

Vector dark solitons in systems with spatiotemporal dispersion and cubic nonlinearity: solutions and stability, transformations and relativity

L. A. Carter¹, J. M. Christian^{1,*}, G. S. McDonald¹, P. Chamorro-Posada²

¹Materials & Physics Research Centre, University of Salford, Greater Manchester M5 4WT, UK

²ETSI Telecomunicacion, Universidad de Valladolid, E-47011 Valladolid, Spain

*Email: j.christian@salford.ac.uk

Abstract

The origin of conventional models for nonlinear optical pulse propagation lies in the ubiquitous slowly-varying envelope approximation (SVEA) accompanied by a Galilean boost to a local-time frame. While such a near-universal procedure typically results in a simpler (parabolic) model of the nonlinear Schrödinger-type, in reality a more subtle but less well-explored class of wave equation underpins the propagation problem.

In reassessing the way conventional models treat the linear part of the wave operator (by omitting the “SVEA + Galilean boost” device), we have uncovered a powerful and elegant framework for describing time-domain nonlinear optical phenomena that has strong overlaps with Einstein’s special theory of relativity. Here, we generalize our scalar modelling to accommodate two coupled waves experiencing spatiotemporal dispersion and cubic nonlinearity. A range of analytical methods has been deployed to derive new families of phase-topological vector solitons (dark-bright and dark-dark), and their stability properties investigated using new numerical algorithms.

Keywords: vector solitons, Galilean transformation, spatiotemporal dispersion.

1 Beyond Slowly-Varying Envelopes

Menyuk’s seminal analysis from 1987 [1] has undeniably helped lay the foundations of today’s understanding of coupled waves in nonlinear optical systems. Formulated in terms of slowly-varying envelopes and Galilean boosts, scores of vectorized Schrödinger-type models have been proposed and studied over nearly three decades. While the SVEA survives as a theoretical mainstay of modelling wave-based nonlinear systems, Biancalana and Creatore [2] have pointed out that there exist modern contexts in condensed-matter physics where its validity may be reassessed. In particular, they assert that spatial dispersion (for example, related to light-exciton

coupling inside superlattice structures) is not necessarily well-described by the SVEA.

In this paper, we generalize our scalar approach to pulse evolution [3] by accommodating the simultaneous propagation of two coupled optical waves which may represent, for instance, the excitations in two orthogonal polarizations of a fibre waveguide whose core has a cubic (Kerr-type) nonlinearity. Moreover, the mathematical context of finding exact solitary solutions to universal hyperbolic or elliptic envelope equations (as generalizations of simpler parabolic models) is both timely and novel.

2 Coupled Governing Equations

We consider a pair of normalized fully second-order (in space and time) coupled equations describing optical waves u_j , where $j = 1$ and 2 :

$$\begin{aligned} \kappa_j \frac{\partial^2 u_j}{\partial \zeta^2} + i \left(\frac{\partial u_j}{\partial \zeta} + \alpha_j \frac{\partial u_j}{\partial \tau} \right) + \frac{s_j}{2} \frac{\partial^2 u_j}{\partial \tau^2} \\ + (|u_j|^2 + \sigma |u_{3-j}|^2) u_j = 0. \end{aligned} \quad (1)$$

Here, τ and ζ are the dimensionless time and (longitudinal) space coordinates in the laboratory frame, respectively, α_j is related to the (linear) group velocity, spatial dispersion is quantified by $|\kappa_j| \ll \mathcal{O}(1)$, group-velocity dispersion (GVD) by s_j [positive and negative values for anomalous- and normal-GVD regimes, respectively, and typically with $|s_j| = \mathcal{O}(1)$], and σ determines the strength of nonlinear coupling (i.e., the cross-phase modulation terms $|u_{3-j}|^2 u_j$).

Frame-of-reference considerations take centre stage in our approach, and space-time coordinate transformations dominate much of the analysis [3]. Conventional pulse theory emerges asymptotically from equation (1) and its solutions, in much the same way that Newtonian mechanics corresponds to the low-speed limit of Einstein’s relativistic physics (e.g., the velocity combination rule for spatiotemporal pulses is akin to that for particles in special relativity).

