Multi-Turing instabilities & spontaneous patterns in discrete nonlinear systems: simplicity and complexity, cavities and counterpropagation

C. Bostock[1], **J. M. Christian**[1,*], **A. B. Leite**[2], **G. S. McDonald**[1], **J. G. Huang**[3]

[1]Materials & Physics Research Centre, University of Salford, Greater Manchester M5 4WT, UK

[2]Departamento de Física, Universidade Federal de São Carlos, CEP 13565-905, Brazil

[3]Faculty of Advanced Technology, University of South Wales, Pontypridd CF37 1DL, UK

*Email: j.christian@salford.ac.uk

Abstract

Alan Turing's profound insight into morphogenesis, published in 1952, has provided the cornerstone for understanding the origin of pattern and form in Nature. When the uniform states of a nonlinear reaction-diffusion system are sufficiently stressed, arbitrarily-small disturbances can drive spontaneous self-organization into simple patterns with finite amplitude. Emergent structures have a universal quality (including hexagons, honeycombs, squares, stripes, rings, spirals, vortices), and they are characterized by a single dominant scalelength that is associated with the most-unstable Fourier component.

In this paper, we extend Turing's ideas to three wave-based discrete nonlinear optical models with a wide range of boundary conditions. In each case, the susceptibility of the uniform states to symmetry-breaking fluctuations is addressed and we predict a threshold instability spectrum for static patterns that comprises a multiple-minimum structure. These Turing systems are also studied numerically, and we uncover examples of simple *and complex* (i.e., fractal, or multi-scale) pattern formation.

Keywords: discrete optical models, Turing instabilities, spontaneous patterns.

1 Nonlinear Fabry-Pérot Cavity

We begin by considering a thin slice of nonlinear (diffusive Kerr-type) material that is sandwiched between two partially reflecting mirrors. Light injected from an external source bounces back and forth between the mirrors, and passes through the slice on each transit. This nonlinear Fabry-Pérot (FP) cavity is the epitome of a complex optical system, involving the interplay between diffraction, diffusion, counterpropagation, and cavity feedback (periodic pumping, mirror losses, interferomic mistuning, time delays).

The Turing threshold instability spectrum [1] for the FP cavity is generally found to possess a *discrete island* structure as opposed to the *lobes* of the closely-related single feedback-mirror (SFM) system [2] [see Fig. 1(a)]. By controlling the spatial frequencies that are allowed to propagate, simulations have predicted a range of simple patterns when the cavity is initialized with a perturbed plane wave solution above threshold [see Figs. 1(b)–(e)]. We will also present evidence of a *spontaneous fractal-generating* capacity [see Figs. 1(f)–(i)]. This strikingly new regime of pattern formation is tightly connected to a hierarchy of comparable minima in the threshold spectrum [3].

2 Discrete Nonlinear Ring Cavity

The first discrete nonlinear-Schrödinger (dNLS) context to consider involves confining a waveguide array to a ring cavity whose total length is L. The complex amplitude $E_{n,m}$ in channel (n,m) is coupled to those in its nearest neighbours, and the host medium has a local cubic (Kerr-type) nonlinearity parametrized by χ:

$$i\partial_z E_{n,m} + c(E_{n+1,m} + E_{n-1,m} + E_{n,m+1} + E_{n,m-1} - 4E_{n,m}) + \chi L|E_{n,m}|^2 E_{n,m} = 0, \tag{1}$$

where $0 \le z \le 1$ is the (local) longitudinal coordinate and c is the coupling constant. Equation (1) is supplemented by a boundary condition applied at the start of each transit, $E_{n,m}(0) = t_1 E_{\text{in}} + r_1 \exp(i\delta) E_{n,m}(1)$, where E_{in} is the external plane wave pump, $r_1^2 + t_1^2 = 1$ connect the (intensity) reflectivity and transmissivity coefficients, and δ determines the linear mistuning.

We will report on our analysis of model (1), and discuss the (periodic in K) multi-Turing threshold spectrum. Results from simulations will be presented, demonstrating simple pattern emergence in arrays with one and two transverse dimensions. Our approach goes beyond mean-field descriptions of related dNLS cavity models

