t) = \{s_2(t)\}^{d_i} \left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - d_i} \tag{3.3.13}$$

for $t \geq 0$, from Equation (3.2.1), since the components are connected in series. Then the reliability function of this series-parallel system takes the form

$$R^{(D)}(t) = 1 - \prod_{i=1}^{n}\left[1 - \{s_2(t)\}^{d_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - d_i}\right] \tag{3.3.14}$$

for $t \geq 0$ and $s_2(t)$ as defined in Equation (3.3.11), since the subsystems are connected in parallel. We can then compute the mean time to failure of this series-parallel system as

$$\begin{aligned} MTTF^{(D)} &= \int_0^\infty R^{(D)}(t)dt \\ &= \int_0^\infty \left(1 - \prod_{i=1}^{n}\left[1 - \{s_2(t)\}^{d_i}\left\{1 - \left(1 - e^{-\alpha t^\beta}\right)^\theta\right\}^{m_i - d_i}\right]\right) dt. \end{aligned} \tag{3.3.15}$$

3.4 Numerical analysis

Suppose that we have a series-parallel system consisting of two subsystems connected in parallel as shown in Figure 3.2. It is easy to imagine systems that display this

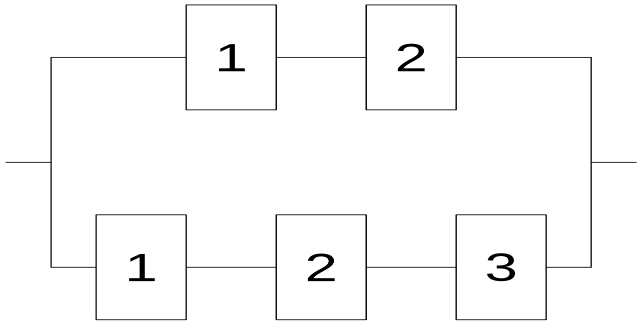


Figure 3.2: Series-parallel system consisting of two subsystems connected in parallel.

structure. For example, one of the authors travels to work by train on one of two routes, which comprise two and three stages respectively, each of which is vulnerable to random failures. The first subsystem that we consider here has two components connected in series and the second subsystem has three components connected in series. This means that $n = 2, m_1 = 2, m_2 = 3$ and the total number of components is $m = 5$. All of the system's components are assumed to be independent and identically distributed, with lifetimes that behave according to an exponentiated Weibull distribution with parameters $\alpha = 1, \beta = 2$ and $\theta = 3$. All $A_k^{(i,j)}$, $B_k^{(i,j)}$, $C_k^{(i,j)}$, and $D_k^{(i,j)}$ were defined in Chapter 2.

For this scenario, in Tables 3.1, 3.2 and 3.3 the SREFs for hot and cold (perfect and imperfect) duplication are calculated using Matlab according to the above formulae where ω is chosen to be $0.1, 0.5, 0.9$ and the imperfect switch has a constant failure rate $\lambda = 0.05$. For more discussions based on the results presented in Tables 3.1, 3.2 and 3.3, it may be observed that:

Table 3.1: Hot survival reliability equivalence factors.

	ω	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
	0.1	0.7238	0.4111	-	-	-	-	-	-	-	-	-
$A_1^{(0,1)}$	0.5	0.6009	-	-	-	-	-	-	-	-	-	-
	0.9	0.4519	-	-	-	-	-	-	-	-	-	-
	0.1	0.8657	0.7330	0.6047	0.6482	0.6108	0.5591	0.4930	0.4250	0.4134	0.3944	0.3648
$A_2^{(0,2)}$	0.5	0.8173	0.6203	0.4006	0.6483	0.5501	0.4239	0.2429	0.2666	0.1961	-	-
	0.9	0.7803	0.4800	-	0.6188	0.4345	-	-	-	-	-	-
	0.1	0.9111	0.8251	0.7445	0.7714	0.7482	0.7167	0.6774	0.6384	0.6320	0.6216	0.6057
$A_3^{(0,3)}$	0.5	0.8807	0.7603	0.6444	0.7767	0.7206	0.6554	0.5836	0.5910	0.5712	0.5444	0.5096
	0.9	0.8597	0.6998	0.5234	0.7675	0.6807	0.5790	0.4623	0.5035	0.4720	0.4312	0.3783
	0.1	0.9182	0.8163	0.6981	0.7403	0.7042	0.6517	0.5804	0.5022	0.4884	0.4654	0.4290
$A_1^{(1,0)}$	0.5	0.8111	0.5830	0.2579	0.6173	0.4929	0.3029	-	-	-	-	-
	0.9	0.7162	-	-	0.4671	-	-	-	-	-	-	-
	0.1	0.9336	0.8459	0.7381	0.7773	0.7438	0.6943	0.6255	0.5487	0.5350	0.5122	0.4760
$A_2^{(1,1)}$	0.5	0.8677	0.6963	0.4697	0.7226	0.6279	0.4953	0.2879	0.3159	0.2322	-	-
	0.9	0.8204	0.5318	-	0.6713	0.4839	-	-	-	-	-	-
	0.1	0.9451	0.8730	0.7848	0.8167	0.7894	0.7491	0.6937	0.6327	0.6219	0.6041	0.5762
$A_3^{(1,2)}$	0.5	0.9013	0.7773	0.6259	0.7959	0.7295	0.6419	0.5283	0.5410	0.5062	0.4552	0.3808
	0.9	0.8732	0.6922	0.3914	0.7749	0.6667	0.5078	0.1574	0.3384	0.2208	-	-
	0.1	0.9537	0.8945	0.8248	0.8497	0.8284	0.7976	0.7565	0.7129	0.7055	0.6932	0.6744
$A_4^{(1,3)}$	0.5	0.9222	0.8286	0.7224	0.8423	0.7940	0.7331	0.6600	0.6679	0.6467	0.6173	0.5780
	0.9	0.9030	0.7753	0.6084	0.8318	0.7587	0.6643	0.5433	0.5876	0.5539	0.5086	0.4473
	0.1	0.9594	0.9095	0.8532	0.8731	0.8560	0.8315	0.7991	0.7647	0.7588	0.7491	0.7341
$A_2^{(2,0)}$	0.5	0.9085	0.8090	0.7070	0.8230	0.7747	0.7169	0.6511	0.6580	0.6395	0.6141	0.5807
	0.9	0.8697	0.7185	0.5488	0.7828	0.7003	0.6026	0.4894	0.5295	0.4988	0.4590	0.4071
	0.1	0.9634	0.9167	0.8617	0.8813	0.8645	0.8401	0.8073	0.7722	0.7661	0.7562	0.7407
$A_3^{(2,1)}$	0.5	0.9235	0.8332	0.7333	0.8463	0.8004	0.7433	0.6757	0.6829	0.6635	0.6366	0.6009
	0.9	0.8954	0.7612	0.5929	0.8201	0.7441	0.6483	0.5297	0.5726	0.5399	0.4966	0.4390
	0.1	0.9669	0.9239	0.8720	0.8907	0.8747	0.8512	0.8193	0.7846	0.7785	0.7685	0.7530
$A_4^{(2,2)}$	0.5	0.9352	0.8563	0.7649	0.8679	0.8268	0.7742	0.7099	0.7169	0.6980	0.6715	0.6355
	0.9	0.9144	0.8008	0.6489	0.8513	0.7859	0.7004	0.5879	0.6296	0.5979	0.5548	0.4952
	0.1	0.9700	0.9308	0.8831	0.9004	0.8856	0.8640	0.8344	0.8020	0.7963	0.7869	0.7723
$A_5^{(2,3)}$	0.5	0.9443	0.8762	0.7968	0.8863	0.8507	0.8050	0.7486	0.7548	0.7381	0.7147	0.6826
	0.9	0.9283	0.8336	0.7071	0.8756	0.8211	0.7500	0.6559	0.6909	0.6644	0.6280	0.5771

- Reducing the failure rate of one component in the second subsystem (which we denote as $A_1^{(0,1)}$) by setting $\rho = 0.7238$ improves the reliability of the system to the same extent as augmenting the second subsystem by adding one component (which we denote as $B_1^{(0,1)}$) according to a hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$, see Table 3.1.

- Reducing the failure rate of each component belonging to the set $A_5^{(2,3)}$ of the

Table 3.2: Cold survival reliability equivalence factors with perfect switch.

	ω	$C_1^{(0,1)}$	$C_2^{(0,2)}$	$C_3^{(0,3)}$	$C_1^{(1,0)}$	$C_2^{(1,1)}$	$C_3^{(1,2)}$	$C_4^{(1,3)}$	$C_2^{(2,0)}$	$C_3^{(2,1)}$	$C_4^{(2,2)}$	$C_5^{(2,3)}$
	0.1	0.1409	-	-	-	-	-	-	-	-	-	-
$A_1^{(0,1)}$	0.5	0.1208	-	-	-	-	-	-	-	-	-	-
	0.9	0.0774	-	-	-	-	-	-	-	-	-	-
	0.1	0.6631	0.1749	-	0.1809	0.1370	-	-	-	-	-	-
$A_2^{(0,2)}$	0.5	0.6984	0.1207	-	0.3541	0.1095	-	-	-	-	-	-
	0.9	0.7302	0.0917	-	0.5010	0.0917	-	-	-	-	-	-
	0.1	0.7808	0.5209	0.2476	0.5230	0.5097	0.4413	0.2470	0.2087	0.2087	0.2085	0.2000
$A_3^{(0,3)}$	0.5	0.8067	0.5580	0.2036	0.6240	0.5568	0.4380	0.2011	0.1779	0.1779	0.1771	0.1550
	0.9	0.8298	0.6054	0.1534	0.7092	0.6054	0.4576	0.1468	0.1388	0.1379	0.1334	0.1015
	0.1	0.7543	0.1654	-	0.1756	0.0853	-	-	-	-	-	-
$A_1^{(1,0)}$	0.5	0.6771	-	-	0.1194	-	-	-	-	-	-	-
	0.9	0.6450	-	-	0.0622	-	-	-	-	-	-	-
	0.1	0.7901	0.2274	-	0.2355	0.1766	-	-	-	-	-	-
$A_2^{(1,1)}$	0.5	0.7680	0.1421	-	0.4174	0.1288	-	-	-	-	-	-
	0.9	0.7756	0.1000	-	0.5535	0.1000	-	-	-	-	-	-
	0.1	0.8272	0.4051	-	0.4097	0.3792	0.1904	-	-	-	-	-
$A_3^{(1,2)}$	0.5	0.8285	0.4817	-	0.5948	0.4794	0.1396	-	-	-	-	-
	0.9	0.8428	0.5539	-	0.7043	0.5539	0.1000	-	-	-	-	-
	0.1	0.8579	0.5689	0.2485	0.5715	0.5546	0.4679	0.2479	0.2089	0.2089	0.2088	0.2001
$A_4^{(1,3)}$	0.5	0.8666	0.6324	0.2076	0.7019	0.6310	0.4928	0.2049	0.1797	0.1797	0.1789	0.1558
	0.9	0.8806	0.6898	0.1654	0.7834	0.6898	0.5380	0.1573	0.1475	0.1464	0.1410	0.1040
	0.1	0.8797	0.6473	0.3151	0.6495	0.6351	0.5567	0.3144	0.2656	0.2656	0.2654	0.2545
$A_2^{(2,0)}$	0.5	0.8482	0.6271	0.2483	0.6884	0.6259	0.5097	0.2453	0.2170	0.2170	0.2161	0.1892
	0.9	0.8416	0.6281	0.1783	0.7275	0.6281	0.4848	0.1711	0.1622	0.1612	0.1562	0.1196
	0.1	0.8879	0.6511	0.3151	0.6533	0.6384	0.5581	0.3144	0.2656	0.2656	0.2654	0.2545
$A_3^{(2,1)}$	0.5	0.8696	0.6504	0.2485	0.7143	0.6492	0.5239	0.2454	0.2171	0.2171	0.2161	0.1893
	0.9	0.8716	0.6739	0.1808	0.7696	0.6739	0.5247	0.1731	0.1637	0.1626	0.1573	0.1198
	0.1	0.8969	0.6614	0.3154	0.6638	0.6483	0.5647	0.3146	0.2656	0.2656	0.2655	0.2545
$A_4^{(2,2)}$	0.5	0.8885	0.6851	0.2517	0.7469	0.6839	0.5553	0.2484	0.2186	0.2186	0.2176	0.1899
	0.9	0.8946	0.7237	0.1939	0.8080	0.7237	0.5829	0.1845	0.1732	0.1720	0.1657	0.1225
	0.1	0.9060	0.6850	0.3344	0.6872	0.6723	0.5904	0.3337	0.2819	0.2819	0.2817	0.2701
$A_5^{(2,3)}$	0.5	0.9040	0.7267	0.3033	0.7811	0.7257	0.6099	0.2995	0.2651	0.2651	0.2639	0.2311
	0.9	0.9117	0.7694	0.2843	0.8396	0.7694	0.6518	0.2731	0.2591	0.2575	0.2495	0.1914

system components by setting $\rho = 0.9040$ improves the reliability of the system like adding a set $C_1^{(0,1)}$ of components to the system according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 3.2.

- Reducing the failure rate of each component belonging to the set $A_5^{(2,3)}$ of the system components by setting factor $\rho = 0.2177$ improves the reliability of the system like adding a set $D_5^{(2,3)}$ of components to the system according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 3.3.

- Missing values of the SREFs mean that it is not possible to reduce the failure rate for the set A of components in order to improve the system reliability to be equivalent with the system reliability that can be obtained by improving the sets B, C, D of component according to duplication methods.

- In the same manner, one can interpret the other results presented in Tables 3.1, 3.2 and 3.3.

Tables 3.4, 3.5 and 3.6 present the MREFs for hot and cold (perfect and imperfect) duplication. Based on the results presented in those tables, we see that:

- The modified system that can be obtained by improving the set $H_1^{(0,1)}$, where $H = B, C, D$ of the system components, according to hot and cold (perfect and imperfect) duplication has the same mean time to failure of that system which can be obtained be reducing the failure rate of each component belonging to the set $A_1^{(0,1)}$ by factors $\rho = 0.614, 0.134, 0.226$ respectively.

- Empty cells of MREFs mean that it is not possible to reduce the failure rate of the set A components in order to improve the mean time to failure of the system to be equivalent with the mean time to failure of the system that can be obtained by improving the sets B, C, D of components according to the duplication methods.

- In the same manner, one can interpret the other results presented in Tables 3.4, 3.5 and 3.6.

Table 3.7 presents the mean time to failure of the modified systems assuming hot and cold duplication methods, the latter with perfect and imperfect switch, assuming a constant failure rate $\lambda = 0.05$. The mean time to failure of the original system is 1.172. From this table, one can conclude that

$$MTTF < MTTF^{(B)} < MTTF^{(D)} < MTTF^{(C)}.$$

Figure 3.3 explains the improvement strategies to calculate the SREFs. Figure 3.4 presents reliability functions of the original and some modified systems. From these figures, one may observe that, for this scenario:

- Improving the reliability of all components according to cold duplication with perfect switch gives the best system.
- For the same number of components

  $$R(t) < R^{(B)}(t) < R^{(D)}(t) < R^{(C)}(t)$$

  where $\lambda = 0.05$.

Figures 3.5 and 3.6 present the behaviour of MTTF against the appropriate reduction factor ρ. It seems from these two figures that the following conditions hold:

- MTTFs are non-decreasing with decreasing ρ for all possible sets A.
- Reducing the failure rate of one or two components from the first subsystem gives a better system than that obtained by reducing the failure rate of one or two components in the second subsystem, see Figure 3.5. This means that improving a component from the subsystem with the smaller number of components is better than improving a component from the subsystem with the larger number of components.

- Reducing the failure rates of all components in the system gives the best system, see Figure 3.6.

- It is not possible to reduce the failure rate of the sets $A_2^{(1,1)}$ or $A_2^{(0,2)}$ of the system components to reach the mean time to failure which we can achieve by improving the sets $B_5^{(2,3)}$ or $C_3^{(1,2)}$ of the system components according to hot duplication and cold duplication with perfect switch respectively, see Figure 3.5.

- Reducing the failure rate of three components in the second subsystem (which we denote as $A_3^{(0,3)}$) by setting $\rho = 0.236$ improves the MTTF of the system like adding three components to the second subsystem (which we denote as $D_3^{(0,3)}$) according to the cold duplication method with imperfect switch, see Figure 3.6 and compare with Table 3.6.

- Reducing the failure rate of one component in the first subsystem and two components in the second subsystem (which we denote as $A_3^{(1,2)}$) by setting $\rho = 0.390$ improves the MTTF of the system like adding two components in the first subsystem and three components in the second subsystem (which we denote as $B_5^{(2,3)}$) according to the hot duplication method, see Figure 3.6 and compare with Table 3.4.

- Improving a number of components selected from two subsystems, with equal numbers if they are even, gives a better system than that obtained by improving the number of components selected from the same subsystem or selected from the two subsystems with unequal numbers, see Figure 3.6.

This example clearly generates interesting conclusions for this particular system and distributional assumptions. More importantly though, it demonstrates the potential for applying these methods to other system structures. It also illustrates how to address specific questions that arise when attempting to improve the reliability of simple systems or simple configurations of possibly complex sub-systems in many diverse applications.

3.5 Conclusions

In this study, we evaluate both the system reliability function and the system mean time to failure in order to study the reliability equivalence factors for series-parallel systems. These system structures arise often in business and industry and the methodology adapts readily for other forms including parallel-series systems and more complex networks. All the system components are assumed to be independent and identically distributed, according to an exponentiated Weibull distribution, on account of its flexibility and tractability for practical purposes. We discuss four different methods to improve such a system: reduction, hot duplication and cold duplication with perfect or imperfect switch.

We derive analytical results for both survival and mean reliability equivalence factors of these systems. Some numerical results are then presented for a representative system in order to illustrate how one can apply the theoretical results obtained and to compare the various approaches in this context. Accordingly, detailed recommendations are discussed for improving the system considered in this study. Although it would be inappropriate to extrapolate these results to other system structures from only this case study, we make some interesting observations which suggest patterns that might arise more generally.

We have also identified several extensions of this study that might be worthy of future exploration, including comparisons with parallel-series formats and analysis of other important system structures, equivalent systems with non-identical components and simpler systems with dependent components. The methods described in this study adapt readily to deal with all these other scenarios.

The GQFRD of Chapter 2 and EWD of Chapter 3 are broad families that cover most common lifetime distributions for practical application. We advocate that, unless a specific distributional form is known, both families should be considered in any given setting. The final choice is then determined using standard goodness of fit measures such as the Bayes information criterion.

Chapter 2 and 3 also differ in the system structure considered. Together, they cover many common forms encountered in practice, and lead nicely to our new developments in Chapter 4.

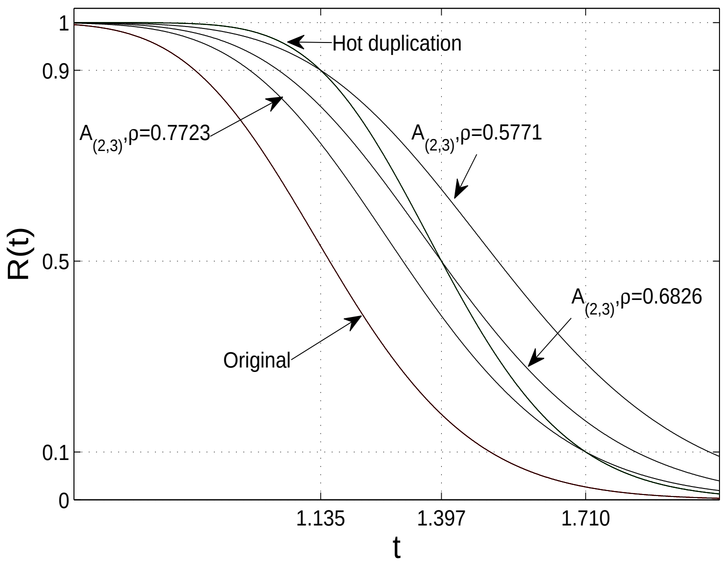


Figure 3.3: Use of survival reliability equivalence factors to recommend system improvement strategies.

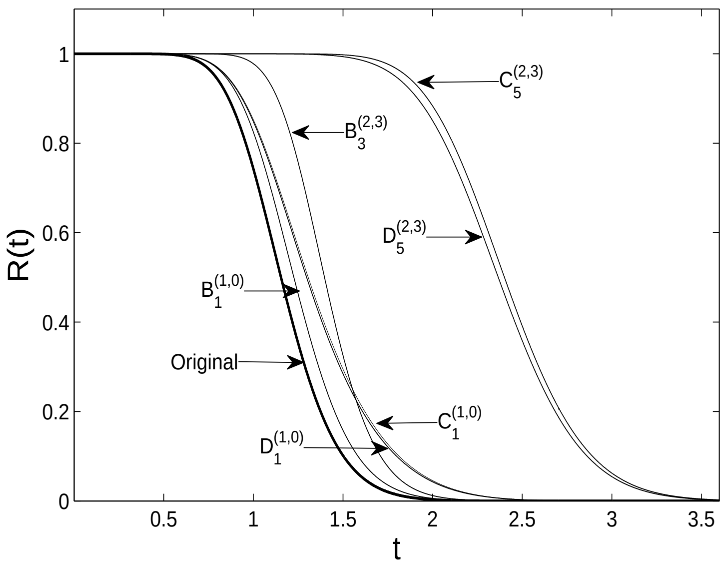


Figure 3.4: Reliability function of the original and some modified systems.

