CAPITAL REPLACEMENT MODELLING WITH A FIXED PLANNING HORIZON

Mohamed Helmy Hashem

School of Accounting, Economics and Management Science

University of Salford, Salford, UK

Business and Informatics Research Institute.

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Abstract

For equipment or plant replacement, when to replace an existing plant, fleet or a part of it, is one of the main concerns in decision-making. The thesis considers this decision-making problem using capital replacement models with a fixed planning horizon, and we look at the behaviour of optimal policy in this context. Application of the models is considered and we compare replacement models with a fixed planning horizon with replacement models with a variable planning horizon models comprising of two cycles. Capital replacement modelling in general and previous work done in the field are reviewed. The main work of this thesis is the study of the behaviour of optimal replacement policy for a single equipment/fleet over a fixed planning horizon, with a numerical investigation of the behaviour for non-like-with-like replacement. This is extended to describe the behaviour of optimal policy for replacement of a mixed fleet. A case study is presented that applies the fixed planning horizon model to a bus fleet; this fleet is operated by a Malaysian inter-city bus company. Finally we consider the challenger problem. Throughout, we recommend the use of a fixed planning horizon model rather than a two cycle variable-horizon model. The rent criterion is also our favoured criterion for decision-making; the rent criterion exists and is well behaved for all the models described. A dynamic programming approach is implemented for the like-with-like replacement problem over a fixed planning horizon for comparison with the economic life modelling approach of this thesis. We discuss the use of the different replacement decision models for supporting replacement decision-making in practical contexts.

CHAPTER 1

CHAPTER 1

Introduction to Capital Replacement Modelling

1.1. Introduction

Capital replacement has long been, and will remain a topic of interest. This is because it is concerned with strategic planning of capital expenditure. The objective is to spend the capital in reasonable manner while minimising (maximising) expenses (profit). Replacement policy in general deals with all sorts of items. The approach, however, is different when dealing with a component than with a plant. In the component replacement case, the factors of interest are generally the distribution of the time to failure, the cost of preventive replacement and the cost of failure, and the long run cost per unit time. For large expensive plant on the other hand, economic factors such as discount factor, rate of inflation, interest rate and tax parameters are considered; the implication is that capital expenditure is planned over a certain specified period, the planning horizon. The planning horizon is expressed in months or years and may be of finite or infinite length.

For plant replacement when to replace a current plant being used, a fleet or a part of it, is one of the main concerns in decision-making. Another concern for the decision-maker or manager is the choice of the new plant to purchase. This of course is an important issue for the decision-makers, but is often out of control of the modellers when the choice is fixed in advance (e.g. political decision). A good policy can lead to large savings in the total cost of operating a plant or a fleet of plant. To achieve this goal different approaches are used, which can be based on either the experience of the operator or modelling approach or a combination of both.

When an equipment continues over an extended time it may deteriorate and a decision regarding the need for replacement should be made supposing that the optimal lifetime ends when the marginal revenues from immediate replacement are the same as those from the later replacement (Verheyen, 1978). This decision is influenced by the increasing maintenance cost per unit time and the effect of spreading the capital cost (replacement cost) over a longer period. Similarly the need for replacement may be due to failure or impending failure; the operating efficiency is not considered to change with use, but replacement is required due to a failure. After failure, no decision is required since repair or replacement is necessary. However, it may be economically advantageous to replace or repair on a scheduled basis before failure occurs, so yearly replacement results in decreased cost. Thus the problem becomes one of determining an optimal replacement interval.

It is a very important fact that technological improvements may render equipment undesirable simply because they are no longer technologically competitive with newer developments. Thus it is feasible to replace equipment currently operating satisfactorily in a mechanical sense with newer, perhaps more expensive equipment. Analysis of problems of this type is simplified if improvement in technology is assumed to occur incrementally each year. In fact this may be appropriate on a short-term basis since technological improvements typically occur at discrete points in time. For example, several years may pass before a significant improvement is made in competing equipment when a newer model is marketed.

We should, however, emphasise that modelling in general can really only support decision makers and guide policy making. We do not claim that modelling can replace the role of the experienced manager/decision-maker. The modeller and decision-maker must work together if such models are to be adopted in practice.