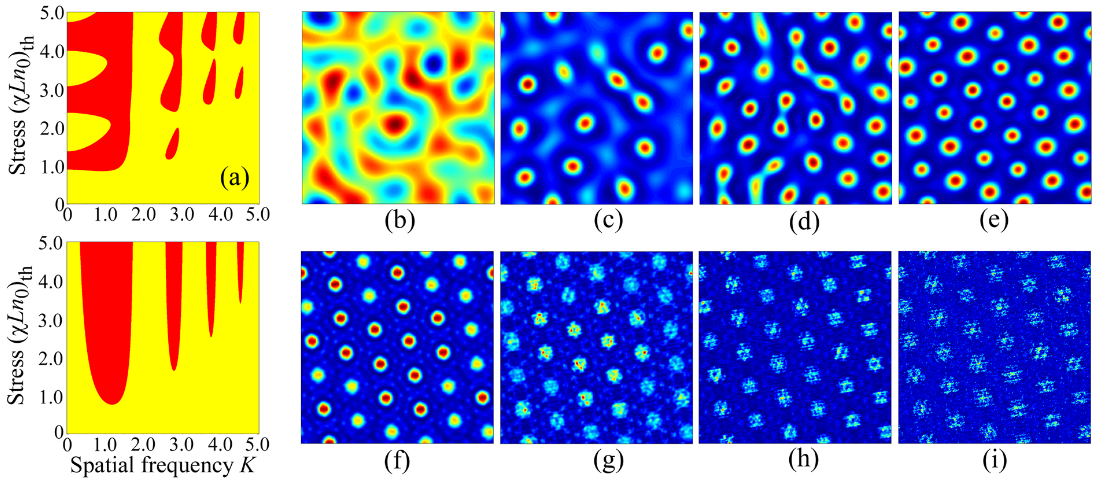

Figure 1: (a) Multi-Turing threshold instability spectrum for an FP cavity (top) and its corresponding SFM system (bottom). Emergence of a static hexagon pattern from a perturbed plane-wave solution in the FP cavity [(b)–(e)], and its transformation towards a volume-filling fractal [(f)–(i)].

[4], which are analytically more tractable at the expense of averaging propagation effects.

3 Discrete Counterpropagating Waves

We have also re-considered the fundamental optical configuration of counterpropagating (CP) laser beams [5] but within the context of nonlinear waveguide arrays. A dNLS-type model has been proposed for describing the evolution of forward and backward envelopes ($F_{n,m}$ and $B_{n,m}$, respectively) in a Kerr-type medium of length L, and which is essentially a discrete analogue of the continuum equations:

$$\begin{aligned} i(\partial_z F_{n,m} + \partial_t F_{n,m}) + c(F_{n+1,m} + F_{n-1,m} \\ + F_{n,m+1} + F_{n,m-1} - 4F_{n,m}) \\ + \chi L(|F_{n,m}|^2 + G|B_{n,m}|^2)F_{n,m} = 0, \end{aligned} \tag{2a}$$

$$\begin{aligned} i(-\partial_z B_{n,m} + \partial_t B_{n,m}) + c(B_{n+1,m} + B_{n-1,m} \\ + B_{n,m+1} + B_{n,m-1} - 4B_{n,m}) \\ + \chi L(|B_{n,m}|^2 + G|F_{n,m}|^2)B_{n,m} = 0, \end{aligned} \tag{2b}$$

where t is the time coordinate and $1 \leq G \leq 2$ characterizes the standing-wave interference grating in the medium [5]. The perturbative method used to investigate the stability of the uniform states of model (2) (subject to equal-intensity constant plane wave pump fields) is reminiscent of that for the continuum model [6], and involves a boundary-value problem whose solution requires the exponentiation of a 4×4 matrix. We will report on the multi-Turing threshold instability spectrum for this new class of dNLS system, and present a set of simulations to illustrate pattern formation.

References

[1] A. M. Turing, The chemical basis of morphogenesis, *Philosophical Transactions of the Royal Society of London B: Biological Sciences* **237** (1952), pp. 37–72.

[2] G. D'Alessandro and W. J. Firth, Spontaneous hexagon formation in a nonlinear optical medium with feedback mirror, *Physical Review Letters* **66** (1991), pp. 2597–2600.

[3] J. G. Huang and G. S. McDonald, Spontaneous optical fractal pattern formation, *Physical Review Letters* **94** (2005), art. no. 174101.

[4] O. A. Egorov, F. Lederer, and Y. S. Kivshar, How does an inclined holding beam affect discrete modulational instability and solitons in nonlinear cavities?, *Optics Express* **15** (2007), pp. 4149–4158.

[5] W. J. Firth and C. Paré, Transverse modulational instabilities for counterpropagating beams in Kerr media, *Optics Letters* **13** (1988), pp. 1096–1098.

[6] J. B. Geddes, R. A. Indik, J. V. Moloney, and W. J. Firth, Hexagons and squares in a passive nonlinear optical system, *Physical Review A* **50** (1994), pp. 3471–3485.