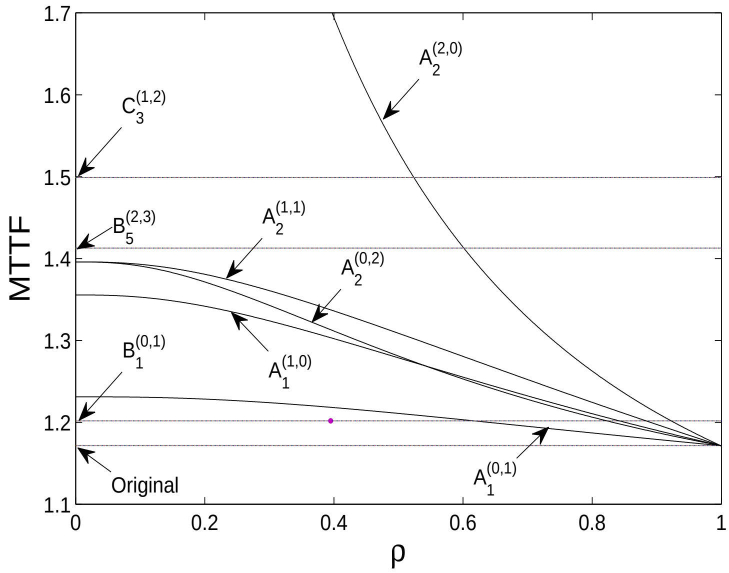


Figure 3.5: The behaviour of MTTF against ρ, when $|A| \leq 2$.

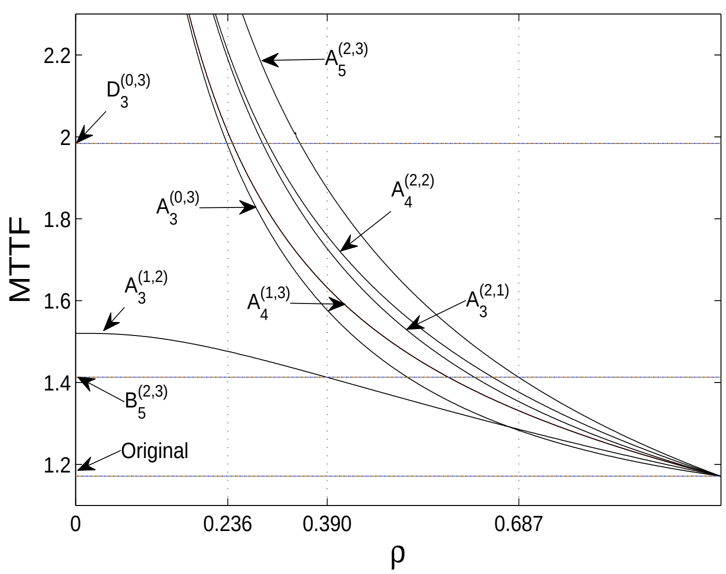


Figure 3.6: The behaviour of MTTF against ρ, when $|A| > 2$.

Table 3.3: Cold survival reliability equivalence factors with imperfect switch ($\lambda = 0.05$).

	ω	$D_1^{(0,1)}$	$D_2^{(0,2)}$	$D_3^{(0,3)}$	$D_1^{(1,0)}$	$D_2^{(1,1)}$	$D_3^{(1,2)}$	$D_4^{(1,3)}$	$D_2^{(2,0)}$	$D_3^{(2,1)}$	$D_4^{(2,2)}$	$D_5^{(2,3)}$
	0.1	0.2157	-	-	-	-	-	-	-	-	-	-
$A_1^{(0,1)}$	0.5	0.2401	-	-	-	-	-	-	-	-	-	-
	0.9	0.2494	-	-	-	-	-	-	-	-	-	-
	0.1	0.6755	0.2153	-	0.2060	0.1666	-	-	-	-	-	-
$A_2^{(0,2)}$	0.5	0.7113	0.2246	-	0.3866	0.1876	-	-	-	-	-	-
	0.9	0.7425	0.2460	-	0.5255	0.2218	-	-	-	-	-	-
	0.1	0.7886	0.5356	0.2578	0.5320	0.5182	0.4506	0.2570	0.2153	0.2153	0.2153	0.2059
$A_3^{(0,3)}$	0.5	0.8146	0.5784	0.2241	0.6381	0.5693	0.4530	0.2192	0.1910	0.1910	0.1895	0.1637
	0.9	0.8370	0.6233	0.2206	0.7205	0.6187	0.4761	0.1945	0.1893	0.1834	0.1679	0.1158
	0.1	0.7657	0.2287	-	0.2149	0.1508	-	-	-	-	-	-
$A_1^{(1,0)}$	0.5	0.6920	-	-	0.2269	-	-	-	-	-	-	-
	0.9	0.6628	-	-	0.2439	-	-	-	-	-	-	-
	0.1	0.8006	0.2816	-	0.2692	0.2162	-	-	-	-	-	-
$A_2^{(1,1)}$	0.5	0.7793	0.2661	-	0.4541	0.2221	-	-	-	-	-	-
	0.9	0.7867	0.2772	-	0.5786	0.2502	-	-	-	-	-	-
	0.1	0.8358	0.4375	-	0.4297	0.3990	0.2217	-	-	-	-	-
$A_3^{(1,2)}$	0.5	0.8367	0.5191	-	0.6164	0.5027	0.2159	-	-	-	-	-
	0.9	0.8503	0.5827	-	0.7187	0.5756	0.2410	-	-	-	-	-
	0.1	0.8647	0.5875	0.2590	0.5829	0.5655	0.4797	0.2582	0.2156	0.2156	0.2156	0.2061
$A_4^{(1,3)}$	0.5	0.8727	0.6545	0.2306	0.7161	0.6447	0.5110	0.2250	0.1938	0.1938	0.1923	0.1648
	0.9	0.8861	0.7066	0.2508	0.7930	0.7023	0.5584	0.2173	0.2106	0.2032	0.1836	0.1202
	0.1	0.8852	0.6631	0.3281	0.6593	0.6444	0.5678	0.3271	0.2740	0.2740	0.2740	0.2621
$A_2^{(2,0)}$	0.5	0.8547	0.6462	0.2730	0.7013	0.6377	0.5248	0.2671	0.2330	0.2330	0.2313	0.1998
	0.9	0.8484	0.6452	0.2494	0.7383	0.6409	0.5029	0.2223	0.2168	0.2107	0.1942	0.1361
	0.1	0.8932	0.6673	0.3281	0.6634	0.6481	0.5694	0.3271	0.2740	0.2740	0.2740	0.2621
$A_3^{(2,1)}$	0.5	0.8755	0.6706	0.2734	0.7275	0.6616	0.5403	0.2674	0.2331	0.2331	0.2313	0.1998
	0.9	0.8774	0.6908	0.2596	0.7796	0.6865	0.5442	0.2291	0.2229	0.2161	0.1979	0.1366
	0.1	0.9019	0.6782	0.3285	0.6741	0.6583	0.5765	0.3274	0.2741	0.2741	0.2741	0.2621
$A_4^{(2,2)}$	0.5	0.8937	0.7049	0.2784	0.7594	0.6961	0.5728	0.2720	0.2353	0.2353	0.2335	0.2007
	0.9	0.8994	0.7389	0.2902	0.8167	0.7350	0.6022	0.2529	0.2454	0.2370	0.2147	0.1414
	0.1	0.9106	0.7011	0.3482	0.6972	0.6820	0.6021	0.3471	0.2908	0.2908	0.2908	0.2781
$A_5^{(2,3)}$	0.5	0.9085	0.7442	0.3334	0.7920	0.7365	0.6259	0.3262	0.2845	0.2845	0.2824	0.2440
	0.9	0.9158	0.7820	0.3892	0.8467	0.7788	0.6679	0.3507	0.3426	0.3335	0.3086	0.2177

Table 3.4: Hot mean equivalence factors.

	$B_1^{(0,1)}$	$B_2^{(0,2)}$	$B_3^{(0,3)}$	$B_1^{(1,0)}$	$B_2^{(1,1)}$	$B_3^{(1,2)}$	$B_4^{(1,3)}$	$B_2^{(2,0)}$	$B_3^{(2,1)}$	$B_4^{(2,2)}$	$B_5^{(2,3)}$
$A_1^{(0,1)}$	0.614	-	-	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.825	0.634	0.421	0.643	0.555	0.441	0.286	0.289	0.241	0.150	-
$A_3^{(0,3)}$	0.885	0.768	0.651	0.773	0.722	0.661	0.590	0.591	0.573	0.548	0.515
$A_1^{(1,0)}$	0.843	0.647	0.387	0.657	0.556	0.415	0.115	0.117	0.066	-	-
$A_2^{(1,1)}$	0.883	0.728	0.513	0.736	0.653	0.536	0.357	0.360	0.301	0.188	-
$A_3^{(1,2)}$	0.910	0.793	0.640	0.799	0.738	0.655	0.540	0.542	0.510	0.462	0.390
$A_4^{(1,3)}$	0.928	0.838	0.728	0.843	0.797	0.739	0.664	0.665	0.645	0.618	0.58
$A_2^{(2,0)}$	0.923	0.834	0.733	0.838	0.796	0.742	0.676	0.677	0.660	0.635	0.602
$A_3^{(2,1)}$	0.934	0.852	0.753	0.856	0.815	0.763	0.696	0.697	0.679	0.654	0.619
$A_4^{(2,2)}$	0.943	0.870	0.779	0.873	0.836	0.788	0.723	0.724	0.707	0.682	0.647
$A_5^{(2,3)}$	0.950	0.886	0.805	0.889	0.856	0.813	0.756	0.757	0.742	0.719	0.687

Table 3.5: Cold mean equivalence factors with perfect switch.

	$C_1^{(0,1)}$	$C_2^{(0,2)}$	$C_3^{(0,3)}$	$C_1^{(1,0)}$	$C_2^{(1,1)}$	$C_3^{(1,2)}$	$C_4^{(1,3)}$	$C_2^{(2,0)}$	$C_3^{(2,1)}$	$C_4^{(2,2)}$	$C_5^{(2,3)}$
$A_1^{(0,1)}$	0.134	-	-	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.692	0.162	-	0.288	0.129	-	-	-	-	-	-
$A_3^{(0,3)}$	0.802	0.549	0.208	0.590	0.543	0.442	0.205	0.181	0.180	0.179	0.157
$A_1^{(1,0)}$	0.710	-	-	0.163	-	-	-	-	-	-	-
$A_2^{(1,1)}$	0.780	0.202	-	0.359	0.162	-	-	-	-	-	-
$A_3^{(1,2)}$	0.832	0.464	-	0.541	0.450	0.167	-	-	-	-	-
$A_4^{(1,3)}$	0.867	0.619	0.214	0.665	0.611	0.490	0.211	0.184	0.184	0.182	0.159
$A_2^{(2,0)}$	0.862	0.636	0.256	0.676	0.630	0.525	0.252	0.222	0.222	0.220	0.193
$A_3^{(2,1)}$	0.878	0.655	0.257	0.696	0.648	0.538	0.253	0.223	0.223	0.221	0.193
$A_4^{(2,2)}$	0.894	0.683	0.263	0.724	0.676	0.564	0.259	0.227	0.227	0.225	0.196
$A_5^{(2,3)}$	0.907	0.720	0.310	0.757	0.714	0.611	0.306	0.270	0.270	0.267	0.234

Table 3.6: Cold mean equivalence factors with imperfect switch ($\lambda = 0.05$).

	$D_1^{(0,1)}$	$D_2^{(0,2)}$	$D_3^{(0,3)}$	$D_1^{(1,0)}$	$D_2^{(1,1)}$	$D_3^{(1,2)}$	$D_4^{(1,3)}$	$D_2^{(2,0)}$	$D_3^{(2,1)}$	$D_4^{(2,2)}$	$D_5^{(2,3)}$
$A_1^{(0,1)}$	0.226	-	-	-	-	-	-	-	-	-	-
$A_2^{(0,2)}$	0.704	0.223	-	0.316	0.179	-	-	-	-	-	-
$A_3^{(0,3)}$	0.810	0.567	0.236	0.602	0.554	0.456	0.229	0.199	0.198	0.195	0.167
$A_1^{(1,0)}$	0.723	-	-	0.224	-	-	-	-	-	-	-
$A_2^{(1,1)}$	0.790	0.280	-	0.393	0.224	-	-	-	-	-	-
$A_3^{(1,2)}$	0.840	0.498	-	0.562	0.473	0.223	-	-	-	-	-
$A_4^{(1,3)}$	0.873	0.639	0.244	0.677	0.624	0.507	0.236	0.204	0.203	0.200	0.170
$A_2^{(2,0)}$	0.867	0.654	0.288	0.688	0.641	0.539	0.280	0.245	0.244	0.240	0.205
$A_3^{(2,1)}$	0.884	0.673	0.291	0.708	0.660	0.553	0.282	0.246	0.245	0.241	0.206
$A_4^{(2,2)}$	0.898	0.701	0.299	0.735	0.688	0.580	0.290	0.251	0.250	0.246	0.209
$A_5^{(2,3)}$	0.911	0.736	0.349	0.767	0.724	0.626	0.339	0.297	0.296	0.291	0.250

Table 3.7: Mean times to failure of the modified systems.

	$\{0_1, 1_2\}$	$\{0_1, 2_2\}$	$\{0_1, 3_2\}$	$\{1_1, 0_2\}$	$\{1_1, 1_2\}$	$\{1_1, 2_2\}$	$\{1_1, 3_2\}$	$\{2_1, 0_2\}$	$\{2_1, 1_2\}$	$\{2_1, 2_2\}$	$\{2_1, 3_2\}$
hot	1.202	1.244	1.305	1.242	1.266	1.299	1.347	1.346	1.360	1.381	1.413
cold perfect	1.230	1.381	2.104	1.347	1.387	1.499	2.120	2.255	2.257	2.266	2.420
cold imperfect	1.228	1.366	1.984	1.338	1.377	1.481	2.013	2.150	2.155	2.173	2.343

Part II

Deriving reliability equivalence factors using survival signature

Chapter 4

Using survival signature to derive the reliability equivalence factors for simple systems

Most studies concerning reliability equivalence factors assume systems with independent and identically distributed components with specific structures. The question is, can we derive the reliability equivalence factors without assuming a system with identically distributed components or specific structure? The answer is yes. Using the recent concept of survival signature we can derive the reliability equivalence factors for any system with any structure with any lifetime distributions as long as the reduction improvement of the component that we need to improve is known.

Samaniego (2007) provided a very good overview about the concept of signature including the theory of system signatures and explained how to calculate the signature for systems with small numbers of components. Coolen and Coolen-Maturi (2012) developed extensions to signature, resulting the new idea of survival signature. Aslett (2012) developed computer packages in R to calculate the system signature and survival signature, which are very useful especially in systems with large numbers of components. Recently Aslett, Coolen and Wilson presented a very good study that applies the survival signature to a complex system in Aslett et al. (2014).

In this part of our research, we present a new technique to derive the reliability equivalence factors for any system using the concept of survival signature. To our knowledge, this is the first attempt to use survival signature to compute the reliability equivalence factors for different systems. The whole information that we need to derive the reliability equivalence factors using survival signature is the system structure and the lifetime distribution of each component in the system. For this reason, using survival signature to derive the reliability equivalence factors is suitable and appropriate for real applications and might offer substantial benefits. To illustrate the idea, deriving the reliability equivalence factors using survival signature is analogous to an adjustable spanner for all of the previous studies but the reliability equivalence factors are analogous to a normal spanner. An adjustable spanner operates on a wide range of bolt sizes, whereas a normal spanner should be used only on a specific bolt size.

One first advantage of using survival signature to derive the reliability equivalence factors is in its flexibility; it provides a general methodology that can be used with different system structures. The survival signature can be used to derive reliability equivalence factors for systems with multiple types of components and different system structures, which matches real applications more than all previous studies in this field. The second reason is that there is a dedicated computer package which facilitates deriving survival signatures for any complex systems as we shall see in the next chapter.

4.1 How survival signature can generate reliability equivalence factors

To derive the reliability equivalence factors using the survival signature the following conditions must be met:

1. The system must be coherent with independent components.
2. The structure of the system is known.
3. The lifetime distribution of each component in the system is known.
4. The reduction improvement of the component which we need to improve is also known.

For any system that satisfies the above four conditions, the survival signature can be used to derive reliability equivalence factors as the following steps:

1. We give each component in the system a serial number, which helps us to derive the survival signature specially for complex systems and systems with large numbers of components.
2. We specify the components that we want to improve.
3. We replace the reliability function for the components that we need to improve with the reliability function for the same components after they are improved.
4. We classify improved system components into different types where each type has one component or several components with identical lifetime distributions.

5. We derive the survival signature for the improved system using Equation (1.4.5) where the improved system has at least two types of components. For complex systems and systems with large numbers of components we use the ReliabilityTheory R package to derive survival signature as we shall see in Chapter 5.

6. We derive the reliability function for the improved system using Equation (1.4.6) for all possible improvements.

7. For the SREFs, we determine the equivalence between the reliability function of that system improved according to the reduction method and the reliability function of that system improved according to any of the duplication methods where the reliability function of the system is chosen to be a fixed value ω.

8. For the MREFs, we determine the equivalence between the mean time to failure of a system improved according to the reduction method and the mean time to failure of the same system improved according to any of duplication methods.

To illustrate how the survival signature can be used and how it is useful to derive the reliability equivalence factors, we present the following:

1. We recalculate the reliability equivalence factors using survival signature for systems that we studied previously in Chapters 2 and 3 to compare the results and methods.

2. We derive the reliability equivalence factors using survival signature for a complex system and a network to demonstrate the usefulness of survival signature and the `ReliabilityTheory` R package in deriving the corresponding reliability equivalence factors.

4.2 Reliability equivalence factors for a parallel-series system with GQFRD using survival signature

We use the concept of survival signature to recalculate the reliability equivalence factors for the parallel-series system that we studied in Chapter 2. We derive the reliability functions and the mean times to failure for this system using survival signature and we use them to calculate both survival reliability equivalence factors and mean reliability equivalence factors for this system. By doing so, we hope to replicate our earlier results and hence confirm the validity of survival signature reliability equivalence in this context. This would then provide a degree of confidence for applying this method in others situations.

In this section we consider the same system which has been studied as an example of a parallel-series system in Chapter 2. All the systems' components are assumed to be independent and follow the generalized quadratic failure rate distribution with identical parameters. First, we compute the survival signature and use it to derive the reliability function and the mean time to failure (MTTF) of the original system. Second, we compute the survival signature to derive the reliability functions and MTTFs of the systems following improvement according to reduction, hot duplication and cold duplication (perfect and imperfect) methods. Third, we match current work with analogies in the previous study. Fourth, we equate the reliability function and the MTTF of the system improved according to the reduction method with the reliability function and the MTTF of the system improved according to each of the duplication methods to determine the reliability equivalence factors. Finally, we compare results and methods for using survival signature to derive reliability equivalence factors with results in the previous chapters.

4.2.1 Properties of the original system in Figure 1.8

The system we consider here is shown in Figure 1.8 and consists of five independent and identically distributed components which follow the generalized quadratic failure rate distribution with identical parameters $\alpha = 0.029, \beta = -1.597 \times 10^{-3}, \gamma = 2.608 \times 10^{-5}$ and $\theta = 0.786$. It is the same system that was investigated in Chapter 2. In fact this system meets all the conditions necessary for using survival signature to derive reliability equivalence factors. It is a coherent system with independent components and the lifetime and reduction improvement are known for all system components. This system consists of two subsystems connected in series, where the first consists of two components connected in parallel and the second consists of three components connected in parallel.

The global structure of the system is more important than that of the individual subsystems in computing the reliability of systems using survival signature. Thus we give components in the first subsystem the serial numbers 1 and 2 and components in the second subsystem the numbers 3, 4 and 5 as shown in Figure 1.8. The reliability function of each component in the system is presented in Chapter 2 in Equation (2.2.1). We summarize the properties of this system in the following points:

- System with five independent and identically distributed components, $m = 5$;
- All the system components follow the generalized quadratic failure rate distribution with identical parameters,
- System with 32 state vectors;

- System with only one type of components;
- System with two minimal cut sets which are $\{1,2\}$ and $\{3,4,5\}$;
- The failure time of this system's components can be ordered in $5! = 120$ ways.

As described in Section 1.3.4, the system signature vector is $S = (\frac{0}{120}, \frac{12}{120}, \frac{36}{120}, \frac{72}{120}, \frac{0}{120})$ $= (0, 0.1, 0.3, 0.6, 0)$. The survival signature of this system can be derived directly from system signature using Relation (1.4.2) as we presented in Table 1.3.