Capital replacement models can be classified (Scarf & Christer, 1997) as: cost limit models (e.g. Jardine et al 1976); or economic life models (e.g. Eilon et al 1966, Christer & Goodbody 1980). Economic life models (our subject of study) may be specified according to the length of the planning horizon. The length of the planning horizon as we mentioned above may be: infinite; finite but variable with a number of cycles determined by the model; or fixed with variable number of cycles (replacements) influenced by the length of the horizon.

Our work is concerned with the modelling aspects of the replacement decision, which can support the experience of the operator. We describe capital replacement models which attempt to reflect the actual replacement problem. We have presented a mathematical model of the replacement decision with a fixed planning horizon. The fixed planning horizon approach is one which more closely resembles the operational context and also overcomes the difficulties in the variable planning horizon approach. In the fixed planning horizon approach, there is a variable number of cycles (replacements) each with a variable length influenced by the specified length of the horizon. In order to gain some insight into the behaviour of models with a fixed planning horizon, we study the behaviour of optimal policy of a simple model. This is done using certain restrictive assumptions regarding the number of replacements, the form of maintenance cost per unit time for current and future equipment, and the replacement cost and resale values. We are particularly concerned with the effect of the planning horizon length on the number of cycles (replacements) and the lengths of these cycles. We would expect the behaviour of the model to carry over into the more complex models, which would perhaps be used in particular applications.

The structure of this work is as follows. In chapter 2 we present some preliminary considerations which are incorporated when establishing capital replacement models: the maintenance cost per unit time and how to model the maintenance cost per unit time in different forms according to the data available; the discount factor either constant or variable and its influence on the optimal policy; tax rates and their effect on the optimal policy; the total discounted cost and rent as two alternative criteria for replacement modelling; the planning horizon over which replacement is performed; the penalty cost (Christer & Scarf, 1994) as a factor in the replacement problem; also resale values for the equipment that becomes obsolete, aged or incurs high maintenance cost per unit time (this resale value can be formulated in different models as a function of age and one of these models is presented).

In chapter 3 we present a review of capital replacement models and considerations concerned with it. In this review most of the authors study the economic life of an equipment under various circumstances. We present capital replacement models in general, considering infinite horizon models and finite horizon models. In the infinite horizon replacement models we present like-with-like replacement as the earliest model developed. Two different criteria of this model are presented; these are the total discounted cost and the equivalent rent. A computation approach is presented to determine the economic life of an equipment in the case of buying new. An extension of this model is also presented for the case of buying an old equipment (second hand). In the finite horizon models we present variable length planning horizon models and fixed horizon models. In the variable length planning horizon models we present the two-cycle model which was developed by Christer & Goodbody (1980). The behaviour of this model is illustrated and a computation approach to determine the average total discounted cost and equivalent rent is described. A numerical example is presented to show the replacement procedure over two cycles. Extensions of this model are also described, namely the case of sub-fleet replacement and retirement of sub-fleet as spares. For the fixed horizon models we consider the fixed length of the planning horizon as a control variable and the variable number of cycles and each of the variable cycle lengths as decision variables. We discuss a capital replacement model with a fixed planning horizon and illustrate the behaviour of the model. We describe how the optimum policy is affected by the length of the horizon.

In chapter 4 we study the behaviour of optimal policy of some simple models. This study is done for the infinite horizon model of like-with-like replacement. Also we study the behaviour of optimal policy for fixed horizon models and variable horizon models of like-with-like replacement and non-like-with-like replacement in different cases. First and second order approximations to the maintenance cost per unit time model are used to obtain optimal closed solutions. Numerical investigation for the behaviour of optimal policy is presented for general cases. The mathematical relationship between the fixed and variable planning horizon models is also presented. Finally, a dynamic programming approach is presented for a fixed planning horizon model which is not restricted to at most two replacements. Computational results are presented for like-with-like replacement.

In chapter 5 we study the behaviour of optimal policy for a mixed fleet. We present the many subfleets case each with a single item. Also we present the two subfleets case numerically in order to study the behaviour of optimal policy for different replacement scenarios. We describe the many subfleets problem with up to two replacements over the fixed planning horizon. Finally, in this chapter, we describe the many subfleets problem with many items in each subfleet.

