# Diffraction of Weierstrass scalar fractal waves by circular apertures: symmetry and patterns, complexity and dimension

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## Abstract

The diffraction of plane waves from simple hard-edged apertures constitutes a class of boundary-value problem that is well understood in optics, at least within the scalar approximation. Similarly, the diffraction of such waves from fractal apertures (amplitude or phase masks possessing structure across decades of spatial scale) has also received much attention in the literature. But the diffraction of *fractal waves* by *simple apertures* constitutes an entirely new paradigm (in optics particularly, and wave physics more generally) that remains largely unexplored.

Here, we consider the diffraction of fractal waves by a hard-edged circular aperture using a range of analytical and semi-analytical methods. Fast computational techniques are used to obtain Fresnel (near field) diffraction patterns, and specialist software assists with the investigation of their properties. Key issues to be addressed include the fractal dimension of diffraction patterns, and the asymptotic emergence of Fraunhofer (far field) predictions in an appropriate limit.

**Keywords:** Fresnel diffraction, fractal waves, Weierstrass function.

## 1 Fractal Diffraction

We have recently proposed fractal diffraction as a context of fundamental physical importance with enormous scope for potential applications [1]. Preliminary analyses investigated a Weierstrass function [2] for modelling fractal illumination of the most elementary aperture imaginable: the infinite single slit. The Weierstrass function has an intuitive interpretation, comprising a set of periodic patterns whose amplitudes and spatial frequencies are connected in a very particular way. Moreover, each constituent pattern scale can be constructed from a superposition of two interfering plane waves.

The diffraction of a uniform wavefront by a circular aperture is another classic wave-based problem that has well-known solutions, both in Fresnel and Fraunhofer regimes [3]. While we consider the optical analogue with fractal illumination, our results are expected to be readily applicable to other fields, such as acoustics [4].

## 2 Diffraction Integral

The diffraction of a scalar optical field $U(r)$ by a circular aperture of radius $a$ is routinely described by the paraxial wave equation. For a hard-edged circular aperture, and where the incident wave $U_{\mathrm{in}}$ is azimuthally invariant, the diffracted wave at a distance $L$ beyond the aperture is given by the formal solution,

$$U(r) = \frac{2\pi N_{\mathrm{F}}}{i} \exp\left(i\pi N_{\mathrm{F}} r^2\right) \times \int_0^{+1} d\rho\, \rho J_0(2\pi N_{\mathrm{F}} r \rho) \exp\left(i\pi N_{\mathrm{F}} \rho^2\right) U_{\mathrm{in}}(\rho), \tag{1}$$

where the radial coordinate $r$ is measured in units of $a$. In this representation, the diffraction pattern is uniquely parametrized by the aperture Fresnel number $N_{\mathrm{F}} \equiv a^2/\lambda L$.

## 3 Weierstrass Illumination

Here, we consider an illuminating field that has the form of an azimuthally invariant Weierstrass wave such that

$$\frac{U_{\mathrm{in}}(r)}{U_0} = 1 + \epsilon \sum_{n=0}^{N} \frac{1}{\gamma^{(2-\mathrm{D}_0)n}} \cos(\kappa_n r + \phi_n), \tag{2}$$

where $U_0$ is a uniform plane-wave amplitude and $\epsilon$ controls the strength of the fractal modulation. The spatial frequencies in Eq. (2) form a Weierstrass spectrum given by $\kappa_n \equiv 2\pi(a/\Lambda)\gamma^n$, where $\gamma > 0$ is a free parameter, and the set of phases $\phi_n$ may be either deterministic or random. When $N \to \infty$, the number $1 < \mathrm{D}_0 \leq 2$ corresponds to the Hausdorff-Besicovich dimension of $U_{\mathrm{in}}$ with values approaching 2 giving an increasingly complex fractal curve.

The illuminating field is bandwidth-limited with a cut-off at $n = N$; the spatial scalelengths in Eq. (2) then range from the largest, $\Lambda$, to the smallest, $\Lambda\gamma^{-N}$. Placing a restriction on the number of spatial scales in $U_{\rm in}$ is important for two principal reasons. Firstly, no physical object can possess structure down to arbitrarily-small scales (since finite-size effects will eventually come into play). Secondly, there tends to exist a high-frequency cut-off beyond which spatial scales cannot contribute to the diffracted *intensity* pattern [5] (so the basis on which one introduces finite-bandwidth considerations, and selects a value for $N$, are rooted in diffraction theory).

## 4 Diffraction of Fractal Waves

Earlier analyses focusing on infinite-slit geometries have tended to use Young's edge waves as convenient spatial structures for understanding and quantifying fractal diffraction phenomena [1]. While edge waves can be used for circular apertures and uniform illumination [3], such a formalism is not quite so readily deployed for fractal illumination (despite the one-dimensional nature of the system) and one must instead consider the diffraction integral more directly. Substitution of Eq. (2) into Eq. (1) yields a formal expression for $U(r)$ as a linear superposition of patterns with different scalelengths,

$$\frac{U(r)}{U_0} = \frac{2\pi N_{\rm F}}{i}\exp\left(i\pi N_{\rm F}r^2\right)\Bigg[P(r;0,0) + \epsilon\sum_{n=0}^{N}\frac{1}{\gamma^{(2-{\rm D}_0)n}}P(r;\kappa_n,\phi_n)\Bigg], \qquad (3a)$$

where $P$ is given by the integral

$$P(r;A,B) \equiv \int_0^{+1} d\rho\,\rho J_0(2\pi N_{\rm F}r\rho) \times \exp\left(i\pi N_{\rm F}\rho^2\right)\cos(A\rho+B) \qquad (3b)$$

and $P(r;0,0)$ fully describes the diffraction pattern in the classic plane-wave problem [3].

A selection of new results will be discussed, with a combination of analytical methods and specialist fractal analysis software [6] identifying trends in the Fresnel patterns (see Fig. 1). Attention is paid to the role played by $N_{\rm F}$ in characterizing these patterns, and we consider different self-affine measures of *dimension* (such as roughness-length, variogram, and rescaled-range). Asymptotic emergence of classic Fraunhofer results will also be demonstrated, and a wide range of potential applications highlighted.

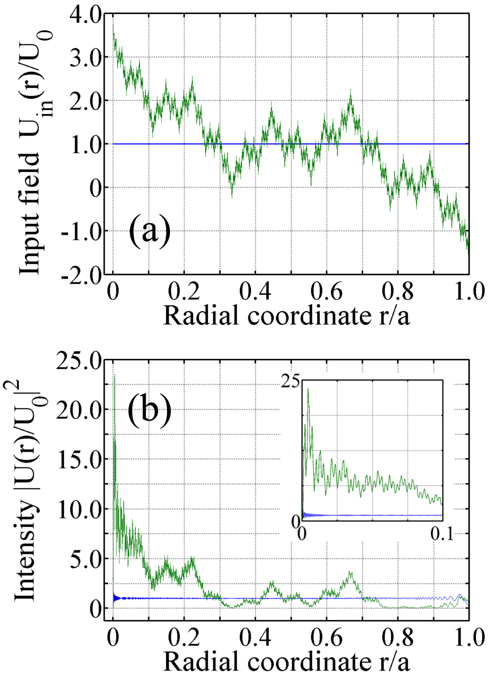


**Figure 1:** (a) Illumination across a circular aperture corresponding to a plane wave (blue) and a Weierstrass function with fractal dimension $D_0 = 1.6$ and $\gamma = 3$ (green). (b) Diffracted intensity patterns when $N_{\rm F} = 1000$.

## References

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