The reliability function of this system is calculated using Equation (1.4.4) as

$$R(t) = \sum_{l=0}^{5} \Phi(l) \binom{5}{l} \left[\left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{\theta} \right]^{5-l} \left[1 - \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{\theta} \right]^{l} \tag{4.2.1}$$

where survival signature $\Phi(l)$ is presented in Table 1.3. The mean time to failure of this system is given by

$$\begin{aligned} MTTF &= \int_0^{\infty} R(t)dt \\ &= \int_0^{\infty} \left\{ \sum_{l=0}^{5} \Phi(l) \binom{5}{l} \left[\left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{\theta} \right]^{5-l} \left[1 - \left\{ 1 - e^{-(\alpha t + \frac{\beta}{2} t^2 + \frac{\gamma}{3} t^3)} \right\}^{\theta} \right]^{l} \right\} dt. \end{aligned} \tag{4.2.2}$$

4.2.2 Properties of the improved systems corresponding to Figure 1.8

Improving one component or more of the system according to any improvement method gives a system with two types of independent components, except in the case of improving all system components. The first type comprises original components that are not improved and their number is m_1. The second type comprises components that are improved according to any improvement method and their number is

m_2 where the total number of system components is $m = m_1 + m_2 = 5$. The main properties of the original system will be retained, such as the number of state vectors, sets of minimal cut and system's component failure time order. We can summarize the properties of the improved system in the following points:

- The reliability function of any component improved according to the reduction method can be written as

$$R_A(t) = 1 - \left\{1 - e^{-\rho(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{\theta}. \tag{4.2.3}$$

- The reliability function of any component improved according to the hot duplication method can be written as

$$R_B(t) = 1 - \left\{1 - e^{-(\alpha t + \frac{\beta}{2}t^2 + \frac{\gamma}{3}t^3)}\right\}^{2\theta}. \tag{4.2.4}$$

- The reliability function of any component improved according to the cold duplication method with perfect switch takes the form

$$R_C(t) = 1 - \int_0^t \frac{-dR(x)}{dx}[1 - R(t-x)]dx \tag{4.2.5}$$

  where $R(t)$ is the reliability of original component which is the reliability of the generalized quadratic failure rate distribution.

- The reliability function of any component improved according to the cold duplication method with imperfect switch takes the form

$$R_D(t) = 1 - \int_0^t \frac{-dR(x)}{dx}[1 - R(t-x)S(x)]dx \tag{4.2.6}$$

  where $S(t)$ is the reliability function for the imperfect switch, which is chosen to have a constant failure rate $\lambda = 0.01$.

- The survival signature of the improved system is derived using Relation (1.4.5) and presented in Tables 4.1 and 4.2.
- The survival signature is not affected by the improvement type for all system components.
- The reliability function of any improved system can be calculated by using Relation (1.4.4) as

$$R^{(H)}(t) = \sum_{l_1=0}^{m_1}\sum_{l_2=0}^{m_2} \Phi(l_1, l_2) \left\{ \binom{m_1}{l_1} [1 - R(t)]^{m_1-l_1} [R(t)]^{l_1} \right. \\ \left. \times \binom{m_2}{l_2} [1 - R_H(t)]^{m_2-l_2} [R_H(t)]^{l_2} \right\} \qquad (4.2.7)$$

  where $\Phi(l_1, l_2)$ is the survival signature of the improved system, $R(t)$ is the reliability function of the original components and $R_H(t)$ is the reliability function of the improved components. $\Phi(l_1, l_2)$ is presented in Tables 4.1 and 4.2. $R(t)$ is the reliability function of GQFRD. $R_H(t)$ is the reliability function of the improved components for all $H = A, B, C, D$, where A is the reduction improvement, B is the hot duplication improvement, C is the cold duplication improvement with perfect switch, and D is the cold duplication improvement with imperfect switch.

- The mean time to failure for any improvement system can be written as

$$MTTF^{(H)} = \int_0^\infty R^{(H)}(t)dt. \qquad (4.2.8)$$

To explain Tables 4.1 and 4.2, we explain the first row of Table 4.1 then the other rows are similar. Improving component number 3 according to any improvement methods, reduction, hot duplication, cold duplication with perfect switch, and cold duplication with imperfect switch, gives rise to a system with two types of components. The first type comprises non-improved components and its number is $m_1 = 4$, while the second type comprises improved components and its number is $m_2 = 1$, so $l_1 = 0, 1, .., 4$ and $l_2 = 0, 1$. The improved system topology is presented in the first cell of the first row of Table 4.1 and the survival signature for the improved system is presented in the second cell of the same row. When we compare improving component number 3 with our previous study in Chapter 2 we find that its effect is similar to improving one component from the second subsystem and no component from the first subsystem, which can be written as presented in the third cell of Table 4.1 as $H_1^{(0,1)}$, $H = A, B, C, D$. Here, A is the reduction method, B is the hot duplication method, C is the cold duplication method with perfect switch, and D is the cold duplication method with imperfect switch.

In Tables 4.3, 4.5 and 4.7 the hot and cold (perfect and imperfect) SREFs are calculated using the definition of the survival equivalence factors where ω is chosen to be $0.1, 0.5, 0.9$ and the imperfect switch has a constant failure rate $\lambda = 0.01$. From those tables we observe that:

- Reducing the failure rate of component number 1 by setting $\rho = 0.6767$ improves the reliability of the system. This is equivalent to improving the same component according to a hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$, see Table 4.3.

- Reducing the failure rate of component number 3 by setting $\rho = 0.2776$ improves the reliability of the system. This is equivalent to improving the same component according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 4.5.

- Reducing the failure rate of component numbers 1, 2 by setting $\rho = 0.2932$ improves the reliability of the system. This is equivalent to improving components 1, 2 according to a cold duplication method with imperfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 4.7.

- Reducing the failure rate of each component belonging to the system by setting $\rho = 0.2278$ improves the reliability of the system. This also is equivalent to improving all system components according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 4.5.

- Improving either component number 1 or 2 according to the same improvement method gives the same result and is true for either components 3, 4 and 5, see Table 4.3.

- In the same manner, one can interpret the other results presented in those tables.

Tables 4.4, 4.6 and 4.8 present the hot and cold (perfect and imperfect) MREFs and the MTTFs for improved systems. Based on the results presented in those tables, we see that:

- Improving component number 1 according to the hot duplication increases the system mean time to failure to be 60.045 and the same system mean time to failure can be obtained by reducing the failure rate of same component by setting $\rho = 0.430$, see Table 4.4. Note that the mean time to failure of the original system is 53.063.

- Improving component 3 according to cold duplication with perfect switch increases the system mean time to failure to be 58.005 and the same mean time to failure can be obtained by reducing the failure rate of component 3 by setting $\rho = 0.257$, see Table 4.6.

- Improving each component belonging to the system according to a cold duplication with perfect switch increases the system mean time to failure to be 99.362 which is the best possible improvement and the same mean time to failure can be obtained by reducing the failure rate of each component in the setting $\rho = 0.196$, see Table 4.6.

- In the same manner, one can interpret the other results presented in those tables.

4.3 Reliability equivalence factors for a series-parallel system with EWD using survival signature

As we demonstrated in the first part of this chapter, we use the concept of survival signature to recalculate the reliability equivalence factors for the series-parallel system that we studied in Chapter 3. We derive the reliability functions and the mean times to failure for this system using survival signature. We calculate both survival reliability equivalence factors and mean reliability equivalence factors for all possible improvements.

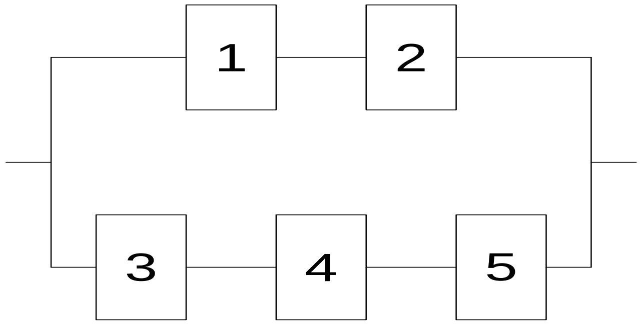


Figure 4.1: Series-parallel system consisting of five identically distributed components.

4.3.1 Properties of the original system in Figure 4.1

The system we consider here is shown in Figure 4.1 and consists of five independent and identically distributed components which follow the exponentiated Weibull lifetime distribution with identical parameters $\alpha = 1, \beta = 2$ and $\theta = 3$. This is the same system that we studied in Chapter 3. This system meets all the conditions to use the survival signature to derive reliability equivalence factors. It is a coherent system with independent components and the lifetime distribution and reduction improvement are known for all system components. This system consists of two subsystems connected in parallel, where the first consists of two components connected in series and the second consists of three components connected in series. We give components in the first subsystem the serial numbers 1 and 2 and components in the second subsystem the numbers 3, 4 and 5 as shown in Figure 4.1. The reliability function of each component in the system is presented earlier in Equation (3.2.1). This system has six minimal cut sets which are $\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\}$ and $\{2,5\}$. The system signature vector is $S = (\frac{0}{120}, \frac{72}{120}, \frac{36}{120}, \frac{12}{120}, \frac{0}{120}) = (0, 0.6, 0.3, 0.1, 0)$. The survival signature of this system can be derived directly from system signature using Relation (1.4.2) and resembles the survival signature when all system components are improved, see the last row of Table 4.10

The reliability function of this system can be calculated by using Equation (1.4.4) as

$$R(t) = \sum_{l=0}^{5} \Phi(l) \binom{5}{l} \left[\left(1 - e^{-\alpha t^{\beta}}\right)^{\theta} \right]^{5-l} \left[1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta} \right]^{l}. \tag{4.3.1}$$

where $\Phi(l)$ is presented in the last row of Table 4.10. The mean time to failure of this system is given by

$$\begin{aligned} MTTF &= \int_{0}^{\infty} R(t)dt \\ &= \int_{0}^{\infty} \left\{ \sum_{l=0}^{5} \Phi(l) \binom{5}{l} \left[\left(1 - e^{-\alpha t^{\beta}}\right)^{\theta} \right]^{5-l} \left[1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{\theta} \right]^{l} \right\} dt. \end{aligned} \tag{4.3.2}$$

4.3.2 Properties of the improved systems corresponding to Figure 4.1

The similarities between the improving steps for this system and the improving steps for the parallel-series in section 4.2.2 are substantial. The differences which we have in this system are the structure of the system and the lifetime distribution of system components. We can summarize the properties of the improved system in the following points:

- The reliability function of any component improved according to the reduction method can be written as

$$R_A(t) = 1 - \left(1 - e^{-\rho\alpha t^{\beta}}\right)^{\theta} \tag{4.3.3}$$

where ρ is the reduction factor.

- The reliability function of any component improved according to the hot duplication method can be written as

$$R_B(t) = 1 - \left(1 - e^{-\alpha t^{\beta}}\right)^{2\theta} \tag{4.3.4}$$

- The reliability function of any component improved according to the cold duplication method with perfect or imperfect switch is the same as Equations (4.2.5) and (4.2.6), where the reliability function for the imperfect switch is chosen to have a constant failure rate $\lambda = 0.05$.

- The survival signature of the improved system is derived using Relation (1.4.5) and is presented in Tables 4.9 and 4.10.

- The survival signature is not affected by the improvement type for all system components.

- The reliability function and the mean time to failure of any improved system can be calculated using Equations (4.2.7) and (4.2.8).

Tables 4.11, 4.13 and 4.15 present the hot and cold (perfect and imperfect) SREFs for some system component improvements for the system in Figure 4.1. In the same manner, one can interpret the results presented in Tables 4.3, 4.5 and 4.7.

Tables 4.12, 4.14 and 4.16 present the hot and cold (perfect and imperfect) MREFs for component improvement for the system in Figure 4.1 and the MTTFs for the improved system. In the same manner, one can interpret the results presented in Tables 4.4, 4.6 and 4.8.

4.4 Comparing SREFs and MREFs derived using survival signature and analytical methods

Subsequent to comparison of the survival reliability equivalence factors (SREFs) and mean reliability equivalence factors (MREFs) for both parallel-series and series-parallel systems, which are derived using survival signature with the results in Chapters 2 and 3, it is found that both methods give the same results. This important finding confirms the apparent validity of survival signature in reliability equivalence testing.

For the parallel-series system, when we compare SREFs, MREFs and MTTFs for the improved system which is derived using survival signature with SREFs, MREFs and MTTFs which were studied in Chapter 2, we find that:

- The hot SREFs for improving component number 1 of the system in Figure 1.8 are equal to the hot SREFs for improving one component in the first subsystem only (which we denote in Chapter 2 as $A_1^{(1,0)}$), where the SREFs in the first row of Table 4.3 are equal to the hot SREFs in the intersection of $A_1^{(1,0)}$ row with $B_1^{(1,0)}$ column in Table 2.1.

- The hot SREFs for improving component number 3 of the system in Figure 1.8 is equal the hot SREFs for improving one component in the second subsystem only (which we denote in Chapter 2 as $A_1^{(0,1)}$), where the SREFs in the third row of Table 4.3 are equal to the SREFs in the intersection of $A_1^{(0,1)}$ row with $B_1^{(0,1)}$ column in Table 2.1.

- The cold SREFs with perfect switch for improving component number 3 of the system in Figure 1.8 are equal to the cold SREFs with perfect switch for improving one component in the second subsystem only (which we denote in Chapter 2 as $A_1^{(0,1)}$), where the SREFs in the second row of Table 4.5 are equal to the SREFs in the intersection of $A_1^{(0,1)}$ row with $C_1^{(0,1)}$ column in Table 2.2.

- The cold SREFs with imperfect switch for improving all system components of the system in Figure 1.8 are equal to the cold SREFs with imperfect switch for improving two components in the first subsystem and three components in the second subsystem (which we denote in Chapter 2 as $A_5^{(2,3)}$), where the SREFs in the last row of Table 4.7 are equal to the SREFs in the intersection of $A_5^{(2,3)}$ row with $D_5^{(2,3)}$ column in Table 2.3.

- The hot MREF and MTTF for improving components number 1 and 3 of the system in Figure 1.8 are equal to the hot MREF and MTTF respectively, for improving one component in the first subsystem and one component in the second subsystem (which we denote in Chapter 2 as $A_2^{(1,1)}$), where the the MREF in the seventh row of Table 4.4 is equal to the MREF in the intersection of $A_2^{(1,1)}$ row with $B_2^{(1,1)}$ column in Table 2.4 and the MTTF in the seventh row of Table 4.4 is equal to the MTTF in Table 2.7 (which we denote in Chapter 2 as $\{1_1, 1_2\}$).

- In the same manner, one can compare SREFs, MREFs and MTTFs of modified systems in this chapter with the analogous results in Chapter 2.

For the series-parallel system, we compare SREFs, MREFs and MTTFs which we derive using the survival signature with SREFs, MREFs and MTTFs which we studied in Chapter 3 and we find that:

- The hot SREFs for improving component number 1 of the system in Figure 4.1 are equal to the hot SREFs for improving one component in the first subsystem only (which we denote in Chapter 3 as $A_1^{(1,0)}$), where the SREFs in the first row of Table 4.11 are equal to the SREFs in the intersection of $A_1^{(1,0)}$ row with $B_1^{(1,0)}$ column in Table 3.1.

- The cold SREFs with perfect switch for improving component number 3 of the system in Figure 4.1 are equal to the cold SREFs with perfect switch for improving one component in the second subsystem only (which we denote in Chapter 3 as $A_1^{(0,1)}$), where the SREFs in the second row of Table 4.13 are equal to the SREFs in the intersection of $A_1^{(0,1)}$ row with $C_1^{(0,1)}$ column in Table 3.2.

- The cold SREFs with imperfect switch for improving all system components of the system in Figure 4.1 are equal to the cold SREFs with imperfect switch for improving two components in the first subsystem and three components in the second subsystem (which we denote in Chapter 3 as $A_5^{(2,3)}$), where the SREFs in the last row of Table 4.15 are equal to the SREFs in the intersection of $A_5^{(2,3)}$ row with $D_5^{(2,3)}$ column in Table 3.3.

- The cold MREF with perfect switch and the MTTF for improving components number 1, 2 and 3 of the system in Figure 4.1 are equal to the cold MREF with perfect switch and the MTTF for improving two components in the first subsystem and one component in the second subsystem respectively (which we denote in Chapter 3 as $A_3^{(2,1)}$), where the MREF in the sixth row of Table 4.14 is equal to the MREF in the intersection of $A_3^{(2,1)}$ row with $C_3^{(2,1)}$ column in Table 3.5 and the MTTF in the sixth row of Table 4.14 is equal to the MTTF in Table 3.7 (which we denote in Chapter 3 as $\{2_1, 1_2\}$).

- In the same manner, one can compare SREFs, MREFs and MTTFs of modified systems in this chapter with the analogous results in Chapter 3.

4.5 Conclusions

This chapter introduces a new technique for deriving the reliability equivalence factors for any system using the concept of survival signature. We present the conditions and steps for using the survival signature to derive the survival reliability equivalence factors (SREFs) and mean reliability equivalence factors (MREFs). The various steps for using survival signature to derive the reliability equivalence are elaborated. To clarify the impact of the newly proposed method, this chapter concludes with a comparison of the SREF and MREF results obtained by applying the new method on the parallel-series and series-parallel systems with the method previously studied. As hoped, the results are in perfect agreement, so leading support to the use survival signature for reliability equivalence factors.

Table 4.1: System topology and survival signature for different improvements to the system in Figure 1.8, when the number of improved components (gray) is $|H| \leq 2$.

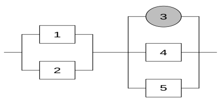
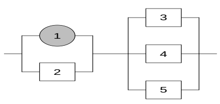
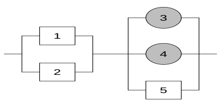
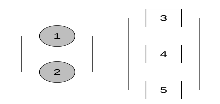
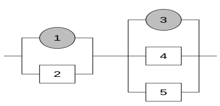
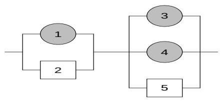
System topology	Survival signature						Analogous system in the previous study
	l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_1	$\Phi(l_1,l_2)$	$H_1^{(0,1)}$, $H = A, B, C, D$
	0	0	0	2	1	5/6	
	0	1	0	3	0	1	A Reduction
	1	0	0	3	1	1	B Hot duplication
	1	1	1/2	4	0	1	C Cold With perfect switch
	2	0	2/3	4	1	1	D Cold With imperfect switch
	l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_1	$\Phi(l_1,l_2)$	$H_1^{(1,0)}$, $H = A, B, C, D$
	0	0	0	2	1	1	
	0	1	0	3	0	3/4	
	1	0	0	3	1	1	
	1	1	3/4	4	0	1	
	2	0	1/2	4	1	1	
	l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_1	$\Phi(l_1,l_2)$	$H_2^{(0,2)}$, $H = A, B, C, D$
	0	0	0	2	0	2/3	
	0	1	0	2	1	1	
	0	2	0	2	2	1	
	1	0	0	3	0	1	
	1	1	2/3	3	1	1	
	1	2	2/3	3	2	1	
	l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_1	$\Phi(l_1,l_2)$	$H_2^{(2,0)}$, $H = A, B, C, D$
	0	0	0	2	0	0	
	0	1	0	2	1	1	
	0	2	0	2	2	1	
	1	0	0	3	0	0	
	1	1	1	3	1	1	
	1	2	1	3	2	1	
	l_1	l_2	$\Phi(l_1,l_2)$	l_1	l_1	$\Phi(l_1,l_2)$	$H_2^{(1,1)}$, $H = A, B, C, D$
	0	0	0	2	0	2/3	
	0	1	0	2	1	5/6	
	0	2	1	2	1	1	
	1	0	0	3	0	1	
	1	1	1/2	3	1	1	
	1	2	1	3	2	1	

Table 4.2: System topology and survival signature for different improvements to the system in Figure 1.8, when the number of improved components (gray) is $|H| > 2$.