In chapter 6 we give an application of the fixed planning horizon model described in chapter 4. The model is applied to a fleet of a large Malaysian inter-city bus company. The fleet is mixed and compromises 5 sub-fleets of different types and ages. Data on maintenance cost per unit time were collected and models for maintenance cost per unit time are fitted.

Finally, in chapter 7 we present the "challenger" problem with a fixed planning horizon and a variable planning horizon. Comparison between the results using a fixed planning horizon and a variable planning horizon is presented in order to illustrate the differences between the alternative replacement decision modelling approaches. For a complete view on the challenger problem, we describe a dynamic programming approach to study the challenger problem numerically.

CHAPTER 2

CHAPTER 2

Preliminary Considerations

In this chapter we discuss a number of factors which are usually considered in capital replacement.

2.1. Maintenance costs

The greatest uncertainty in many maintenance and replacement decision problems lies with the prediction of future maintenance costs, and with the adequacy of the data relating to the maintenance history. For example, for vehicle replacement modelling, it would be an ideal situation where we have sufficient data for building a mathematical model in which maintenance cost is related to age and usage. This is often difficult in practice. The under estimation of costs for older vehicles is inevitable due to selection bias (Scarf, 1994); this is because the older vehicles currently on the road must by definition be the "good" ones. Maintenance cost in general comprises the cost of, for example, parts, labour and lubricants. Other operating costs would include costs of fuel, for example. In our work, we assume that the operating cost is included in the maintenance cost. In order to model the maintenance cost per unit time for a plant, data for the old and the new plant need to be available and as reliable as possible. This will reduce the uncertainty in maintenance and replacement decision policies. Typically, simple linear, exponential or power law type regression models are fitted to the data and used for simple prediction. When maintenance costs associated with new technology may be unknown, a simple way of proceeding would be to model the maintenance costs per unit time of new plant as old except for a multiplicative factor, the ratio of the cost of new plant to old one. A refinement of this approach was used by Christer (1988).

An example of a model of maintenance cost per unit time is the power law type which we use in this work for the most part. The model is expressed as a power law function by

$$M(t) = \alpha t^{\beta}, \tag{2.1}$$

or can be expressed linearly as

$$\log M(t) = \log\alpha + \beta \log t, \tag{2.2}$$

where t is the age/usage of plant, $M(t)$ the maintenance cost per unit time at age t and the parameter β is the slope (on the log-log-scale) of the regression and represents the increase in the log of the age-dependent maintenance cost per unit of log time. The coefficient α is the intercept (on the log-log scale) of the regression.

This model is fitted to the data in the case study in chapter 6. These data relate to the late 1992 maintenance costs of a fleet of five models of Malaysian buses as illustrated in Figure 2.1. Here a common α with different βs was fitted, although other maintenance cost per unit time models may be appropriate.

Other possible maintenance cost per unit time models are the simple linear and exponential regression types. Eilon et al. (1966) used a linear regression to fit the maintenance cost data for a sample of 10 fork lift trucks, giving in the following expression

$$f(t) = a + bt, \tag{2.3}$$

where t is the age of the truck after its purchase new, $f(t)$ is the maintenance cost per unit time at age t ; the parameter b is the slope of the regression and represents the increase in the age-dependent maintenance cost per unit time. In his paper Scarf (1994) described an exponential form for the maintenance costs for Ford Escorts (mark III and IV), given in the following expression

$$M(t) = \exp(a + bx), \tag{2.4}$$

where x is the total mileage of the vehicle.

Christer (1988) used a different method and this method consisted of collecting data on the maintenance cost of old plant and more limited data on that of new plant, and then the ratio of the average cumulative cost of the new plant to the old plant was determined. The estimate of the jth quarter for the new vehicle was obtained using a sample of 8 vehicles from each type and was formulated as follows

$$C_1(j) = y(j) \times C_0(j), \tag{2.5}$$

where $y(j)$ is the value of the quadratic function fitted to the ratio of the maintenance cost of new vehicle to the old in the jth quarter, $C_1(j)$ and $C_0(j)$ are respectively the quarterly estimate of the maintenance cost per unit time of the new and the maintenance cost per unit time of the old vehicle in the jth quarter.