Operationally, one can recover (a generalized version of) Menyuk's classic vector model [1] alongside all its predictions by: (i) assuming $|\kappa_j \partial_{\zeta} u_j| \ll |\partial_{\zeta} u_j|$, and (ii) Galilean-boosting to a local-time frame moving at an *averaged* group speed by introducing a new pair of coordinates $\tau_{\text{loc}} = \tau - \alpha\zeta$ and $\zeta_{\text{loc}} = \zeta$, where $\alpha \equiv (\alpha_1 + \alpha_2)/2$:

$$i \left(\frac{\partial u_1}{\partial \zeta_{\text{loc}}} + \delta \frac{\partial u_1}{\partial \tau_{\text{loc}}} \right) + \frac{s_1}{2} \frac{\partial^2 u_1}{\partial \tau_{\text{loc}}^2} + (|u_1|^2 + \sigma |u_2|^2) u_1 \simeq 0, \quad (2a)$$

$$i \left(\frac{\partial u_2}{\partial \zeta_{\text{loc}}} - \delta \frac{\partial u_2}{\partial \tau_{\text{loc}}} \right) + \frac{s_2}{2} \frac{\partial^2 u_2}{\partial \tau_{\text{loc}}^2} + (|u_2|^2 + \sigma |u_1|^2) u_2 \simeq 0, \quad (2b)$$

where $\delta \equiv (\alpha_1 - \alpha_2)/2$ parametrizes a mismatch in the group velocity. Implementing such a transformation *without* first making the SVEA hinders rather than helps the analysis of spatiotemporal effects (e.g., by generating mixed-derivative terms that can be awkward to interpret physically) [3].

3 Dark-Bright & Dark-Dark Solitons

Exact analytical dark-bright and dark-dark solitons of Eq. (1) will be presented, derived by combining ansatz methods with linear transformations in the space-time plane. Such solutions (with their nontrivial phase topology) offer the greatest potential impact in the arena of future optical device designs when their continuous-wave (cw) backgrounds are not susceptible to spontaneous fluctuations.

A vector generalization of our scalar linear analysis [4] has been deployed to quantify the modulational instability spectrum for cw solutions (obtained by solving an eighth-degree polynomial characteristic equation) that have been subjected to small disturbances. Computations have confirmed theoretical predictions for the most-unstable frequency in the system.

Finally, we will summarize results from numerical perturbative analyses demonstrating the instability of exact conventional dark-bright and dark-dark solitons [5] when used as initial conditions in Eq. (1) (see Fig. 1). In contrast, we fully expect the more general spatiotemporal dark-type vector solitons to be relatively robust entities exhibiting *waves of invariant form* characteristics in regimes where the cw backgrounds are modulationally stable.

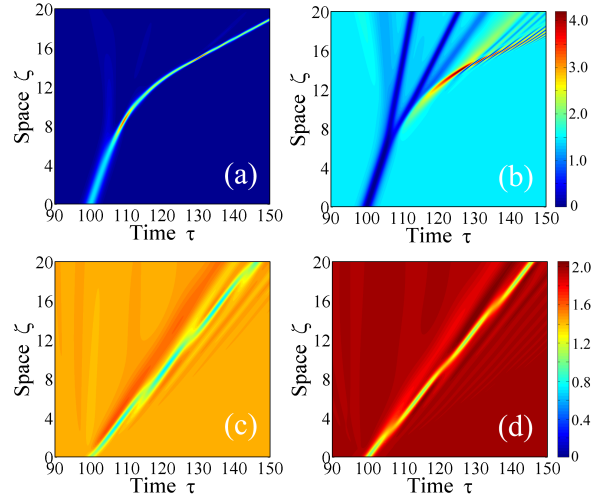


Figure 1: Instability of exact conventional dark-bright [(a) bright $|u_1|^2$, (b) dark $|u_2|^2$] and dark-dark [(c) $|u_1|^2$, (d) $|u_2|^2$] solitons when used as initial conditions in equation (1) [$\kappa_1 = 1.0 \times 10^{-3}$, $\kappa_2 = 2.5 \times 10^{-3}$, and $\sigma = 2/3$]. Pulse splitting, snaking, and radiation shedding are observed.

References

- [1] C. R. Menyuk, Nonlinear pulse propagation in birefringent optical fibres, *IEEE Journal of Quantum Electronics* **QE-23** (1987), pp. 174–176.
- [2] F. Biancalana and C. Creatore, Instabilities and solitons in systems with spatiotemporal dispersion, *Optics Express* **16** (2008), pp. 14882–14893.
- [3] J. M. Christian, G. S. McDonald, T. F. Hodgkinson, and P. Chamorro-Posada, Spatiotemporal dispersion and wave envelopes with relativistic and pseudorelativistic characteristics, *Physical Review Letters* **108** (2012), art. no. 034101.
- [4] J. M. Christian, G. S. McDonald, T. F. Hodgkinson, and P. Chamorro-Posada, Wave envelopes with second-order spatiotemporal dispersion II. Modulational instabilities and dark Kerr solitons, *Physical Review A* **86** (2012), art. no. 023839.
- [5] Y. S. Kivshar and S. K. Turitsyn, Vector dark solitons, *Optics Letters* **18** (1993), pp. 337–339.