<table>
<tr><th>System topology</th><th>Survival signature</th><th>Analogous system
in the previous study</th></tr>
<tr><td></td><td>
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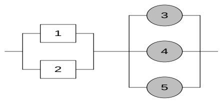
<tr><th>l_1</th><th>l_2</th><th>$\Phi(l_1, l_2)$</th><th>l_1</th><th>l_1</th><th>$\Phi(l_1, l_2)$</th></tr>
<tr><td>0</td><td>0</td><td>0</td><td>1</td><td>2</td><td>5/6</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>3</td><td>1</td></tr>
<tr><td>0</td><td>2</td><td>2/3</td><td>2</td><td>0</td><td>1</td></tr>
<tr><td>0</td><td>3</td><td>1</td><td>2</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>2</td><td>2</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>1/2</td><td>2</td><td>3</td><td>1</td></tr>
</table>
</td><td>$H_3^{(1,2)}$, $H = A, B, C, D$</td></tr>
<tr><td></td><td>
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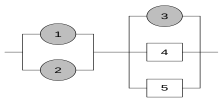
</td><td>$H_3^{(0,3)}$, $H = A, B, C, D$</td></tr>
<tr><td></td><td>
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<tr><td>0</td><td>0</td><td>0</td><td>1</td><td>2</td><td>1</td></tr>
<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>3</td><td>1</td></tr>
<tr><td>0</td><td>2</td><td>2/3</td><td>2</td><td>0</td><td>0</td></tr>
<tr><td>0</td><td>3</td><td>1</td><td>2</td><td>1</td><td>2/3</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>2</td><td>2</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>2/3</td><td>2</td><td>3</td><td>1</td></tr>
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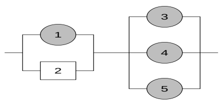
</td><td>$H_3^{(2,1)}$, $H = A, B, C, D$</td></tr>
<tr><td></td><td>
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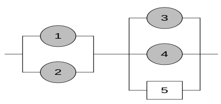
<tr><td></td><td>
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<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1/2</td></tr>
<tr><td>0</td><td>2</td><td>2/3</td><td>1</td><td>2</td><td>5/6</td></tr>
<tr><td>0</td><td>3</td><td>1</td><td>1</td><td>3</td><td>1</td></tr>
<tr><td>0</td><td>4</td><td>1</td><td>1</td><td>4</td><td>1</td></tr>
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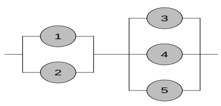
<tr><td></td><td>
<table>
<tr><th>l</th><th>$\Phi(l)$</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>2</td><td>6/10</td></tr>
<tr><td>3</td><td>9/10</td></tr>
<tr><td>4</td><td>1</td></tr>
<tr><td>5</td><td>1</td></tr>
</table>
</td><td>$H_5^{(2,3)}$, $H = A, B, C, D$</td></tr>
</table>

Table 4.3: Hot $SREF$ for system in Figure 1.8 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.6767	0.4828	0.2516
2	0.6767	0.4828	0.2516
3	0.6681	0.4590	0.1993
4	0.6681	0.4590	0.1993
5	0.6681	0.4590	0.1993
1,2	0.6927	0.5120	0.2863
1,3	0.6927	0.5120	0.2863
3,4	0.6803	0.4764	0.2056
1,2,3	0.7979	0.5423	0.3329
1,3,4	0.7040	0.5312	0.3045
1,2,3,4	0.7187	0.5622	0.3598
1,2,3,4,5	0.7268	0.5765	0.3761

Table 4.4: Hot $MREF$ and $MTTF$ for modified system in Figure 1.8 derived using survival signature

Component	MREF	MTTF
1	0.430	60.045
2	0.430	60.045
3	0.465	56.068
4	0.465	56.068
5	0.465	56.068
1,2	0.462	63.672
1,3	0.467	63.672
3,4	0.493	57.744
1,2,3	0.492	67.697
1,3,4	0.488	65.746
1,2,3,4	0.512	70.042
1,2,3,4,5	0.527	71.530

Table 4.5: Cold $SREF$ with perfect switch for system in Figure 1.8 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.2511	0.2783	0.2077
3	0.2577	0.2776	0.1391
1,2	0.2422	0.2777	0.2517
1,3	0.1832	0.2751	0.2558
3,4	0.2508	0.2777	0.1466
1,2,3	0.1512	0.2667	0.2713
1,3,4	0.1512	0.2691	0.2656
3,4,5	0.2472	0.2776	0.1557
1,2,3,4	0.1366	0.2474	0.2751
1,3,4,5	0.1419	0.2634	0.2680
1,2,3,4,5	0.1329	0.2278	0.2768

Table 4.6: Cold $MREF$ with perfect switch and $MTTF$ for modified system in Figure 1.8 derived using survival signature

Component	MREF	MTTF
1	0.253	63.516
3	0.257	58.005
1,2	0.255	68.017
1,3	0.227	75.421
3,4	0.258	60.250
1,2,3	0.217	84.259
1,3,4	0.217	82.003
3,4,5	0.256	61.312
1,2,3,4	0.205	93.639
1,3,4,5	0.209	85.862
1,2,3,4,5	0.196	99.362

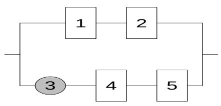
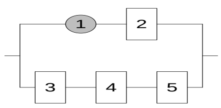
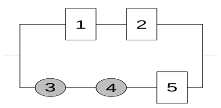
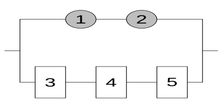
Table 4.7: Cold $SREF$ with imperfect switch for system in Figure 1.8 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.4308	0.4129	0.2395
3	0.4344	0.4021	0.1673
1,2	0.4220	0.4232	0.2932
1,3	0.4086	0.4255	0.2920
3,4	0.4299	0.4099	0.1737
1,2,3	0.3787	0.4345	0.3365
1,3,4	0.3819	0.4319	0.3129
3,4,5	0.4254	0.4136	0.1806
1,2,3,4	0.3250	0.4382	0.3551
1,3,4,5	0.3564	0.4352	0.3216
1,2,3,4,5	0.2828	0.4385	0.3661

Table 4.8: Cold $MREF$ with imperfect switch and $MTTF$ for modified system in Figure 1.8 derived using survival signature

Component	MREF	MTTF
1	0.359	61.316
3	0.376	58.816
1,2	0.372	65.572
1,3	0.366	67.425
3,4	0.385	58.887
1,2,3	0.369	73.401
1,3,4	0.367	71.216
3,4,5	0.392	60.102
1,2,3,4	0.366	78.488
1,3,4,5	0.364	73.728
1,2,3,4,5	0.362	82.000

Table 4.9: System topology and survival signature for different improvements to the system in Figure 4.1, when the number of improved components (gray) is $|H| \leq 2$.

System topology	Survival signature						Analogous system in the previous study
	l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_1	$\Phi(l_1, l_2)$	
	0	0	0	2	1	2/6	$H_1^{(0,1)}$, $H = A, B, C, D$
	0	1	0	3	0	1/2	A Reduction
	1	0	0	3	1	1	B Hot duplication
	1	1	0	4	0	1	C Cold With perfect switch
	2	0	1/6	4	1	1	D Cold With imperfect switch
	l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_1	$\Phi(l_1, l_2)$	
	0	0	0	2	1	3/6	$H_1^{(1,0)}$, $H = A, B, C, D$
	0	1	0	3	0	1/4	
	1	0	0	3	1	1	
	1	1	1/4	4	0	1	
	2	0	0	4	1	1	
	l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_1	$\Phi(l_1, l_2)$	
	0	0	0	2	0	1/3	$H_2^{(0,2)}$, $H = A, B, C, D$
	0	1	0	2	1	1/3	
	0	2	0	2	2	1	
	1	0	0	3	0	1	
	1	1	0	3	1	1	
	1	2	1/3	3	2	1	
	l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_1	$\Phi(l_1, l_2)$	
	0	0	0	2	0	0	$H_2^{(2,0)}$, $H = A, B, C, D$
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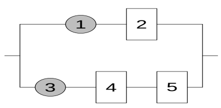
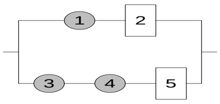
	0	2	1	2	2	1	
	1	0	0	3	0	1	
	1	1	0	3	1	1	
	1	2	1	3	2	1	
	l_1	l_2	$\Phi(l_1, l_2)$	l_1	l_1	$\Phi(l_1, l_2)$	
	0	0	0	2	0	0	$H_2^{(1,1)}$, $H = A, B, C, D$
	0	1	0	2	1	3/6	
	0	2	0	2	1	1	
	1	0	0	3	0	0	
	1	1	1/6	3	1	1	
	1	2	1/3	3	2	1	

Table 4.10: System topology and survival signature for different improvements to the system in Figure 4.1, when the number of improved components (gray) is $|H| > 2$.

<table>
<tr><th>System topology</th><th>Survival signature</th><th>Analogous system in the previous study</th></tr>
<tr><td>
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<table>
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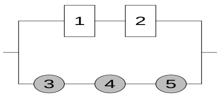
<tr><td>
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<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>3</td><td>1</td></tr>
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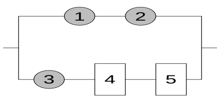
<tr><td>
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<table>
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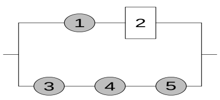
<tr><td>0</td><td>3</td><td>1</td><td>2</td><td>1</td><td>2/3</td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>2</td><td>2</td><td>1</td></tr>
<tr><td>1</td><td>1</td><td>2/3</td><td>2</td><td>3</td><td>1</td></tr>
</table></td><td>$H_3^{(2,1)}$, $H = A, B, C, D$</td></tr>
<tr><td>
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<tr><td>0</td><td>1</td><td>0</td><td>1</td><td>1</td><td>1/4</td></tr>
<tr><td>0</td><td>2</td><td>0</td><td>1</td><td>2</td><td>3/6</td></tr>
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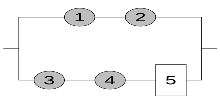
<tr><td>
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<tr><th>l_1</th><th>l_2</th><th>$\Phi(l_1,l_2)$</th><th>l_1</th><th>l_1</th><th>$\Phi(l_1,l_2)$</th></tr>
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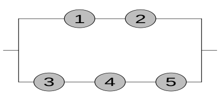
<tr><td>
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<tr><th>l</th><th>$\Phi(l)$</th></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td>1</td><td>0</td></tr>
<tr><td>2</td><td>1/10</td></tr>
<tr><td>3</td><td>4/10</td></tr>
<tr><td>4</td><td>1</td></tr>
<tr><td>5</td><td>1</td></tr>
</table></td><td>$H_5^{(2,3)}$, $H = A, B, C, D$</td></tr>
</table>

Table 4.11: Hot $SREF$ for system in Figure 4.1 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.7403	0.6173	0.4671
3	0.7238	0.6009	0.4519
1,2	0.7647	0.6580	0.5295
1,3	0.7438	0.6279	0.4839
3,4	0.7330	0.6203	0.4800
1,2,3	0.7661	0.6635	0.5399
1,3,4	0.7491	0.6419	0.5078
3,4,5	0.7445	0.6444	0.5234
1,2,3,4	0.7685	0.6715	0.5548
1,3,4,5	0.7556	0.6600	0.5433
1,2,3,4,5	0.7723	0.6826	0.5771

Table 4.12: Hot $MREF$ and $MTTF$ for modified system in Figure 4.1 derived using survival signature

Component	MREF	MTTF
1	0.657	1.242
3	0.614	1.202
1,2	0.677	1.346
1,3	0.653	1.266
3,4	0.634	1.244
1,2,3	0.679	1.360
1,3,4	0.655	1.299
3,4,5	0.651	1.305
1,2,3,4	0.682	1.381
1,3,4,5	0.664	1.347
1,2,3, 4,5	0.687	1.413

Table 4.13: Cold $SREF$ with perfect switch for system in Figure 4.1 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.1756	0.1194	0.0622
3	0.1409	0.1208	0.0774
1,2	0.2656	0.2170	0.1622
1,3	0.1766	0.1288	0.1000
3,4	0.1749	0.1207	0.0917
1,2,3	0.2656	0.2171	0.1626
1,3,4	0.1904	0.1396	0.1000
3,4,5	0.2476	0.2036	0.1534
1,2,3,4	0.2655	0.2176	0.1657
1,3,4,5	0.2479	0.2049	0.1573
1,2,3,4,5	0.2701	0.2311	0.1914

Table 4.14: Cold $MREF$ with perfect switch and $MTTF$ for modified system in Figure 4.1 derived using survival signature

Component	MREF	MTTF
1	0.163	1.347
3	0.134	1.230
1,2	0.222	2.255
1,3	0.162	1.387
3,4	0.162	1.381
1,2,3	0.223	2.257
1,3,4	0.167	1.499
3,4,5	0.208	2.104
1,2,3,4	0.225	2.266
1,3,4,5	0.211	2.120
1,2,3,4,5	0.234	2.420

Table 4.15: Cold $SREF$ with imperfect switch for system in Figure 4.1 derived using survival signature.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.2149	0.2269	0.2439
3	0.2157	0.2401	0.2494
1,2	0.2740	0.2330	0.2168
1,3	0.2162	0.2221	0.2502
3,4	0.2153	0.2246	0.2460
1,2,3	0.2740	0.2331	0.2161
1,3,4	0.2217	0.2159	0.2410
3,4,5	0.2578	0.2241	0.2206
1,2,3,4	0.2741	0.2335	0.2147
1,3,4,5	0.2582	0.2250	0.2173
1,2,3,4,5	0.2781	0.2440	0.2177

Table 4.16: Cold $MREF$ with imperfect switch and $MTTF$ for modified system in Figure 4.1 derived using survival signature

Component	MREF	MTTF
1	0.224	1.338
3	0.226	1.228
1,2	0.245	2.150
1,3	0.224	1.377
3,4	0.223	1.366
1,2,3	0.245	2.155
1,3,4	0.223	1.481
3,4,5	0.236	1.984
1,2,3,4	0.246	2.173
1,3,4,5	0.236	2.013
1,2,3,4,5	0.250	2.343

Chapter 5

Reliability equivalence factors for complex systems and networks using the survival signature

In this study, we present a new methodology to derive the reliability equivalence factors for any coherent system with any structure and with any lifetime distributions. We use the concept of survival signature and the `ReliabilityTheory` R package to derive reliability equivalence factors for complex systems with independent components. Using the `ReliabilityTheory` package, we derive reliability functions and the mean times to failure for systems improved according to (a) reduction method; (b) duplication methods: (i) hot duplication; (ii) cold duplication with perfect switch; (iii) cold duplication with imperfect switch. For consistency with our preceding analyses, two measures for comparing system improvements are considered in this study, survival reliability equivalence factors and mean reliability equivalence factors. Numerical examples for complex systems and networks are presented, to explain the new reliability equivalence factors technique and to illustrate how to apply the theoretical results and demonstrate the relative benefits of various system improvements.

5.1 Introduction

Improving a system's design can be preformed using a redundancy duplication method, which involves adding extra components in parallel to existing system components. As discussed earlier, there are three ways to add extra components to the system: hot duplication; cold duplication with perfect switch; cold duplication with imperfect switch. Sometimes, and for many different reasons such as high cost and space limitation, it is impossible to improve the reliability of the system by the redundancy duplication method. For example a satellite system's design has expensive units and limited space. These constraints can be overcome using a reduction method, which involves improving the reliability of the system by reducing the failure rate by a factor ρ for some of the system components, where $\rho \in (0, 1)$.

In this study, we extend our previous analyses of simple systems by considering a complex system and a network with different structures and with multiple types of components. First, we compute the reliability function (RF) and the mean time to failure (MTTF) of the original system using the `ReliabilityTheory` R package of Aslett (2012). Second, using the same package we compute the RFs and MTTFs of the systems following improvement according to reduction, hot duplication and cold duplication (perfect and imperfect) methods. Third, we separately equate the RF and MTTF of the system improved according to the reduction method with the RF and MTTF respectively of the system improved according to each of the duplication methods, in order to determine the corresponding reliability equivalence factors. Finally, we illustrate the results obtained with an application example by presenting summary tables and figures.

5.2 Survival signature

As mentioned earlier, Coolen and Coolen-Maturi (2012) introduced the concept of survival signature. They studied survival signature for a system with identical components and systems with multiple types of components. They defined the survival signature as the probability that a system functions given that a specified number of its components function.

For any coherent system with m independent and identically distributed components with continuous lifetime distribution. let $\Phi(l)$ for $l = 0, 1, ..., m$ be the probability that the system functions given that precisely l of its components function. The system will not function when all system components fail, which means $\Phi(0) = 0$ and the system should function when all system components function, which means $\Phi(m) = 1$. There are $\binom{m}{l}$ state vectors $\underline{x}$ in which precisely l components function (l components with state $x_i = 1$), so $\sum_{i=1}^{m} x_i = l$; we will denote the set of these vectors by X_l. The system survival signature $\Phi(l)$ can be written as:

$$\Phi(l) = \binom{m}{l}^{-1} \sum_{\underline{x} \in X_l} \phi(\underline{x}) \tag{5.2.1}$$

For a system with multiple types of component these authors considered a coherent system with m independent components classified into n types of components where type i has m_i identical components for $i = 1, 2, .., n$. Let $\Phi(l_1, l_2, ..., l_n)$, for $l_i = 0, 1, ..., m_i$, be the probability that a system functions given that precisely l_i of its components of type i function, for $i = 1, 2, .., n$. There are $\binom{m_i}{l_i}$ state vectors $\underline{x}^i$ where precisely l_i components of type i function (l_i of its m_i components have the state $x_j^i = 1$), so $\sum_{j=1}^{m_i} x_j^i = l_i$. Let $X_{l_1,...,l_n}$ be the set of all state vectors for the whole system for which $\sum_{j=1}^{m_i} x_j^i = l_i$, $i = 1, 2, ..., n$. Then the survival signature of such a system is

$$\Phi(l_1, l_2, ..., l_n) = \left[\prod_{i=1}^{n} \binom{m_i}{l_i}^{-1}\right] \times \sum_{\underline{x} \in X_{l_1,...,l_n}} \phi(\underline{x}) \tag{5.2.2}$$

5.3 Original system

Consider any coherent system with m independent components classified into n different types where type i consists of m_i identical components for $i = 1, 2, .., n$. The total number of system components is $\sum_{i=1}^{n} m_i = m$. The survival signature of the system $\Phi(l_1, l_2, ..., l_n)$, for $l_i = 0, 1, ..., m_i$, is defined as the probability that a system functions given that precisely l_i of its components of type i function at time t. If the lifetime distribution of component j $(j = 1, 2, ..., m_i)$ of type i $(i = 1, 2, ..., n)$ are known and has the reliability function $R_i(t)$ then according to Coolen and Coolen-Maturi (2012) and Aslett et al. (2014) the reliability function for this system can be written as

$$R(t) = \sum_{l_1=0}^{m_1} ... \sum_{l_n=0}^{m_n} \left[\Phi(l_1, ..., l_n) \prod_{i=1}^{n} \left\{ \binom{m_i}{l_i} [1 - R_i(t)]^{m_i - l_i} [R_i(t)]^{l_i} \right\}\right]. \tag{5.3.1}$$

We can then compute the mean time to failure of this system as

$$\begin{aligned} MTTF &= \int_0^{\infty} R(t)dt \\ &= \int_0^{\infty} \left\{ \sum_{l_1=0}^{m_1} ... \sum_{l_n=0}^{m_n} \left[\Phi(l_1, ..., l_n) \prod_{i=1}^{n} \left\{ \binom{m_i}{l_i} [1 - R_i(t)]^{m_i - l_i} [R_i(t)]^{l_i} \right\}\right]\right\} dt. \end{aligned} \tag{5.3.2}$$

5.4 Designs of improved systems

The two main approaches for improving a system are reduction and standby redundancy (duplication). The latter comprises two variations, hot duplication and cold duplication. Furthermore, cold duplication can be performed with perfect switch or imperfect switch. In this section, we derive the reliability function and the mean time to failure for a complex system and a network improved according to the methods identified above.

5.4.1 Reduction method

As mentioned in the introduction, the reliability of a system can be improved by scaling the hazard function for some of the system's components by a factor $\rho \in (0,1)$. In order to improve the original system by improving one or more of its components according to the reduction method, we need to know the reduction improvement strategies for this type of component. The reduction improvement strategies for most common lifetime distributions have been discussed in depth within previous studies in this field. Reduction improvement strategies for components with exponential lifetime distributions were presented in many papers including Råde (1993a,b); Sarhan (2000, 2002, 2004, 2005, 2009b); Sarhan and Mustafa (2006); Sarhan et al. (2008a). Reduction improvement strategies were also presented for components with non-constant failure rate lifetime distributions, including the gamma distribution in Xia and Zhang (2007), Wiebull distribution in El-Damcese (2009), exponentiated exponential distribution in Abdelkader et al. (2013), exponentiated Weibull distribution in Alghamdi and Percy (2014, 2015), and Burr type X distribution in Migdadi and Al-Batah (2014). The remarkable point here is that all aforementioned papers and all previous studies in this field assume a system with specific structure and most of them assume a system with identically distributed components. By adopting a generic framework, we overcome both of these constraints in this study.

5.4.1.1 System component improvement

For any coherent system with m independent components divided into n different types, we can derive the reliability equivalence factors for component j ($j = 1, 2, ..., m_i$) of type i ($i = 1, 2, ..., n$) if the reduction improvement strategy for this component is specified and known. Define $R_i^A(t)$ as the reliability function of component j when it improves according to this reduction method. By improving k_i components, for $k_i \in \{1, .., m_i - 1\}$ according to the reduction method, the number of system component types of the improved system becomes $n + 1$. The improved system has all the properties of the original system except that the number of system types is now $n+1$ instead of n, and the number of components of type i is now $m_i - k_i$ instead of m_i. The new type of improved system has k_i components with reliability function $R_i^A(t)$.

The reliability function and the mean time to failure of the system improved according to the reduction method can then be derived by applying the properties of the improved system given by Equations (5.3.1) and (5.3.2) respectively.

5.4.1.2 System type improvement

When we improve all components of type i according to the reduction method, the improved system has all the properties of the original system except that we replace the reliability function $R_i(t)$ of type i with $R_i^A(t)$. In the case of improving more than one type of system component or improving components from different types, we repeat those steps for each improvement.

The reliability function and the mean time to failure of the system improved according to the reduction method can then be derived by applying the properties of the improved system given by Equations (5.3.1) and (5.3.2) respectively.

5.4.2 Duplication methods

Now we obtain the corresponding reliability measures of the system when it is improved by duplication. We derive the reliability function and the mean time to failure of a complex system or network improved according to the hot duplication method and the cold duplication methods with perfect and imperfect switches.

5.4.2.1 Hot duplication

This means that some of the system components are duplicated in parallel by similar components. We assume that in the hot duplication method each component is augmented by introducing a new but identical component.

For system component improvement, we again consider any coherent system with m independent components divided into n different types. If the lifetime distribution of component j $(j = 1, 2, ..., m_i)$ of type i $(i = 1, 2, ..., n)$ is known and has the reliability function $R_i(t)$ then the reliability function of the component improved according to hot duplication takes the form

$$R_i^B(t) = 1 - [1 - R_i(t)]^2. \tag{5.4.1}$$

By improving k_i components, for $k_i \in \{1, .., m_i - 1\}$ according to the hot duplication method, the number of component types in the improved system becomes $n+1$. The improved system has all the properties of the original system except that the number of system types is now $n+1$ instead of n, and the number of components of type i is now $m_i - k_i$ instead of m_i. The new type of improved system has k_i components with reliability function $R_i^B(t)$.