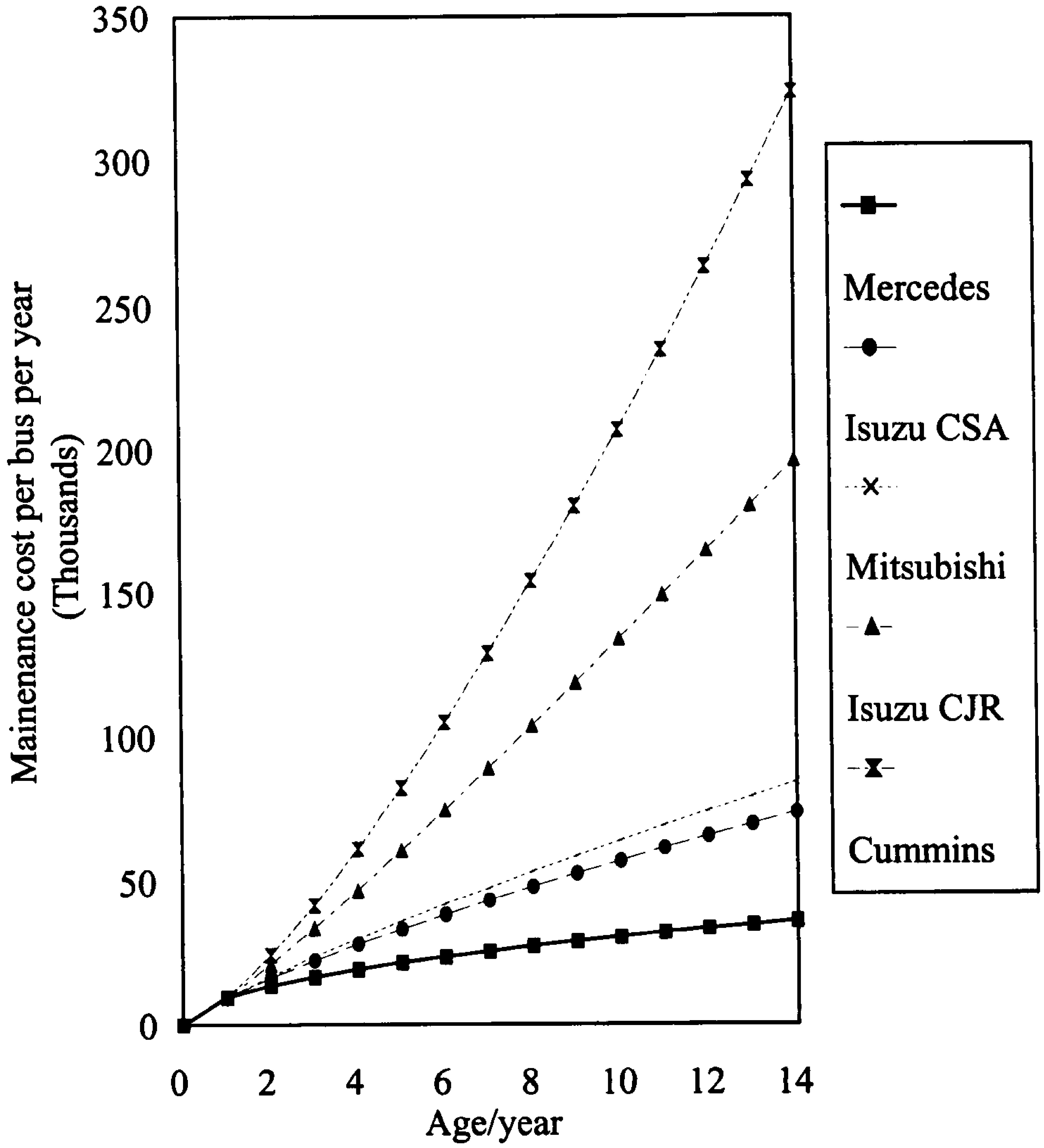


Figure 2.1. Maintenance cost rates for Mercedes $M(t) = 9680\,t^{0.50}$, Isuzu CSA $M(t) = 9680\,t^{0.77}$, Mitsubishi $M(t) = 9680\,t^{0.82}$, Isuzu CJR $M(t) = 9680\,t^{1.14}$, Cummins $M(t) = 9680\,t^{1.33}$.