For system type improvement, we consider what would happen if we were to improve all components of type i according to the hot duplication method. The improved system has all the properties of the original system except that we replace the reliability function $R_i(t)$ of type i with $R_i^B(t)$. In the case of improving more than one type of system component or improving components from different types, we repeat those steps for each improvement.

The reliability function and the mean time to failure of the system improved according to the hot duplication method can then be derived by applying the properties of the improved system given by Equations (5.3.1) and (5.3.2) respectively.

5.4.2.2 Cold duplication with perfect switch

This approach to improving system reliability means that a similar component is connected with an original component in such a way that it is activated immediately upon failure of the original component.

For system component improvement, again consider any coherent system with m independent components divided into n different types. If the lifetime distribution of component j $(j = 1, 2, ..., m_i)$ of type i $(i = 1, 2, ..., n)$ is known and has the reliability function $R_i(t)$ then regarding a definition of cold duplication with perfect switch, we can describe this improvement as a renewal process with only one renewal, Alghamdi and Percy (2014). Using the convolution technique, the reliability function of the component whose reliability is improved according to cold duplication with perfect switch can be derived as

$$R_i^C(t) = 1 - \int_0^t \frac{-dR_i(x)}{dx}[1 - R_i(t-x)]dx. \tag{5.4.2}$$

By improving k_i components, for $k_i \in \{1, .., m_i - 1\}$ according to cold duplication with perfect switch, the number of types of component in the improved system again becomes $n + 1$. The improved system has all the properties of the original system except that the number of component types is now $n+1$ instead of n, and the number of components of type i is now $m_i - k_i$ instead of m_i. The new type of improved system has k_i components with the reliability function $R_i^C(t)$.

For system type improvement, we consider what would happen if we were to improve all components of type i according to the cold duplication method with perfect switch. The improved system has all the properties of the original system except that we replace the reliability function $R_i(t)$ of type i with $R_i^C(t)$. In the case of improving more than one type of system component or improving components from different types, we repeat those steps for each improvement.

The reliability function and the mean time to failure of the system improved according to the cold duplication method with perfect switch by improving some of its components or some system types can be derived by applying the properties of the improved system given by Equations (5.3.1) and (5.3.2) respectively.

5.4.2.3 Cold duplication with imperfect switch

This approach to improving system reliability means that a similar component is connected with an original component by a cold standby via a random switch having a constant failure rate. For this aspect of our analysis, the cold duplication method assumes that each component is improved by introducing a new but identical component with an imperfect switch.

For system component improvement, again consider any coherent system with m independent components divided into n different types. If the lifetime distribution of component j ($j = 1, 2, ..., m_i$) of type i ($i = 1, 2, ..., n$) is known and has the reliability function $R_i(t)$ then following the same technique that we used for cold duplication with perfect switch but with the extra condition that the switch is not 100% reliable Billinton and Allan (1992), the reliability function of the component whose reliability is improved according to cold duplication with imperfect switch can be derived as

$$R_i^D(t) = 1 - \int_0^t \frac{-dR_i(x)}{dx}[1 - R_i(t-x)s(x)]dx \tag{5.4.3}$$

where $s(x)$ is the reliability function for the imperfect switch. The imperfect switch is chosen to have a constant failure rate ν, which means that it has an exponential lifetime distribution with parameter ν and reliability function

$$s(t) = e^{-\nu t}. \tag{5.4.4}$$

By improving k_i components, for $k_i \in \{1, .., m_i - 1\}$ according to the cold duplication method with imperfect switch, the number of component types in the improved system becomes $n + 1$. The improved system has all the properties of the original system except that the number of component types is now $n + 1$ instead of n, and the number of components of type i is now $m_i - k_i$ instead of m_i. The new type of component in the improved system has k_i components with reliability function $R_i^D(t)$.

For system type improvement, consider what happens if we improve all components of type i according to the cold duplication method with imperfect switch. The improved system has all the properties of the original system except that we replace the reliability function $R_i(t)$ of type i with $R_i^D(t)$. In the case of improving more than one type of system component or improving components from different types, we repeat those steps for each improvement.

The reliability function and the mean time to failure of the system improved according to the cold duplication method with imperfect switch by improving some of its components or some of its types can be derived by applying the properties of the improved system given by Equations (5.3.1) and (5.3.2) respectively.

5.5 Reliability equivalence factors

We compute two measures of reliability equivalence as in previous chapters. The first involves survival reliability equivalence factors (SREFs) and these are determined from the reliability function. The second involves mean reliability equivalence factors (MREFs) and these are determined from the mean time to failure.

5.5.1 Survival reliability equivalence factors

To derive the SREFs, we have to solve the following set of equations

$$R_r(t) = R_d(t) = \omega \tag{5.5.1}$$

where $R_r(t)$ is the reliability function of the system improved according to the reduction method and $R_d(t)$ is the reliability function of the system improved according to one of the duplication methods.

5.5.2 Mean reliability equivalence factors

To derive the MREFs, we have to solve the following set of equations

$$MTTF_r = MTTF_d \tag{5.5.2}$$

where $MTTF_r$ is the mean time to failure of the system improved according to the reduction method and $MTTF_d$ is the mean time to failure of the system improved according to one of the duplication methods.

5.6 Numerical results and analysis

To illustrate how to apply the preceding theory, suppose that we have a coherent system with 11 independent components divided into 4 types where the links between system components are 100% reliable. The system that we consider here is shown in Figure 5.1, as this particular system structure and system survival signature were presented and discussed by Aslett et al. (2014). The properties of this system that we analyse now are presented in Table 5.1.

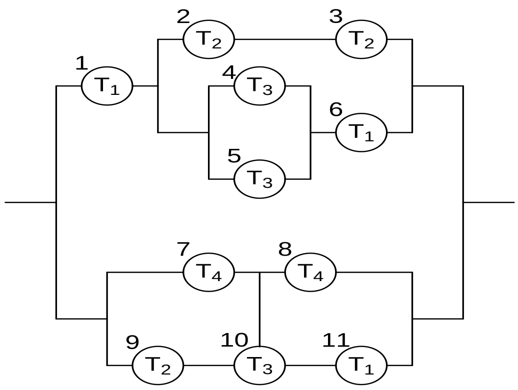
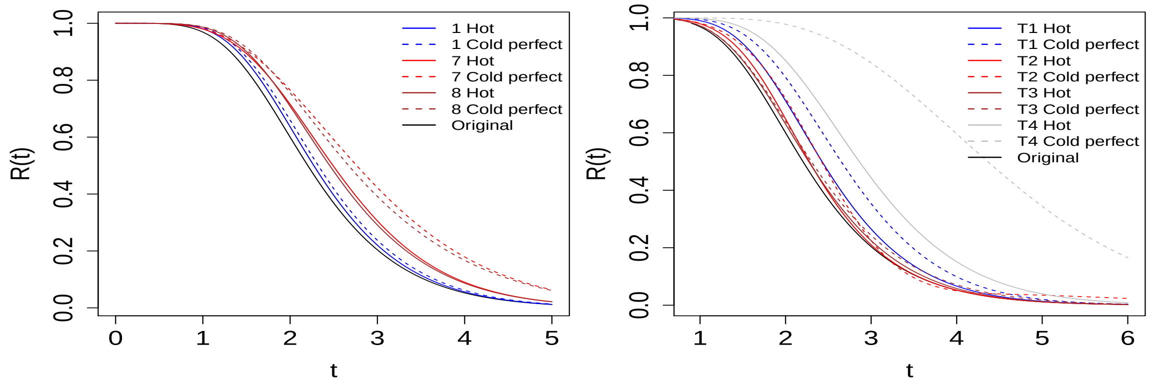


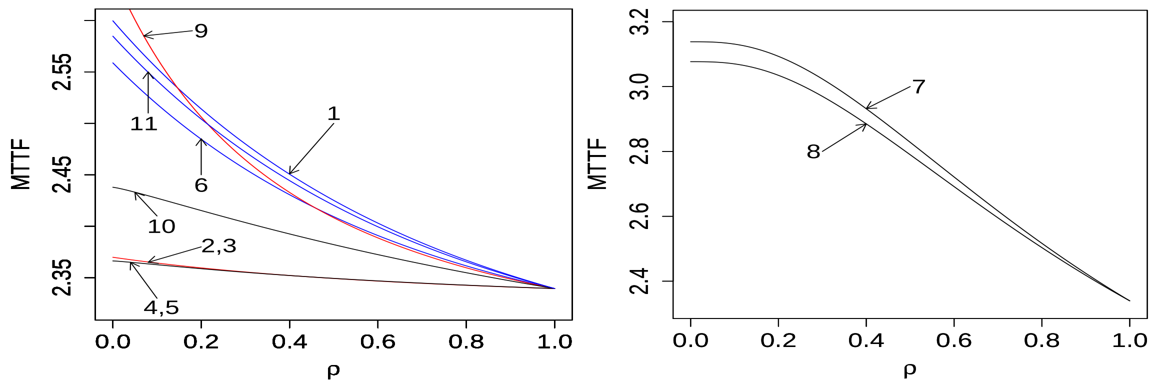
Figure 5.1: System with 11 components divided into 4 different types. Component type is inside the circle while component number is above to the left.

For this scenario, the SREFs for the system components for hot and cold (perfect and imperfect) duplication are calculated using the `ReliabilityTheory` R package



(a) Best three components improvement. (b) System types improvement.

Figure 5.2: Reliability function of the original and some modified systems for the system in Figure 5.1.



(a) (b)

Figure 5.3: The behaviour of MTTF against ρ, for the components of the system in Figure 5.1

Table 5.1: Properties of the complex system in Figure 5.1.

Types of system components	System component lifetimes	Reduction improvement strategy
$T_1 = \{1, 6, 11\}$	$T_1 \sim$ Exponential$(\lambda = 0.55)$	$R_1^{(A)}(t) = e^{-\rho\lambda t}$, see Sarhan (2000, 2002)
$T_2 = \{2, 3, 9\}$	$T_2 \sim$ Weibull$(\alpha = 0.274, \beta = 2.2)$	$R_2^{(A)}(t) = e^{-\rho\alpha t^{\beta}}$, see El-Damcese (2009)
$T_3 = \{4, 5, 10\}$	$T_3 \sim$ Exponentiated Weibull $(\alpha = 0.111, \beta = 2, \theta = 1.2)$	$R_3^{(A)}(t) = 1 - (1 - e^{-\rho\alpha t^{\beta}})^{\theta}$, see Alghamdi and Percy (2014, 2015)
$T_4 = \{7, 8\}$	$T_4 \sim$ Gamma$(n = 3.2, \lambda = 1.111)$	$R_4^{(A)}(t)=\int_t^{\infty} \frac{(\rho\lambda)^n t^{n-1}}{\Gamma n} e^{-\rho\lambda t} dt$, see Xia and Zhang (2007)

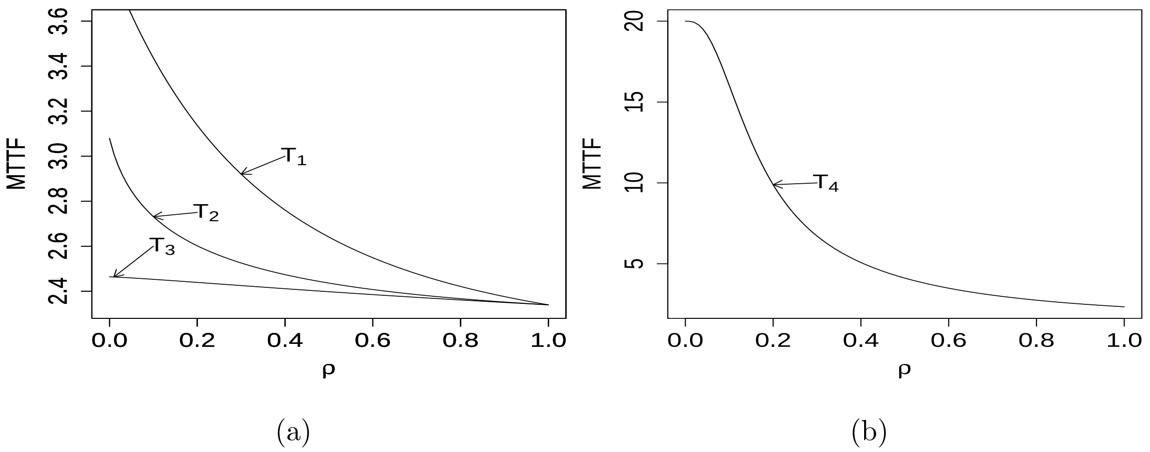


Figure 5.4: The behaviour of MTTF against ρ, for the component types for the system in Figure 5.1

according to the above formulae; see Appendices C.1 and C.2. The results are presented in Tables 5.2, 5.6 and 5.10 where ω is chosen to be $0.1, 0.5, 0.9$ and the imperfect switch has a constant failure rate $\nu = 0.05$. The same measures for the SREFs for component types are presented in Tables 5.4, 5.8 and 5.12. For more discussions based on the results presented in the those tables, it may be observed that:

- Reducing the failure rate of component number 1 by setting $\rho = 0.6858$ improves the reliability of the system like adding an extra component in parallel to component 1 according to a hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$, see Table 5.2.

- Reducing the failure rate of component 7 by setting $\rho = 0.3327$ improves the reliability of the system like adding an extra component in parallel to component 7 according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 5.6.

Table 5.2: Hot $SREF$ for components of the system in Figure 5.1.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.6858	0.5690	0.4419
2	0.8504	0.6289	0.3439
3	0.8504	0.6289	0.3439
4	0.6003	0.3552	0.1615
5	0.6003	0.3552	0.1615
6	0.6858	0.5680	0.4333
7	0.7675	0.6458	0.4936
8	0.7655	0.6414	0.4989
9	0.8455	0.6407	0.3634
10	0.6041	0.3656	0.1690
11	0.6855	0.5691	0.4394

Table 5.3: Hot $MREF$ and $MTTF$ for components of the system in Figure 5.1.

Component	MREF	MTTF
1	0.5726	2.4088
2	0.5427	2.3482
3	0.5427	2.3482
4	0.5431	2.3483
5	0.5431	2.3483
6	0.5942	2.3920
7	0.6985	2.6179
8	0.6956	2.5993
9	0.6176	2.3856
10	0.4459	2.3877
11	0.5748	2.4045

Table 5.4: Hot $SREF$ for component types of the system in Figure 5.1.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.6941	0.5863	0.4659
T_2	0.8456	0.6453	0.3795
T_3	0.6086	0.3679	0.1693
T_4	0.7907	0.6848	0.5571

Table 5.5: Hot $MREF$ and $MTTF$ for component types of the system in Figure 5.1.

Type	MREF	MTTF
T_1	0.5982	2.5502
T_2	0.6076	2.4054
T_3	0.4762	2.4009
T_4	0.7182	2.9942

- Reducing the failure rate of the component 11 by setting factor $\rho = 0.2803$ improves the reliability of the system like adding an extra component to component 11 according to a cold duplication method with imperfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 5.10.

- Reducing the failure rate of each component belonging to the first type of system component T_1 by setting $\rho = 0.3830$ improves the reliability of the system like adding extra component in parallel to each component in type T_1 according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 5.8.

- Reducing the failure rate of each component belonging to the the fourth type of system component T_4 by setting $\rho = 0.3891$ improves the reliability of the

Table 5.6: Cold $SREF$ with perfect switch for components of the system in Figure 5.1.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.4521	0.3556	0.2637
2	0.3924	0.2693	0.1061
3	0.3924	0.2693	0.1061
4	0.1429	0.0685	0.0271
5	0.1429	0.0685	0.0271
6	0.4521	0.3544	0.2550
7	0.4361	0.3327	0.2261
8	0.4339	0.3261	0.2295
9	0.4021	0.2998	0.1193
10	0.1468	0.0723	0.0287
11	0.4516	0.3558	0.2612

Table 5.7: Cold $MREF$ with perfect switch and $MTTF$ for components of the system in Figure 5.1.

Component	MREF	MTTF
1	0.3664	2.4600
2	0.2646	2.3569
3	0.2646	2.3569
4	0.1543	2.3605
5	0.1543	2.3605
6	0.3817	2.4349
7	0.3939	2.9376
8	0.3951	2.8895
9	0.4744	2.4140
10	0.1182	2.4251
11	0.3670	2.4531

Table 5.8: Cold $SREF$ with perfect switch for component types of the system in Figure 5.1.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.4709	0.3830	0.2928
T_2	0.6147	0.3164	0.1325
T_3	0.1501	0.0730	0.0288
T_4	0.5128	0.4260	0.3276

Table 5.9: Cold $MREF$ with perfect switch and $MTTF$ for component types of the system in Figure 5.1.

Type	MREF	MTTF
T_1	0.3966	2.7653
T_2	0.2414	2.5660
T_3	0.1270	2.4491
T_4	0.4508	4.5310

system like adding an extra component in parallel to each component in type T_4 according to a cold duplication method with imperfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 5.12.

- In the same manner, one can interpret the other results presented in those tables.

Tables 5.3, 5.7 and 5.11 present the MREFs and MTTFs for the system components, for hot and cold (perfect and imperfect) duplication. The MREFs for component types are presented in Tables 5.5, 5.9 and 5.13. Based on the results presented in those tables, we see that:

- Improving component number 1 according to hot duplication increases the system mean time to failure to be 2.4088 and the same system mean time to failure

Table 5.10: Cold $SREF$ with imperfect switch for components of the system in Figure 5.1.

Component	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
1	0.4810	0.3800	0.2828
2	0.4361	0.3455	0.1671
3	0.4361	0.3455	0.1671
4	0.2085	0.1390	0.0866
5	0.2085	0.1390	0.0866
6	0.4810	0.3787	0.2740
7	0.4845	0.4071	0.3427
8	0.4826	0.4131	0.3446
9	0.4415	0.3732	0.1805
10	0.2115	0.1428	0.0889
11	0.4805	0.3801	0.2803

Table 5.11: Cold $MREF$ with imperfect switch and $MTTF$ for components of the system in Figure 5.1.

Component	MREF	MTTF
1	0.3904	2.4532
2	0.3364	2.3543
3	0.3364	2.3543
4	0.2124	2.3583
5	0.2124	2.3583
6	0.4067	2.4291
7	0.4507	2.8800
8	0.4514	2.8369
9	0.5896	2.3905
10	0.1795	2.4177
11	0.3912	2.4467

Table 5.12: Cold $SREF$ with imperfect switch for component types of the system in Figure 5.1.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.4987	0.4066	0.3118
T_2	0.7301	0.3874	0.1930
T_3	0.2142	0.1435	0.0890
T_4	0.5430	0.4658	0.3891

Table 5.13: Cold $MREF$ with imperfect switch and $MTTF$ for component types of the system in Figure 5.1.

Type	MREF	MTTF
T_1	0.4207	2.7330
T_2	0.2811	2.5374
T_3	0.1880	2.4407
T_4	0.4919	4.1762

can be obtained by reducing the failure rate of the same component by setting $\rho = 0.5726$, see Table 5.3. Note that the mean time to failure of the original system is 2.3395.

- Improving component 8 according to cold duplication with perfect switch increases the system mean time to failure to be 2.8895 and the same mean time to failure can be obtained by reducing the failure rate of component 8 by setting $\rho = 0.3951$, see Table 5.7.

- Improving each component belonging to the first type T_1 according to hot duplication increases the system mean time to failure to be 2.5502 and the same mean time to failure can be obtained by reducing the failure rate of each component in type T_1 by setting $\rho = 0.5982$, see Table 5.5.

- Improving the reliability of component 7 gives the best possible component improvement, see Tables 5.3, 5.7 and 5.11.

- Improving the reliability of component 8 gives the second best possible component improvement, see Tables 5.3, 5.7 and 5.11.

- Improving either component 2 or component 3 gives the same result as for component 4 or component 5, see Tables 5.3, 5.7 and 5.11.

- Improving the reliability of each component belonging to the fourth type T_4 gives the best possible type of improvement, see Tables 5.5, 5.9 and 5.13.

- Improving the reliability of each component belonging to the first type T_1 gives the second best possible type of improvement, see Tables 5.5, 5.9 and 5.13.

- In a similar manner, one can interpret the other results presented in those tables.

Figure 5.2 presents reliability functions of the original and some modified systems. From this figure, one may observe that, for this scenario:

- Component 7 is the best component to be improved by either hot or cold duplication, see Figure 5.2a.

- Component 8 is the second best component that can be improved by either hot or cold duplication, then component number 1, see Figure 5.2a and compare with Tables 5.3, 5.7 and 5.9.

- The best type of component that can be improved is type T_4, then type T_1 by either hot or cold duplication methods, see Figure 5.2b.