2.2. Discounting

The discounting of future costs (French, 1988) is necessary when the future costs are required to be presented in terms of present values. Although discounting of future costs is not necessary for the determination of optimum policy, it may influence the replacement decision (Scarf, 1994). The same conclusion was obtained by Kobbacy and Nicol (1994). Discounting is appropriate because, when an individual retains a capital sum for a certain period before purchasing, the value (present value) of the sum would be different from its value if purchase was made immediately. It is clear that the future value of an amount of money differs from its present value. This is due to the combination of the inflation and the interest rate or any other rate of return on investment, provided the money is invested sensibly. Thus it is necessary to scale all future costs to their present value by means of a discount factor or rate. Decision makers would normally be in a position to decide a discount rate appropriate to themselves (Ehrhardt, 1994). The discount factor υ may be defined by $(100+i)/(100+j)$ where i and j are the inflation and the internal rate of return respectively (Christer and Goodbody, 1980). It should be noted that, although the discount factor is assumed to be constant, it corresponds, in fact, to different inflating and discounting situations. In practice, the discount factor is known to be time dependent and does not remain constant for any length of time (Kobbacy and Nicol, 1994), since inflation, interest rate or any internal rate of return vary constantly. As in Christer and Waller (1987a) the discount factor in any year t, $\upsilon(t)$, may be expressed as

$$\upsilon(t) = (100+i_t)/(100+j_t), \tag{2.6}$$

where i_t and j_t are the inflation rate and any rate of return in year t, respectively. The discount factor over K years, taken to end of the year r_K, is given by

$$r_K = \prod_{t=1}^{K} \upsilon(t). \tag{2.7}$$

The discount factor to the midpoint of the year K, denoted by d_K, is given by

$$d_K = \prod_{t=1}^{K-1} \upsilon(t)\sqrt{\upsilon(K)}\,. \qquad (2.8)$$

These discount factors (equations 2.7 and 2.8) can be easily introduced into the models (Christer and Goodbody, 1980). Kobbacy and Nicol (1994) used the same discount factor (equation 2.8) for a study related to replacement of commercial vehicles (tractor units) in the UK. It was shown that the discount factor had no influence on the optimal age (age at which replacement takes place). In their paper, Hawkins and Nasoni (1977) developed variable discounting as an approach to dealing with uncertainty. They introduced a linear approximation to the variable discount function. It was concluded by authors that it is better to use a relatively simple replacement model than to introduce complicated changes such as variable discounting which do not usually have a significant effect on the optimal policy.

2.3. Tax considerations

Tax rates are determined by the appropriate tax authority and are subject to changing laws on taxation; change occurs for different reasons which are either political or economic. Therefore modellers should be aware of tax factors, especially when new tax legislation is introduced. Tax considerations have not been taken into account in our work, although it may be incorporated where appropriate. Eilon et al. (1966) considered corporation tax and the system of allowances for capital expenditure on equipment; these tax considerations had an influence upon the optimal decision. Tax was also considered by Christer and

Waller (1987a), but it appeared that following the 1984 Finance Act, which simplified the tax allowance scheme, there was no major effect on the optimal decision. Tax allowances were also encompassed within the models of Christer and Goodbody (1980), but appeared to have no influence upon the replacement decision. It is worth pointing out that generally maintenance costs are considered as expenses against profit, therefore tax is not paid on them (Eilon et al., 1966). Also depreciation is taken into account in tax payment.

2.4. Total discounted cost versus rent criteria

Various criteria for replacement may be considered in the modelling. Policy is then optimised with respect to the chosen criterion. One of these criteria may be the total discounted cost; this means that all future costs are discounted to present value using a certain discount factor over a certain period. Another is the rent criterion where rent is the sum payable, per unit time interval over the length of a certain horizon, which should be necessary to meet the total discounted cost over that horizon (Christer & Goodbody, 1980). The two criteria above are often used in capital replacement modelling and the total discounted cost per unit time may be considered over a finite horizon. Over an infinite horizon, the total discounted cost criterion is only valid for discount factor, υ, less than 1 because as υ tends to 1 the total discounted cost tends to infinity (described latter in chapter 3). It is recommended to use the rent criterion when the discount factor approaches 1 because as υ tends to 1 the rent tends to the cost per unit time. This leads to a discounted cost replacement criterion with a simple interpretation. For a replacement policy over a finite period both criteria can be used in a similar way without distinction, provided that usage is at least reasonably constant. The term "rent criterion" was first used by Christer and Waller (1987a) for tax adjustment replacement models, though this type of criteria was referred to by Churchman et al. (1966) and used by Russel (1982) in vehicle replacement. The two criteria have a continuous as well as discrete representation, with the latter used for computational convenience.