Figures 5.3 and 5.4 present the behaviour of MTTF against the appropriate reduction factor ρ for system components and system types. It seems from these two figures that:

- MTTFs are non-decreasing with decreasing ρ for all possible reduction improvement.
- Reducing the failure rate of component 7 gives the best possible reduction component improvement, see Figure 5.3b.
- Reducing the failure rate of component 8 gives the second best possible reduction component improvement, see Figure 5.3b.
- Reducing the failure rate of component 1 gives the third best possible reduction component improvement for all $\rho \geq 0.2$, see Figure 5.3a.
- Reducing the failure rate of any component 2 or component 3 gives the same improvement and same is true for component 4 or component 5, see Figures 5.3a and compare with Tables 5.3, 5.7 and 5.11.
- Reducing the failure rate of type T_4 gives the best possible reduction type of improvement. It gives a huge improvement for the mean time to failure of the system, see Figure 5.4b.
- Reducing the failure rate of type T_1 gives the second best possible reduction type improvement, see Figure 5.4a.
- Reducing the failure rate of type T_2 gives the third best possible reduction type improvement, see Figure 5.4a.

- Reducing the failure rate of type T_3 gives the worst reduction type improvement, see Figure 5.4a.

5.7 Reliability equivalence factors for networks

In all our previous studies we consider systems with unreliable components and reliable links. In this approach we consider a coherent system with unreliable components and unreliable links. These types of structure are generally known as networks. Studying the reliability of networks is very important nowadays because they have many applications in different fields related to digital communication. In this approach we provide a new technique to derive reliability equivalence factors for networks using the survival signature. This technique allows us to see how improving links between system components affects the reliability of the system. Using the same steps that we used to derive the reliability equivalence factors for a complex system, we can derive the reliability equivalence factors for networks. We treat links between components as independent components with individual lifetime distributions. To illustrate how to derive the reliability equivalence factors for networks we present a simple yet representative example.

The network system we consider here is shown in Figure 5.5 and consists of 4 components and 7 links. The system structure and system survival signature were presented by Aslett et al. (2014). We assume that all system links are independent and identically distributed with an exponential lifetime distribution, and we put them in one set $T_1 = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$. Also we assume that components 1 and 4 are independent and identically distributed with a Weibull lifetime distribution and we put them in the second set $T_2 = \{1, 4\}$. We put component 2 in the third set

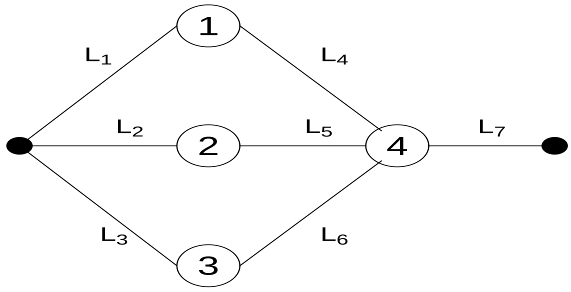


Figure 5.5: Network of 4 components and 7 links.

$T_3 = \{2\}$ and finally we put component 3 in the fourth set $T_4 = \{3\}$. All these properties of the network system are presented in Table 5.14.

For this scenario, in Tables 5.15, 5.17 and 5.19 the SREFs for hot and cold (perfect and imperfect) duplication are calculated using the `ReliabilityTheory` R package according to the above formulae where ω is chosen to be $0.1, 0.5, 0.9$ and the imperfect switch has a constant failure rate $\nu = 0.05$. For more discussions based on the results presented in the Tables 5.15, 5.17 and 5.19, it may be observed that:

- Reducing the failure rate of each link in the system (set T_1) by setting $\rho = 0.5172$ improves the reliability of the system like adding an extra link in parallel to each link according to the hot duplication method where the reliability function of the system is chosen to be $\omega = 0.1$, see Table 5.15.

- Reducing the failure rate of each component belonging to the set T_2 of the system components by setting $\rho = 0.0381$ improves the reliability of the system like adding an extra component in parallel to each component of the same set

Table 5.14: Properties of network system in Figure 5.5.

Network system sets	Sets lifetime	Reduction improvement strategy
$T_1 = \{L_1, L_2, L_3, L_4, L_5, L_6, L_7\}$	$T_1 \sim$ Exponential($\lambda = 0.55$)	$R_1^{(A)}(t) = e^{-\rho\lambda t}$, see Sarhan (2000, 2002)
$T_2 = \{1, 4\}$	$T_2 \sim$ Weibull($\alpha = 0.274, \beta = 2.2$)	$R_2^{(A)}(t) = e^{-\rho\alpha t^{\beta}}$, see El-Damcese (2009)
$T_3 = \{2\}$	$T_3 \sim$ Exponentiated Weibull ($\alpha = 0.111, \beta = 2, \theta = 1.2$)	$R_3^{(A)}(t) = 1 - (1 - e^{-\rho\alpha t^{\beta}})^{\theta}$, see Alghamdi and Percy (2014)
$T_4 = \{3\}$	$T_4 \sim$ Gamma($n = 3.2, \lambda = 1.111$)	$R_4^{(A)}(t) = \int_t^{\infty} \frac{(\rho\lambda)^n t^{n-1}}{\Gamma n} e^{-\rho\lambda t} dt$, see Xia and Zhang (2007)

Table 5.15: Hot $SREF$ for sets of the network system in Figure 5.5.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.5172	0.3842	0.2047
T_2	0.4450	0.1403	0.0063
T_3	0.1880	0.0507	0.3232
T_4	0.4878	0.2994	0.0961

Table 5.16: Hot $MREF$ with perfect switch and $MTTF$ for sets of the network system in Figure 5.5.

Type	MREF	MTTF
T_1	0.3988	1.1491
T_2	0.3500	0.8546
T_3	0.2015	0.7639
T_4	0.4960	0.7647

Table 5.17: Cold $SREF$ with perfect switch for sets of the network system in Figure 5.5.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.3334	0.2394	0.1229
T_2	0.1733	0.0381	0.0152
T_3	0.0323	0.0079	0.0037
T_4	0.2179	0.1222	0.0530

Table 5.18: Cold $MREF$ with perfect switch and $MTTF$ for sets of the network system in Figure 5.5.

Type	MREF	MTTF
T_1	0.2484	1.2893
T_2	0.1544	0.8989
T_3	0.0380	0.7650
T_4	0.2375	0.7662

according to a cold duplication method with perfect switch where the reliability function of the system is chosen to be $\omega = 0.5$, see Table 5.17.

- Reducing the failure rate of the component in set T_3 of the system components by setting factor $\rho = 0.0144$ improves the reliability of the system like adding an extra component to the component in set T_3 according to a cold duplication method with imperfect switch where the reliability function of the system is chosen to be $\omega = 0.9$, see Table 5.19.

- In the same manner, one can interpret the other results presented in Tables 5.15, 5.17 and 5.19.

Tables 5.16, 5.18 and 5.20 present the MREFs for hot and cold (perfect and imperfect) duplication. Based on the results presented in those tables, we see that:

Table 5.19: Cold $SREF$ with imperfect switch for sets of the network system in Figure 5.5.

Type	$\omega = 0.1$	$\omega = 0.5$	$\omega = 0.9$
T_1	0.3545	0.2554	0.1315
T_2	0.2350	0.0814	0.0126
T_3	0.0942	0.0482	0.0144
T_4	0.3390	0.2848	0.0530

Table 5.20: Cold $MREF$ with imperfect switch and $MTTF$ for sets of the network system in Figure 5.5.

Type	MREF	MTTF
T_1	0.2651	1.2729
T_2	0.2075	0.8856
T_3	0.0966	0.7647
T_4	0.3447	0.7656

- Improving each link of the network system according to hot duplication increases the system mean time to failure to be 1.1491 and the same system mean time to failure can be obtained by reducing the failure rate of each link by setting $\rho = 0.3988$. Note that the mean time to failure of the original network is 0.7575.

- Improving each component in set T_2 according to a cold duplication with perfect switch increases the system mean time to failure to be 0.8989 and the same mean time to failure can be obtained by reducing the failure rate of each component in the same set by setting $\rho = 0.1544$.

- Improving the reliability of the set of the links of the network gives the best possible set improvement.

- In the same manner, one can interpret the other results presented in Tables 5.16, 5.18 and 5.20.

Figure 5.6 presents reliability functions of the original and some modified systems. From this figure, one may observe that, for this scenario:

- Improving the reliability of the set of system links according to any duplication method gives the best choice (solid and dotted blue curves).

- Improving set T_2 of network components (components 1 and 4) according to any duplication method gives the second best choice (solid and dotted red curves).

Figures 5.7 presents the behaviour of MTTF against the appropriate reduction factor ρ. It seems from these two figures that:

- MTTFs are non-decreasing with decreasing ρ for all possible reduction improvement.

- Reducing the failure rate of the set of system links T_1 gives the best possible reduction improvement, then reducing the failure rate of the set T_2 (components 1 and 4) gives the second best possible reduction improvement, see Figures 5.7a.

- Reducing the failure rate of the set T_4 (component 3) gives the third best possible reduction improvement, then reducing the failure rate of the set T_3 (component 2) gives the worst possible reduction improvement, see Figures 5.7b.

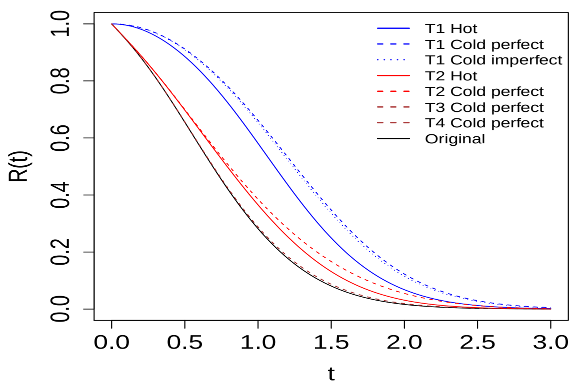
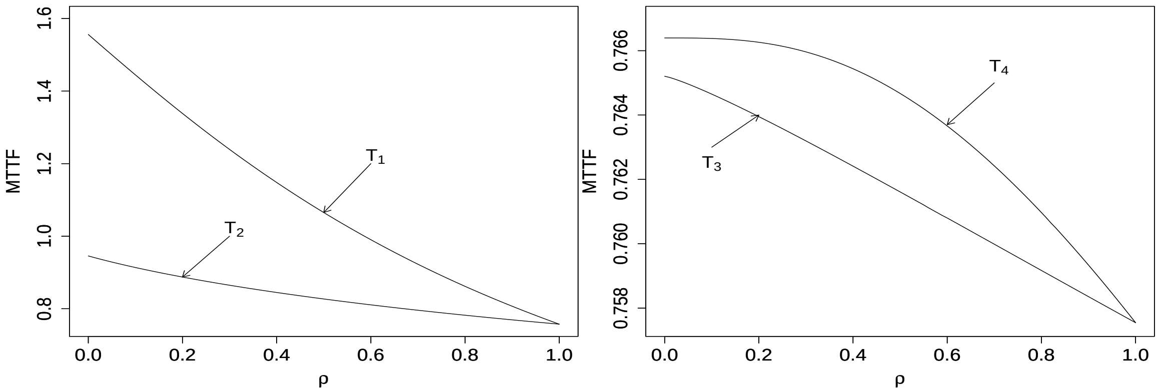


Figure 5.6: Reliability function of the original and some modified network systems.



(a) (b)

Figure 5.7: The behaviour of MTTF against ρ, for the illustrative network system

5.8 Conclusions

Chapter 5 is the most important chapter of this thesis, as it presents the main new contribution arising from this research project. Although earlier chapters present novel developments relating to reliability equivalence analysis, the original use of survival signature presented in Chapters 4 and 5 has the potential both to generalize and to standardize the method for application to complex systems and networks. In addition to the study of its theoretical properties, this technique might prove to be of value for improving system reliability in a cost-effective manner and for many different application areas that include engineering, government, communications, management, manufacturing, servicing, commerce and health care. In this chapter, we consider specific examples to demonstrate the broad applicability of this approach and to illustrate the type of inference and range of benefits that it might offer in practical scenarios.

Part III

Conclusions, discussion and future work

Chapter 6

Conclusions and future work

6.1 Conclusions and discussion

We investigate the reliability equivalence factors for some simple systems including parallel-series and series-parallel systems with generalized quadratic failure rate and exponentiated Weibull lifetime distributions in the first part of this thesis. The reduction improvement method and duplication improvement methods including hot, cold with perfect switch and cold with imperfect switch are considered for all improved systems. Generalized quadratic failure rate and exponentiated Weibull distributions are chosen because they are flexible and enable informative comparisons with other reliability equivalence studies. For comparing system improvements we use two measures which are survival reliability equivalence factors (SREF) and mean reliability equivalence factors (MREF).

In the second part of this thesis, we present a new method for deriving reliability equivalence factors using the concept of survival signature. We then apply survival signature to derive the reliability equivalence factors for networks and complex systems with multiple types of components. We use the `ReliabilityTheory` R package to derive survival reliability equivalence factors and mean reliability equivalence factors for simple and complex systems.

This thesis generalizes several previous studies about the reliability equivalence factors as presented in Tables 6.1 and 6.2. The usefulness of analysing survival signature to derive the reliability equivalence factors is prominent in these tables. All the previous studies that are mentioned in these tables are special cases of the reliability equivalence factors for complex systems and networks using the survival signature which is presented in Chapter 5.

Finally, the theoretical research that we present in this thesis has a wide range of real-life applications. We can envisage real applications for reliability equivalence factors in industry, business, science, public health care, etc. We now present some hypothetical and real application examples to illustrate how to apply the theoretical results on real applications as follows:

- New drug approval process

  According to Guarino and Guarino (2009) the process for a new drug approval takes on average 12 years and over US $ 350 million to get a new drug from the laboratory onto the pharmacy shelf. Ensuring drugs are safe and effective requires several phases of clinical trial to be approved in series where each phase has several steps that should also be approved. The first step for the new drug approval process is the drug discovery which takes between 2 and 10 years. The second step comprises preclinical research and development which include initial synthesis of substance, laboratory studies and animal testing and this phase takes between 3 and 6 years. The third step is asking for approval where the pharmaceutical industry takes the investigational new drug application (IND) to the food and drug administration in the United States (FDA). If the FDA decides that it is reasonably safe for the company to move forward with testing the drug in humans this will be the third step and it is called clinical trials. Clinical trials include three phases. Phase 1 studies the drug's side effects and involves between 20 and 80 people. If Phase 1 is safe, Phase 2 next studies whether the drug works in people who have a certain disease and typically involves about 300 people. If Phase 2 demonstrates sufficient effectiveness, then Phase 3 aims to measure these effects more generally and typically involves about 3000 people. If the drug passes all previous steps then the FDA approves the product for marketing. The final step is manufacturing and and so the drug appears on the market. For our purposes, the new drug approval process can be described as a parallel-series system where each improvement step involves several phases. We can envisage that the new drug approval process can be improved using:

1. Duplication methods.

   – Increase numbers of members in each phase of the clinical trials (hot duplication). This improves the robustness of model estimation and power for predictive inference.

   – Add extra members as standby to the members of the phases of the clinical trials (cold duplication). This avoids common problems that arise from patients dropping out of clinical trials for various reasons.

   – Increase numbers of animals in the animal testing stages.

2. Reduction method.

- Using different types of experimental animal such as mouse, rat, pig, rabbit, dog, and horse in the animal testing stages could improve the new drug approval process and reduce the risk associated with new drugs failing to gain approval. For example testing eyes drugs on a rabbet is better than using anther animal such as mouse, see Guarino and Guarino (2009).
- Using samples from different countries and different authenticities could be better than using samples with similar members. This would then require meta-analysis to combine the results of dissimilar trials.

We envisage that if real data are available the concept of reliability equivalence factors can be used for the new drug approval process. For example, using samples from different places can improve the new drug approval process as much as duplicating a sample which is chosen from a single area. Also using a type of experimental animal can improve the new drug approval process like using a double of another type of experimental animal.

- Vehicle inspection centres

Samah (2010) studied the vehicle inspection stations in Malaysia. The inspection stations have several lanes in parallel where some lanes are reserved for light vehicles and some lanes are reserved for heavy vehicles in addition to some lanes that are treated as universal. Each lane has several test machines including smokemeter machine, side slip testers machine, brake testers and headlight testers. Improving any unit of these machines can make the corresponding lane of vehicle inspection centres more reliable and improve the whole station. Vehicle inspection centres consist of several parallel lanes where each lane consists of several testing machines so the whole system can be described as a series-parallel system. We envisage that the vehicle inspection centres can be improved by:

1. Duplication methods.
   - Increase numbers of some test machines in all or some lanes of the vehicle inspection centres (hot duplication).
   - Use mobile machines which can be moved easily to be used instead of a broken machine and keeping the corresponding lane of the station in serviceable working order (cold duplication).
   - Increase the number of universal lanes.
2. Reduction method.
   - Using different types of smokemeter machine, side slip testers machine, brake testers and headlight testers which are better and more reliable can improve the vehicle inspection centres.

We envisage that if real data are available the concept of reliability equivalence factors can be used for improving the vehicle inspection centres. For example, by modifying some units such as the smokemeter machine or the side slip testers machine the reliability of the lane can be improved like using a mobile smokemeter machine in standby state or like adding an extra side slip testers machine.

- **Police and ambulance response times**

  Police in the UK use a wide range of four-wheel vehicles including hatchbacks, trucks and 4x4 cars. Other vehicles used by police include motorcycles, helicopters and boats. Ambulances also can be based on many types of vehicle including vans, cars, motorcycles and bicycles. For improving response times police and ambulance forces sometimes use small sizes of vehicles which can reach target destinations faster than four-wheel vehicles. We envisage that police and ambulance response times can be improved by:

1. Duplication methods.
   - Increasing numbers of police cars and ambulances can improve police and ambulance response times.
2. Reduction method.
   - Using different types of vehicle like motorcycles can improve police and ambulance response times.
   - Using horses on some occasions and in some places can improve police response times.

We envisage that if real data are available the concept of reliability equivalence factors can be used for improving police and ambulance response times. Instead of increasing the numbers of police and ambulance cars to improve average response times an option is to use motorcycles. The size of a motorcycle allows it arrive at accident scenes more quickly when incidents such as traffic collisions slow down access by four-wheeled vehicles. Using a motorcycle allows medics to reach patients quickly and start to give life-saving treatment while an ambulance is still on the way. In big cities like London and New York police and ambulance services also use helicopters to improve their performance.

- New car manufacturing process

  We can describe the new cars manufacturing process as a series-parallel system. There are several production lanes working in parallel where each production lane has several units working in series. Improving any unit can improve the global process of the factory. Indeed, we consider that most factories nowadays, including electronic factories, pharmaceutical factories, food factories and recycling factories, can be described as series-parallel systems. We envisage that the new car manufacturing process can be improved by:

  1. Duplication methods.
     - Increase numbers of machines in the production lanes such as painting machine and pressing machine (hot duplication).
     - Preparing flexible and removable machines which can be moved between production lanes (cold duplication).
  2. Reduction method.
     - Using a modified pressing machine which gives double the power of the current pressing machine.
     - Using a modified painting machine which reduces the painting time to half that of the current painting machine.

  We envisage that if real data are available the concept of reliability equivalence factors can be used for improving the new car manufacturing process. For example, using a modified pressing machine system can result in a system improvement that is like adding an extra pressing machine to the current pressing machine. Also using a new or modified painting machine can improve the time of painting to be like the time of using two painting machines of the same specification as the current painting machine.

6.2 Future work

There are several possible extensions for our current research. In this section, we present some further research challenges which can be considered for future work including:

- Availability equivalence factors

  All systems that are studied in this thesis are assumed to be with non-repairable components, so for this type of system one of two methods which are reduction and duplication can be used to improve system reliability. The other type of systems are systems with repairable components. For a system with repairable components, the system can be improved according to the methods mentioned previously, and can also be improved by increasing the maintenance of its components which accordingly increases the repair rates, see Hu et al. (2012). Availability is defined as the probability that the system is functioning when it is requested for use which means the probability that a system is functioning at a given time. According to this definition of availability, there is a clear relationship between availability and reliability.

  Repairable systems can improved by: (1) improving the times between failures which means modifying system components reliability using reduction and

Table 6.1: Previous studies as special cases of our current study (1).

Current study / Previous studies	Reliability equivalence factors for a parallel-series system assuming a generalized quadratic failure rate distribution	Reliability equivalence factors for a series-parallel system of components with exponentiated Weibull lifetimes	Reliability equivalence factors for complex systems and networks using the survival signature
Reliability equivalence, Råde (1993a)	✓	✓	✓
Reliability survival equivalence, Råde (1993b)	✓	✓	✓
Reliability equivalence of independent and non-identical components series systems, Sarhan (2000)			✓
Reliability equivalence with a basic series/parallel system, Sarhan (2002)	✓		✓
Reliability equivalence factors of a bridge network system, Sarhan (2004)			✓
Reliability equivalence factors of a parallel system, Sarhan (2005)	✓	✓	✓
Reliability equivalences of a series system consists of n independent and non-identical components, Sarhan and Mustafa (2006)			✓
Equivalence factors of a parallel-series system, Sarhan et al. (2008a)	✓		✓
Reliability equivalence factors of a general series-parallel system, Sarhan (2009b)		✓	✓

Table 6.2: Previous studies as special cases of our current study (2).

Current study / Previous studies	Reliability equivalence factors for a parallel-series system assuming a generalized quadratic failure rate distribution	Reliability equivalence factors for a series-parallel system of components with exponentiated Weibull lifetimes	Reliability equivalence factors for complex systems and networks using the survival signature
Reliability equivalence factors in Gamma distribution, Xia and Zhang (2007)			✓
Reliability equivalence Factors of a series-parallel system in Weibull distribution, El-Damcese (2009)		✓	✓
Reliability equivalence factors in exponentiated exponential distribution, Abdelkader et al. (2013)	✓	✓	✓
Testing reliability equivalence factors of a series-parallel systems in Burr type X distribution, Migdadi and Al-Batah (2014)			✓

duplication methods, or (2) improving the repair time by modifying the maintenance using an increase method. Reduction and duplication methods are defined previously in this thesis and the increase method means that the component can be improved by increasing its repair rate by a factor τ where $\tau > 1$.

Availability equivalence factors of a repairable series-parallel system were published by Hu et al. (2012) and the authors in this paper assumed a simple repairable series-parallel system with constant failure rate and repair rate. Sarhan and Mustafa (2013) studied the availability equivalence factors of a repairable parallel-series system and they also assumed that the life and repair times of the system components are exponentially distributed.

For possible further research one can derive the availability equivalence factors for simple repairable systems with components with non-constant failure rates and non-constant repair rates. The concept of availability equivalence factors can be applied on repairable systems under the assumption that the system components' life and repair times are gamma, Weibull or exponentiated Weibull lifetime distributions. The availability equivalence factors might be derived for repairable systems with multiple types of components and multiple types of repair times.

- ## Cold duplication with imperfect switch and imperfect storage environment

  In this thesis we consider that the switch in the method involving cold duplication with imperfect switch is not 100% reliable which means that the switch can fail to transfer the load to the standby components. We assume that the switch immediately transfers load to the standby component when the original component fails, which means the switch operation time is negligible. We assume the standby component does not fail when in the standby position. It can only fail given that the original component has already failed. Under the assumption that the standby component can fail during storage time a new structure for the cold duplication with imperfect switch methods can be studied. For this scenario, there are three possible characteristics for system failure, see Pan (1997).

1. When the main component fails and the switch successfully switches to the standby component which is in a good standby state. In this case, the system fails when the standby component fails.

2. When the main component fails and the switch successfully switches to the standby component but the standby component fails in its standby state. In this case, the system fails when the main component fails.

3. When the main component fails and the switch fails switching to the standby component which is in a good standby state. In this case the system fails when main component fails.

Under this assumption one can derive the reliability equivalence factors for simple systems, complex systems and networks using the survival signature.

- ## Reliability equivalence factors for systems with dependent components

All systems that are studied in this thesis are assumed to be with independent components. All previous studies in this field (reliability equivalence factors) also assumed systems with independent components. For systems with dependent components when the load is shared by several components the reliability function of such types of system can be derived using special methods. The challenge for deriving reliability equivalence factors for systems with dependent components is that the survival signature cannot be used in this case.

The author hopes to investigate extensions of the theory in this thesis to tackle this problem in the near future, possibly by applying the theory of copulae and vines as described by Bedford and Cooke (2002).

Appendix A

Appendix for Chapter 2

We used Matlab software to derive the hot survival reliability equivalence factors in Table 2.1. We define a function (Hot) to solve the set of equations in (2.4.1), then we use this function to derive the hot SREF as follows:

```
%We defined function Hot first
function F=Hot(z,a,b,c,d,m1,m2,r1,r2,a1,a2,alpha)
t=z(1);
q=z(2);
F(1)=(1-(((1-exp(-q*(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^r1).*...
(((1-exp(-(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^(m1-r1))).*...
(1-(((1-exp(-q*(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^r2).*...
(((1-exp(-(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^(m2-r2)))-alpha;
F(2)=(1-((1-exp(-(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^(m1+a1)).*...
(1-((1-exp(-(a*t+(b/2)*t.^2+(c/3)*t.^3))).^d).^(m2+a2))-alpha;
%%%%%%%%
clear all;
a=0.029;%alpha
b=-1.597*10^-3;%beta
c=2.608*10^-5;%gamma
d=0.786;%theta
m1=2;%number of components in the subsystem one.
m2=3;%number of components in the subsystem two.
q=0.1;% reduction factor.
mu=.2;% Imperfect switch failure rate.
r1=0; %Components improved according to reduction method (First system).
r2=1; %Components improved according to reduction method (second system).
```

```
a1=2; %Components improved according to hot duplication method (First system).
a2=3; %Components improved according to hot duplication method (second system).
r1=0;
    A1=[];
A2=[];
    r2=-1;
    for z=1:4;
        r2=r2+1;
a1=-1;
for s=1:3;
    a1=a1+1;
    a2=-1;
    for k=1:4;
        a2=a2+1;
alpha=-0.3;
for i=1:3;
    alpha=alpha+.4;
a0=[50 5];
%options = optimset('Display','off');
options=optimset('Display','off','MAXITER',10000,'MaxFunevals',20000);
[solution,fval,ExitFlag]=fsolve(@Hot,a0,options,a,b,c,d,m1,m2,r1,r2,a1,a2,alpha);
t(i)=solution(1);
q(i)=solution(2);
if ExitFlag==1;
t(i)=real(solution(1));
q(i)=real(solution(2));
else
    t(i)=0;
    q(i)=0;
end
end
tt(:,(s-1)*4+k)=t';
qq(:,(s-1)*4+k)=q';
    end
end
A1=[A1;tt];
A2=[A2;qq];
    end
    A2(A2<=0)=0;
X=A2;
```

```
X(1:3,:) = [];
X(:,1)=[];
U=[0.1 0.5 0.9 0.1 0.5 0.9 0.1 0.5 0.9]';
X=[U X];
Q1=X
Y=A1;
Y(1:3,:) = [];
Y(:,1)=[];
T1=Y;
r1=1;
    B1=[];
B2=[];
    r2=-1;
    for z=1:4;
        r2=r2+1;
a1=-1;
for s=1:3;
    a1=a1+1;
    a2=-1;
    for k=1:4;
        a2=a2+1;
alpha=-0.3;
for i=1:3;
    alpha=alpha+.4;
a0=[50 5];
%options = optimset('Display','off');
options=optimset('Display','off','MAXITER',10000,'MaxFunevals',20000);
[solution,fval,ExitFlag]=fsolve(@Hot,a0,options,a,b,c,d,m1,m2,
                                     r1,r2,a1,a2,alpha);
solution;
ExitFlag;
t(i)=solution(1);
q(i)=solution(2);
if ExitFlag==1;
t(i)=real(solution(1));
q(i)=real(solution(2));
else
    t(i)=0;
    q(i)=0;
end
```

```
end
tt(:,(s-1)*4+k)=t';
qq(:,(s-1)*4+k)=q';
    end
end
B1=[B1;tt];
B2=[B2;qq];
    end
    B2(B2<=0)=0;
X1=B2;
X1(:,1)=[];
U1=[0.1 0.5 0.9 0.1 0.5 0.9 0.1 0.5 0.9 0.1 0.5 0.9]';
X1=[U1 X1];
Q2=X1;
Y2=B1;
Y2(:,1)=[];
T2=Y2;
r1=2;
    C1=[];
C2=[];
    r2=-1;
    for z=1:4;
        r2=r2+1;
a1=-1;
for s=1:3;
    a1=a1+1;
    a2=-1;
    for k=1:4;
        a2=a2+1;
alpha=-0.3;
for i=1:3;
    alpha=alpha+.4;
a0=[.5 .5];
%options = optimset('Display','off');
options=optimset('Display','off','MAXITER',10000,'MaxFunevals',20000);
[solution,fval,ExitFlag]=fsolve(@Hot,a0,options,a,b,c,d,m1,m2,r1,r2,a1,a2,alpha);
t(i)=solution(1);
q(i)=solution(2);
if ExitFlag==1;
t(i)=real(solution(1));
```

```
q(i)=real(solution(2));
else
    t(i)=0;
    q(i)=0;
end
end
tt(:,(s-1)*4+k)=t';
qq(:,(s-1)*4+k)=q';
    end
end
C1=[C1;tt];
C2=[C2;qq];
    end
    C2(C2<=0)=0;
X2=C2;
X2(:,1)=[];
U2=[0.1 0.5 0.9 0.1 0.5 0.9 0.1 0.5 0.9 0.1 0.5 0.9]';
X2=[U2 X2];
Q3=X2;

Y3=C1;
Y3(:,1)=[];
T3=Y3;
T=[T1;T2;T3]
Q=[Q1;Q2;Q3]
size(Q);
```

All results in Chapter 2 and 3 are derived using Matlab software and checked using Mathcad software.

Appendix B

Appendix for Chapter 4

We used R software to derive the SREF and MREF and MTTF in Chapter 4. We used `computeSystemSurvivalSignature` function in `ReliabilityTheory` R package to derive the hot SREF in Table 4.3. We present an example and others much the same. For example, in Table 4.3 when all system components are improved except component number 5 $A_4^{(2,2)}$ we compute the factors as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  #########
gD <- graph.formula(s -- 1:2--3--t,s -- 1:2--4--t,
                    s -- 1:2--5--t,1:2--3:4:5)
V(gD)$compType <- NA
V(gD)$compType[match(c("1","2","3","4"), V(gD)$name)] <- "T1"
V(gD)$compType[match(c("5"), V(gD)$name)] <- "T2"
plot(gD)
computeSystemSurvivalSignature(gD)
sigD <- computeSystemSurvivalSignature(gD)
########### Hot duplication  ##########
sysSurvSD <- function(t, a, b, c, d) {
  res <- 0
  for(l1 in 0:4) {
    for(l2 in 0:1) {
        res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
          choose(4,l1) * (((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^2)^(4-l1)
           * (1-(((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^2))^l1*
```

```
          choose(1,l2) *  ((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(1-l2)
           * (1-((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l2
         }
  }
  res
}
sysSurvD <- Vectorize(sysSurvSD, vectorize.args=c("t"))
t <- seq(0, 120, length.out=100)
plot(t, sysSurvD(t, 0.029,-1.597*10^-3,2.608*10^-5,0.786), type="l",
            xlab="t", ylab="R(t)")
########  Reduction method  ############
sysSurvR <- function(t, rho, a,b,c,d) {
  res <- 0
  for(l1 in 0:4) {
    for(l2 in 0:1) {
        res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
          choose(4,l1) * ((1-exp(-rho*(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(4-l1)
           * (1-((1-exp(-rho*(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l1 *
          choose(1,l2) * ((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(1-l2)
           * (1-((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l2
    }
  }
  res
}
# Equivalence match at p for given Tx parameters
# p=0.1
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.1
  a <- 0.029
  b <- (-1.597*10^-3)
  c <- 2.608*10^-5
  d <- 0.786
  abs(sysSurvD(par[1], a,b,c,d)-p)+abs(sysSurvR(par[1], par[2], a,b,c,d)-p)
}
res <- optim(c(82,.5), objF)
A1 <- res$par[2]
B1 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 0.029,-1.597*10^-3,2.608*10^-5,0.786),
      type="l", lty=2,xlab="t", ylab="R(t)")
```

```
# p=0.5
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.5
  a <- 0.029
  b <- (-1.597*10^-3)
  c <- 2.608*10^-5
  d <- 0.786
  abs(sysSurvD(par[1], a,b,c,d)-p)+abs(sysSurvR(par[1], par[2], a,b,c,d)-p)
}
res <- optim(c(62,.5), objF)
A2 <- res$par[2]
B2 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 0.029,-1.597*10^-3,2.608*10^-5,0.786),
 type="l", lty=3, xlab="t", ylab="R(t)")

# p=0.9
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.9
  a <- 0.029
  b <- (-1.597*10^-3)
  c <- 2.608*10^-5
  d <- 0.786
  abs(sysSurvD(par[1], a,b,c,d)-p)+abs(sysSurvR(par[1], par[2], a,b,c,d)-p)
}
res <- optim(c(11,.2), objF)
A3 <- res$par[2]
B3 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 0.029,-1.597*10^-3,2.608*10^-5,0.786),
 type="l", lty=4,xlab="t", ylab="R(t)")
c(B1,B2,B3)
# Hot SREF when all system components improved except component number 5.
c(A1,A2,A3)
```

The MREF in Table 4.4 when all system components are improved except component number 5 $A_4^{(2,2)}$ can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
```

```
############# Survival signature  #############
gD <- graph.formula(s -- 1:2--3--t,s -- 1:2--4--t,
s -- 1:2--5--t,1:2--3:4:5)
V(gD)$compType <- NA
V(gD)$compType[match(c("1","2","3","4"), V(gD)$name)]<-"T1"
V(gD)$compType[match(c("5"), V(gD)$name)] <- "T2"
plot(gD)
computeSystemSurvivalSignature(gD)
sigD <- computeSystemSurvivalSignature(gD)
############# Hot Duplication  #################
sysSurvSD <- function(t, a, b, c, d) {
  res <- 0
  for(l1 in 0:4) {
    for(l2 in 0:1) {
        res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
        choose(4,l1) * (((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^2)^(4-l1)
        * (1-(((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^2))^l1 *
        choose(1,l2) *   ((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(1-l2)
        * (1-((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l2
         }
  }
  res
}

integrand <- function(t) {sysSurvSD(t, 0.029,-1.597*10^-3,
                                  2.608*10^-5,0.786)}
MTTFD <- integrate(integrand, lower = 0, upper = Inf)$value
MTTFD
############# Hot Duplication  ##################
sysSurvR <- function(t, rho, a,b,c,d) {
res <- 0
  for(l1 in 0:4) {
    for(l2 in 0:1) {
    res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
    choose(4,l1) * ((1-exp(-rho*(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(4-l1)
     * (1-((1-exp(-rho*(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l1    *
     choose(1,l2) * ((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d)^(1-l2)
     * (1-((1-exp(-(a*t+(b/2)*t^2+(c/3)*t^3)))^d))^l2
    }
  }
```

```
  res
}
F2 <- function(rho){
    f2 <- function(t){
       sysSurvR(t, rho, 0.029,-1.597*10^-3,2.608*10^-5,0.786)
   }
return(f2)
}
MTTFr <- function (rho) {
    integrate(F2(rho),  lower=0, upper=Inf)$value
}
MTTFrd <- function(par) { # par[1] == rho,
      abs(MTTFr(par[1])-MTTFD)
}
rHo <- optim(0.43, MTTFrd, method = c( "Brent"),
lower = 0, upper = 200)$par
rHo
# Mean time to failure for improved system.
MTTFD
# Hot MREF when all system component improved except component 5.
MTTFr(rHo)
```

The SREF in Table 4.11 when component number 1 and 2 are improved $A_2^{(2,0)}$ can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  ##################
# Duplication method system and survival function
gD <- graph.formula(s -- 3--4--5--t,s -- 1--2--t,1:3,2:4)
V(gD)$compType <- NA
V(gD)$compType[match(c("3","5","4"), V(gD)$name)] <- "T1"
V(gD)$compType[match(c("1","2"), V(gD)$name)] <- "T2"
plot(gD)
computeSystemSurvivalSignature(gD)
sigD <- computeSystemSurvivalSignature(gD)
sigD
############# Hot Duplication  ##################
sysSurvSD <- function(t, a, b, c) {
  res <- 0
```

```
  for(l1 in 0:3) {
    for(l2 in 0:2) {
  res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
  choose(3,l1) * ((1-exp(-a*t^b))^c)^(3-l1)
   * (1-((1-exp(-a*t^b))^c))^l1  *
  choose(2,l2) * ((1-exp(-a*t^b))^(2*c))^(2-l2)
   * (1-((1-exp(-a*t^b))^(2*c)))^l2
        }
  }
  res
}
sysSurvD <- Vectorize(sysSurvSD, vectorize.args=c("t"))
t <- seq(0, 3, length.out=100)
plot(t, sysSurvD(t, 1,2,3), type="l",xlab="t", ylab="R(t)")
#################  Reduction method  #############
sysSurvR <- function(t, rho, a,b,c) {
res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:2) {
   res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
    choose(3,l1) * ((1-exp(-a*t^b))^c)^(3-l1) *
    (1-((1-exp(-a*t^b))^c))^l1 *
    choose(2,l2) * ((1-exp(-rho*a*t^b))^c)^(2-l2)
     * (1-((1-exp(-rho*a*t^b))^c))^l2
    }
  }
  res
}
# Equivalence match at p for given Tx parameters
# p=0.1
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.1
  a <- 1
  b <- 2
  c <- 3
  abs(sysSurvD(par[1], a,b,c)-p)+abs(sysSurvR(par[1],
   par[2], a,b,c)-p)
}
res <- optim(c(1,.5), objF)
A1 <- res$par[2]
```

```
B1 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 1,2,3), type="l",
 lty=2,xlab="t",
 ylab="R(t)")
# p=0.5
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.5
  a <- 1
  b <- 2
  c <- 3
  abs(sysSurvD(par[1], a,b,c)-p)+abs(sysSurvR(par[1],
   par[2], a,b,c)-p)
}
res <- optim(c(1,.5), objF)
A2 <- res$par[2]
B2 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 1,2,3), type="l",
 lty=3, xlab="t", ylab="R(t)")

# p=0.9
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.9
  a <- 1
  b <- 2
  c <- 3
  abs(sysSurvD(par[1], a,b,c)-p)+abs(sysSurvR(par[1],
   par[2], a,b,c)-p)
}
res <- optim(c(1,.2), objF)
A3 <- res$par[2]
B3 <- res$par[1]
print(res[2])
points(t, sysSurvR(t, res$par[2], 1,2,3), type="l", lty=4,xlab="t",
 ylab="R(t)")
c(B1,B2,B3)
# Hot SREF when only components number 1 and 2 are improved.
c(A1,A2,A3)
```

The MREF in Table 4.12 when component number 3 and 4 are improved $A_2^{(0,2)}$ can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  ###############
# Duplication method system and survival function
gD <- graph.formula(s -- 1--2--t,s -- 3--4--5--t,1:3,2:4)
V(gD)$compType <- NA
V(gD)$compType[match(c("1","2","4"), V(gD)$name)] <- "T1"
V(gD)$compType[match(c("3","5"), V(gD)$name)] <- "T2"
plot(gD)
computeSystemSurvivalSignature(gD)
sigD <- computeSystemSurvivalSignature(gD)
sigD
############# Hot Duplication  #############
sysSurvSD <- function(t, a, b, c) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:2) {
    res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
    choose(3,l1) * ((1-exp(-a*t^b))^c)^(3-l1)
    * (1-((1-exp(-a*t^b))^c))^l1  *
    choose(2,l2) * ((1-exp(-a*t^b))^(2*c))^(2-l2)
     * (1-((1-exp(-a*t^b))^(2*c)))^l2
    }
  }
  res
}
integrand <- function(t) {sysSurvSD(t, 1,2,3)}
MTTFD <- integrate(integrand, lower = 0, upper = Inf)$value
MTTFD
#######################  Reduction method  #################
sysSurvSR <- function(t, rho, a,b,c) {
res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:2) {
    res <- res+with(sigD, sigD[T1==l1 & T2==l2 ,"Probability"]) *
    choose(3,l1) * ((1-exp(-a*t^b))^c)^(3-l1) *
    (1-((1-exp(-a*t^b))^c))^l1 *
```

```
    choose(2,l2) * ((1-exp(-rho*a*t^b))^c)^(2-l2)
     * (1-((1-exp(-rho*a*t^b))^c))^l2
    }
  }
  res
}
F2 <- function(rho){
    f2 <- function(t){
       sysSurvSR(t, rho, 1,2,3)
   }
return(f2)
}
MTTFr <- function (rho) {
    integrate(F2(rho),  lower=0, upper=Inf)$value
}
MTTFrd <- function(par) { # par[1] == rho,
      abs(MTTFr(par[1])-MTTFD)
}
rHo <- optim(0.5, MTTFrd, method = c( "Brent"),lower = 0,
 upper = 200)$par
rHo
# The mean time to improved when components 3 and 4 are improved.
MTTFD
# Hot MREF when components 3 and 4 are improved.
MTTFr(rHo)
```

Appendix C

Appendix for Chapter 5

We use `computeSystemSurvivalSignature` function in `ReliabilityTheory` R package to derive SREF MREF and MTTF for complex systems and networks in Chapter 5. The hot SREF in Table 5.2 for the best component that can be improved (component 7) can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  #########
g <- graph.formula(s -- 1 -- 2:4:5, 2 -- 3 -- t, 4:5 -- 6 -- t,
        s -- 7 -- 8 -- t, s -- 9 -- 10 -- 11 -- t, 7 -- 10 -- 8)
V(g)$compType <- NA
V(g)$compType[match(c("1","6","11"), V(g)$name)]<-"T1"
V(g)$compType[match(c("2","3","9"), V(g)$name)]<-"T2"
V(g)$compType[match(c("4","5","10"), V(g)$name)]<-"T3"
V(g)$compType[match(c("7"), V(g)$name)] <- "T4"
V(g)$compType[match(c("8"), V(g)$name)] <- "T5"
#plot(g)
sig <- computeSystemSurvivalSignature(g)
#sig
##########Hot duplication ################
sysSurvSD <- function(t, T1r, T2sc, T2sh,a,b,c, T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
```

```
        for(l4 in 0:1) {
           for(l5 in 0:1) {
            res <- res+with(sig, sig[T1==l1 & T2==l2 & T3==l3 & T4==l4 &
            T5==l5,"Probability"]) *
            choose(3,l1) * pexp(t, rate=T1r)^(3-l1) * pexp(t, rate=T1r,
             lower.tail=FALSE)^l1 *
            choose(3,l2) * pweibull(t, scale=T2sc, shape=T2sh)^(3-l2) *
             pweibull(t, scale=T2sc, shape=T2sh, lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3) *
            (1-((1-exp(-a*t^b))^c))^l3 *
            choose(1,l4) * ((pgamma(t, shape=T4sh, scale=T4sc))^2)^(1-l4)
             * (1-(pgamma(t, shape=T4sh, scale=T4sc))^2)^l4*
            choose(1,l5) * pgamma(t, shape=T4sh, scale=T4sc)^(1-l5) *
             pgamma(t, shape=T4sh, scale=T4sc, lower.tail=FALSE)^l5
      }
    }
  }
 }
}
res
}
sysSurvD <- Vectorize(sysSurvSD, vectorize.args=c("t"))
t <- seq(0, 5, length.out=100)
#points(t, sysSurvD(t, 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9),
 type="l", lty=2 ,xlab="t", ylab="R(t)")
plot(t, sysSurvD(t, 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9),
type="l",xlab="t", ylab="R(t)")
###Reduction component 7##################
sysSurvSR <- function(t,rho, T1r, T2sc, T2sh,a,b,c,
 T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
        for(l4 in 0:1) {
           for(l5 in 0:1) {
          res <- res+with(sig, sig[T1==l1 & T2==l2 & T3==l3 & T4==l4 & T5==l5,
          "Probability"]) *
            choose(3,l1) * pexp(t, rate=T1r)^(3-l1) * pexp(t, rate=T1r,
            lower.tail=FALSE)^l1 *
```

```
            choose(3,l2) * pweibull(t, scale=T2sc, shape=T2sh)^(3-l2) *
             pweibull(t, scale=T2sc, shape=T2sh, lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3) *
            (1-((1-exp(-a*t^b))^c))^l3 *
            choose(1,l4) * pgamma(t, shape=T4sh, scale=(1/rho)*T4sc)^(1-l4) *
             pgamma(t, shape=T4sh, scale=(1/rho)*T4sc, lower.tail=FALSE)^l4*
            choose(1,l5) * pgamma(t, shape=T4sh, scale=T4sc)^(1-l5) *
            pgamma(t, shape=T4sh, scale=T4sc, lower.tail=FALSE)^l5
      }
    }
  }
 }
}
res
}
sysSurvR <- Vectorize(sysSurvSR, vectorize.args=c("t"))

# Equivalence match at p for given Tx parameters
objF <- function(par) { # par[1] == t, par[2] == rho
  p <- 0.9
  T1r <- 0.55
  T2sc <- 1.8
  T2sh <- 2.2
  a  <- 0.111
  b  <- 2
  c  <- 1.2
  T4sh <- 3.2
  T4sc <- 0.9
  abs(sysSurvD(par[1], T1r, T2sc, T2sh,a,b,c, T4sh, T4sc)-p)+
  abs(sysSurvR(par[1], par[2], T1r, T2sc, T2sh, a,b,c,T4sh,
   T4sc)-p)
}
res <- optim(c(.7, 0.3), objF)
A11 <- res$par[2]
B11 <- res$par[1]
A11
B11
print(res[2])
points(t, sysSurvR(t, res$par[2], 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9),
 type="l", lty=2)
```

The hot MREF in Table 5.3 for the best component that can be improved (component 7) can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  #########
g <- graph.formula(s -- 1 -- 2:4:5, 2 -- 3 -- t, 4:5 -- 6 -- t,
 s -- 7 -- 8 -- t, s -- 9 -- 10 -- 11 -- t, 7 -- 10 -- 8)
V(g)$compType <- NA
V(g)$compType[match(c("1","6","11"), V(g)$name)] <- "T1"
V(g)$compType[match(c("2","3","9"), V(g)$name)] <- "T2"
V(g)$compType[match(c("4","5","10"), V(g)$name)] <- "T3"
V(g)$compType[match(c("7"), V(g)$name)] <- "T4"
V(g)$compType[match(c("8"), V(g)$name)] <- "T5"
#plot(g)
sig <- computeSystemSurvivalSignature(g)
#sig
###Hot duplication component 7
sysSurvSD <- function(t, T1r, T2sc, T2sh,a,b,c, T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
        for(l4 in 0:1) {
           for(l5 in 0:1) {
            res <- res+with(sig, sig[T1==l1 & T2==l2 &
             T3==l3 & T4==l4 &
             T5==l5,"Probability"]) *
            choose(3,l1) * pexp(t, rate=T1r)^(3-l1)
             * pexp(t, rate=T1r,
             lower.tail=FALSE)^l1 *
            choose(3,l2) * pweibull(t, scale=T2sc,
             shape=T2sh)^(3-l2) *
             pweibull(t, scale=T2sc, shape=T2sh,
             lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3) *
            (1-((1-exp(-a*t^b))^c))^l3 *
            choose(1,l4) * ((pgamma(t, shape=T4sh,
            scale=T4sc))^2)^(1-l4)
            * (1-(pgamma(t, shape=T4sh, scale=T4sc))^2)^l4*
```

```
            choose(1,l5) * pgamma(t, shape=T4sh,
            scale=T4sc)^(1-l5)
             * pgamma(t, shape=T4sh, scale=T4sc,
             lower.tail=FALSE)^l5
      }
    }
  }
 }
}
res
}
integrand<-function(t){sysSurvSD(t, 0.55, 1.8, 2.2,
0.111,2,1.2,3.2, 0.9)}
MTTFD <- integrate(integrand, lower = 0,
upper = Inf)$value
MTTFD
###Reduction component 1
sysSurvSR <- function(t,rho, T1r, T2sc, T2sh,a,b,c,
 T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
        for(l4 in 0:1) {
           for(l5 in 0:1) {
          res <- res+with(sig, sig[T1==l1 & T2==l2 & T3==l3
          & T4==l4 &
          T5==l5,"Probability"]) *
            choose(3,l1) * pexp(t, rate=T1r)^(3-l1) * pexp(t, rate=T1r,
             lower.tail=FALSE)^l1 *
            choose(3,l2) * pweibull(t, scale=T2sc, shape=T2sh)^(3-l2) *
            pweibull(t, scale=T2sc, shape=T2sh, lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3) *
            (1-((1-exp(-a*t^b))^c))^l3 *
            choose(1,l4) * pgamma(t, shape=T4sh,
            scale=(1/rho)*T4sc)^(1-l4) *
             pgamma(t, shape=T4sh, scale=(1/rho)*T4sc,
              lower.tail=FALSE)^l4*
            choose(1,l5) * pgamma(t, shape=T4sh, scale=T4sc)^(1-l5) *
            pgamma(t, shape=T4sh, scale=T4sc, lower.tail=FALSE)^l5
```

```
      }
    }
  }
 }
}
res
}
F2 <- function(rho){
    f2 <- function(t){
       sysSurvSR(t, rho, 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9)
   }
return(f2)
}
MTTFr <- function (rho) {
    integrate(F2(rho),  lower=0, upper=Inf)$value
}
MTTFrd <- function(par) { # par[1] == rho,
      abs(MTTFr(par[1])-MTTFD)
}
rHo <- optim(0.5, MTTFrd, method = c( "Brent"),lower = 0,
 upper = 200)$par
rHo

MTTFD
MTTFr(rHo)

#par(new=TRUE)
plot(Vectorize(MTTFr),ylim=c(2.32, 3.2),
xlab=expression(rho), ylab="MTTF",
col = "black",cex.lab=1.5, cex.axis=1.5, cex.main=1.5,
cex.sub=1.5)
text(0.52,3.0,expression(7),cex=1.5)
arrows(0.5,3,0.4,2.932, length=0.1)
```

The hot SREF in Table 5.4 for the best type that can be improved (component T_4) can be computed as follows:

```
library(ReliabilityTheory)
?computeSystemSurvivalSignature
############# Survival signature  #########
```

```
g <- graph.formula(s -- 1 -- 2:4:5, 2 -- 3 -- t, 4:5 -- 6 -- t,
 s -- 7 -- 8 -- t, s -- 9 -- 10 -- 11 -- t, 7 -- 10 -- 8)
V(g)$compType <- NA
V(g)$compType[match(c("1","6","11"), V(g)$name)] <- "T1"
V(g)$compType[match(c("2","3","9"), V(g)$name)] <- "T2"
V(g)$compType[match(c("4","5","10"), V(g)$name)] <- "T3"
V(g)$compType[match(c("7","8"), V(g)$name)] <- "T4"
#plot(g)
sig <- computeSystemSurvivalSignature(g)
#sig
###Hot duplication type 4
sysSurvSD <- function(t, T1r, T2sc, T2sh, a, b, c,
 T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
        for(l4 in 0:2) {
          res <- res+with(sig, sig[T1==l1 & T2==l2
           & T3==l3 &
           T4==l4,"Probability"]) *
            choose(3,l1) * ((pexp(t, rate=T1r))^1)^
            (3-l1) *
             (1-(pexp(t, rate=T1r))^1)^l1 *
            choose(3,l2) * pweibull(t, scale=T2sc,
            shape=T2sh)^(3-l2)
             * pweibull(t, scale=T2sc, shape=T2sh,
             lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3)
            * (1-((1-exp(-a*t^b))^c))^l3 *
            choose(2,l4) * ((pgamma(t, shape=T4sh,
             scale=T4sc))^2)^(2-l4)
             * (1-(pgamma(t, shape=T4sh,
              scale=T4sc))^2)^l4
      }
    }
  }
 }
res
}
```

```
sysSurvD <- Vectorize(sysSurvSD, vectorize.args=c("t"))
t <- seq(0, 5, length.out=100)
#points(t, sysSurvD(t, 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9),
 type="l", lty=2 ,xlab="t", ylab="R(t)")
plot(t, sysSurvD(t, 0.55, 1.8, 2.2, 0.111,2,1.2,3.2, 0.9),
 type="l",xlab="t", ylab="R(t)")
###Reduction Type 4

sysSurvSR <- function(t,rho, T1r, T2sc, T2sh,a,b,c, T4sh, T4sc) {
  res <- 0
  for(l1 in 0:3) {
    for(l2 in 0:3) {
      for(l3 in 0:3) {
        for(l4 in 0:2) {
          res <- res+with(sig, sig[T1==l1 & T2==l2 & T3==l3 &
           T4==l4,"Probability"]) *
            choose(3,l1) * pexp(t, rate=T1r)^(3-l1) *
            pexp(t, rate=T1r, lower.tail=FALSE)^l1 *
            choose(3,l2) * pweibull(t, scale=T2sc, shape=T2sh)^(3-l2) *
            pweibull(t, scale=T2sc, shape=T2sh, lower.tail=FALSE)^l2 *
            choose(3,l3) * ((1-exp(-a*t^b))^c)^(3-l3)
            * (1-((1-exp(-a*t^b))^c))^l3 *
            choose(2,l4) * pgamma(t, shape=T4sh, scale=(1/rho)*T4sc)^(2-l4)
            * pgamma(t, shape=T4sh, scale=(1/rho)*T4sc, lower.tail=FALSE)^l4
      }
    }
  }
 }
res
}

sysSurvR <- Vectorize(sysSurvSR,
 vectorize.args=c("t"))

# Equivalence match at p for given Tx parameters
objF <- function(par) { # par[1] == t,
 par[2] == rho
  p <- 0.9
  T1r <- 0.55
  T2sc <- 1.8
```

```
  T2sh <- 2.2
  a  <- 0.111
  b  <- 2
  c  <- 1.2
  T4sh <- 3.2
  T4sc <- 0.9
  abs(sysSurvD(par[1], T1r, T2sc, T2sh,a,b,c, T4sh,
   T4sc)-p)+abs(sysSurvR(par[1], par[2], T1r, T2sc,
    T2sh,a,b,c,T4sh, T4sc)-p)
}
res <- optim(c(.7, 0.3), objF)
A11 <- res$par[2]
B11 <- res$par[1]
A11
B11
print(res[2])
points(t, sysSurvR(t, res$par[2], 0.55, 1.8, 2.2,
 0.111,2,1.2,3.2, 0.9), type="l", lty=2)
```

Table C.1: The survival signature of improved system when component 1 of the system in Figure 5.1 is improved. The components of improved system are classified into 5 types which are $T_1 = \{1\}$, $T_2 = \{2, 3, 9\}$, $T_3 = \{4, 5, 10\}$, $T_4 = \{6, 11\}$ and $T_5 = \{7, 8\}$.

l_1	l_2	l_3	l_4	l_5	$\Phi(.)$
0	0	0	0	0	0
0	0	0	0	1	0
0	0	0	0	2	1
0	0	0	1	0	0
0	0	0	1	1	0
0	0	0	1	2	1
0	0	0	2	0	0
0	0	0	2	1	0
0	0	0	2	2	1
0	0	1	0	0	0
0	0	1	0	1	0
0	0	1	0	2	1
0	0	1	1	0	0
0	0	1	1	1	1/12
0	0	1	1	2	1
0	0	1	2	0	0
0	0	1	2	1	1/6
0	0	1	2	2	1
0	0	2	0	0	0
0	0	2	0	1	0
0	0	2	0	2	1
0	0	2	1	0	0
0	0	2	1	1	1/6
0	0	2	1	2	1
0	0	2	2	0	0
0	0	2	2	1	1/3
0	0	2	2	2	1
0	0	3	0	0	0
0	0	3	0	1	0
0	0	3	0	2	1
0	0	3	1	0	0
0	0	3	1	1	1/4
0	0	3	1	2	1
0	0	3	2	0	0
0	0	3	2	1	1/2
0	0	3	2	2	1
0	1	0	0	0	0
0	1	0	0	1	0
0	1	0	0	2	1
0	1	0	1	0	0
0	1	0	1	1	0
0	1	0	1	2	1
0	1	0	2	0	0
0	1	0	2	1	0
0	1	0	2	2	1
0	1	1	0	0	0
0	1	1	0	1	1/18
0	1	1	0	2	1
1	0	0	0	0	0
1	0	0	0	1	0
1	0	0	0	2	1
1	0	0	1	0	0
1	0	0	1	1	0
1	0	0	1	2	1
1	0	0	2	0	0
1	0	0	2	1	0
1	0	0	2	2	1
1	0	1	0	0	0

l_1	l_2	l_3	l_4	l_5	$\Phi(.)$
0	1	1	1	0	1/18
0	1	1	1	1	5/36
0	1	1	1	2	1
0	1	1	2	0	1/9
0	1	1	2	1	2/9
0	1	1	2	2	1
0	1	2	0	0	0
0	1	2	0	1	1/9
0	1	2	0	2	1
0	1	2	1	0	1/9
0	1	2	1	1	5/18
0	1	2	1	2	1
0	1	2	2	0	2/9
0	1	2	2	1	4/9
0	1	2	2	2	1
0	1	3	0	0	0
0	1	3	0	1	1/6
0	1	3	0	2	1
0	1	3	1	0	1/6
0	1	3	1	1	5/12
0	1	3	1	2	1
0	1	3	2	0	1/3
0	1	3	2	1	2/3
0	1	3	2	2	1
0	2	0	0	0	0
0	2	0	0	1	0
0	2	0	0	2	1
0	2	0	1	0	0
0	2	0	1	1	0
0	2	0	1	2	1
0	2	0	2	0	0
0	2	0	2	1	0
0	2	0	2	2	1
0	2	1	0	0	0
0	2	1	0	1	1/9
0	2	1	0	2	1
0	2	1	1	0	1/9
0	2	1	1	1	7/36
0	2	1	1	2	1
0	2	1	2	0	2/9
0	2	1	2	1	5/18
0	2	1	2	2	1
0	2	2	0	0	0
0	2	2	0	1	2/9
0	2	2	0	2	1
0	2	2	1	0	2/9
0	2	2	1	1	7/18
0	2	2	1	2	1
1	1	1	1	0	7/18
1	1	1	1	1	17/36
1	1	1	1	2	1
1	1	1	2	0	7/9
1	1	1	2	1	8/9
1	1	1	2	2	1
1	1	2	0	0	0
1	1	2	0	1	1/9
1	1	2	0	2	1
1	1	2	1	0	11/18

l_1	l_2	l_3	l_4	l_5	$\Phi(.)$
0	2	2	2	0	4/9
0	2	2	2	1	5/9
0	2	2	2	2	1
0	2	3	0	0	0
0	2	3	0	1	1/3
0	2	3	0	2	1
0	2	3	1	0	1/3
0	2	3	1	1	7/12
0	2	3	1	2	1
0	2	3	2	0	2/3
0	2	3	2	1	5/6
0	2	3	2	2	1
0	3	0	0	0	0
0	3	0	0	1	0
0	3	0	0	2	1
0	3	0	1	0	0
0	3	0	1	1	0
0	3	0	1	2	1
0	3	0	2	0	0
0	3	0	2	1	0
0	3	0	2	2	1
0	3	1	0	0	0
0	3	1	0	1	1/6
0	3	1	0	2	1
0	3	1	1	0	1/6
0	3	1	1	1	1/4
0	3	1	1	2	1
0	3	1	2	0	1/3
0	3	1	2	1	1/3
0	3	1	2	2	1
0	3	2	0	0	0
0	3	2	0	1	1/3
0	3	2	0	2	1
0	3	2	1	0	1/3
0	3	2	1	1	1/2
0	3	2	1	2	1
0	3	2	2	0	2/3
0	3	2	2	1	2/3
0	3	2	2	2	1
0	3	3	0	0	0
0	3	3	0	1	1/2
0	3	3	0	2	1
0	3	3	1	0	1/2
0	3	3	1	1	3/4
0	3	3	1	2	1
0	3	3	2	0	1
0	3	3	2	1	1
0	3	3	2	2	1
1	2	2	2	0	1
1	2	2	2	1	1
1	2	2	2	2	1
1	2	3	0	0	1/3
1	2	3	0	1	2/3
1	2	3	0	2	1
1	2	3	1	0	1
1	2	3	1	1	1
1	2	3	1	2	1
1	2	3	2	0	1

Table C.2: Continued from previous page.

l_1	l_2	l_3	l_4	l_5	$\Phi(.)$	l_1	l_2	l_3	l_4	l_5	$\Phi(.)$	l_1	l_2	l_3	l_4	l_5	$\Phi(.)$
1	0	1	0	1	0	1	1	2	1	1	13/18	1	2	3	2	1	1
1	0	1	0	2	1	1	1	2	1	2	1	1	2	3	2	2	1
1	0	1	1	0	1/3	1	1	2	2	0	1	1	3	0	0	0	1
1	0	1	1	1	5/12	1	1	2	2	1	1	1	3	0	0	1	1
1	0	1	1	2	1	1	1	2	2	2	1	1	3	0	0	2	1
1	0	1	2	0	2/3	1	1	3	0	0	0	1	3	0	1	0	1
1	0	1	2	1	5/6	1	1	3	0	1	1/6	1	3	0	1	1	1
1	0	1	2	2	1	1	1	3	0	2	1	1	3	0	1	2	1
1	0	2	0	0	0	1	1	3	1	0	2/3	1	3	0	2	0	1
1	0	2	0	1	0	1	1	3	1	1	5/6	1	3	0	2	1	1
1	0	2	0	2	1	1	1	3	1	2	1	1	3	0	2	2	1
1	0	2	1	0	1/2	1	1	3	2	0	1	1	3	1	0	0	1
1	0	2	1	1	2/3	1	1	3	2	1	1	1	3	1	0	1	1
1	0	2	1	2	1	1	1	3	2	2	1	1	3	1	0	2	1
1	0	2	2	0	1	1	2	0	0	0	1/3	1	3	1	1	0	1
1	0	2	2	1	1	1	2	0	0	1	1/3	1	3	1	1	1	1
1	0	2	2	2	1	1	2	0	0	2	1	1	3	1	1	2	1
1	0	3	0	0	0	1	2	0	1	0	1/3	1	3	1	2	0	1
1	0	3	0	1	0	1	2	0	1	1	1/3	1	3	1	2	1	1
1	0	3	0	2	1	1	2	0	1	2	1	1	3	1	2	2	1
1	0	3	1	0	1/2	1	2	0	2	0	1/3	1	3	2	0	0	1
1	0	3	1	1	3/4	1	2	0	2	1	1/3	1	3	2	0	1	1
1	0	3	1	2	1	1	2	0	2	2	1	1	3	2	0	2	1
1	0	3	2	0	1	1	2	1	0	0	1/3	1	3	2	1	0	1
1	0	3	2	1	1	1	2	1	0	1	4/9	1	3	2	1	1	1
1	0	3	2	2	1	1	2	1	0	2	1	1	3	2	1	2	1
1	1	0	0	0	0	1	2	1	1	0	2/3	1	3	2	2	0	1
1	1	0	0	1	0	1	2	1	1	1	13/18	1	3	2	2	1	1
1	1	0	0	2	1	1	2	1	1	2	1	1	3	2	2	2	1
1	1	0	1	0	0	1	2	1	2	0	1	1	3	3	0	0	1
1	1	0	1	1	0	1	2	1	2	1	1	1	3	3	0	1	1
1	1	0	1	2	1	1	2	1	2	2	1	1	3	3	0	2	1
1	1	0	2	0	0	1	2	2	0	0	1/3	1	3	3	1	0	1
1	1	0	2	1	0	1	2	2	0	1	5/9	1	3	3	1	1	1
1	1	0	2	2	1	1	2	2	0	2	1	1	3	3	1	2	1
1	1	1	0	0	0	1	2	2	1	0	8/9	1	3	3	2	0	1
1	1	1	0	1	1/18	1	2	2	1	1	8/9	1	3	3	2	1	1
1	1	1	0	2	1	1	2	2	1	2	1	1	3	3	2	2	1

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