2.5. Planning horizon

The planning horizon is that interval over which we wish to consider the formulation of replacement policy and in particular, over which we take account of costs or cash-flows.

In capital replacement policy, the planning horizon may be either finite or infinite, fixed or variable. An infinite planning horizon is used for simplifying the modelling process. Sethi and Chand (1979) have developed a forecast horizon for the optimal replacement decision which frees the solution from an arbitrary horizon. They have shown that there exists a forecast horizon T such that the optimal replacement decision for the first machine (new or existing), based on the forecast of machine technology until period T, remains optimal for any longer horizon than T, and for that matter, the infinite horizon problem. The infinite planning horizon implies that if replacement has to be made on the basis of non-like-with-like, then the new model of equipment, as well as economic factors and failure costs need to be predicted in an objective fashion. This, of course, is rather difficult, if not impossible to realise in practice. The infinite horizon model with no technological change is invalid for many real situations. In their paper, Elton and Gruber (1976), proved that an equal life over an infinite horizon is optimum for assets with linear rates of technological improvements. In real world applications, finite horizon models are desired and accepted, especially for cost prediction, and for considering factors such as inflation rate or discount factor. With a finite planning horizon, the prediction of the costs of the new model of equipment is made simpler.

When the planning horizon is finite it may be either variable or fixed. For a finite planning horizon, replacement decision policy can in certain circumstances lead to the realisation of assets. This can impose replacement when it is not necessary (end-of-horizon-effects) and also the sale of the "best" plant at the end of the planning horizon. Therefore care must be taken. A finite variable planning horizon model was first introduced by Christer and Goodbody (1980), and later refined (Christer and Waller, 1987a, Christer, 1988; Christer and Scarf, 1994). This model has two replacement cycles. The length of the horizon is variable and depends on the length of the two cycles. The term cycle is a time interval over which an equipment is bought, operated and sold. In this case the lengths of the cycles are decision variables.

A fixed planning horizon model has a variable number of cycles influenced by the specified length of the horizon. The important decision variable is the time to first replacement since this is the immediate decision problem. Thus, the choice for the length of the horizon (chosen by the modeller and the plant owner) should be made adequately in order not to impose a poor replacement schedule. It is recommended that optimum policy be determined for a range of values of the fixed planning horizon provided that the horizon length is not too large, but large enough in order not to increase costs by imposing a poorly scheduled replacement. However as the decision-maker will not have a firm value for the horizon length, the optimum policy must be "robust" to variation in the horizon length. Based on experience, the operator can plan replacement by considering a certain horizon which is appropriate in a strategic sense that is consistent with the time scale of the strategic planning of the organisation.

2.6. Penalty cost

We suppose that a penalty cost arises when equipment fails and causes a stoppage of production or service. This can lead to a financial consequence for the manager or operator. This financial consequence may be the cost of inconvenience or loss of opportunity. The notion of penalty cost appears to be readily recognised and accepted, but it is difficult to quantify. By taking into account and accepting the notation of penalty cost, the operator may reduce the risk of paying a high price when failure and unavailability occur. The modelling of the penalty cost is not an easy task, because of the subjectivity of this cost factor. This difficulty can be overcome by considering a wide range of acceptable values of the penalty cost per breakdown for the operator. This will enable the modellers to establish the influence of this parameter on the decision variables through sensitivity analysis. Christer and Scarf (1994) showed the strong influence of the penalty cost on the decision variables. Lake and Muhleman (1979) developed a simulation model for the cost resulting from plant stoppages (plant down time). It is important to note, however, that it is not proposed that penalty cost be estimated, but that the influence of penalty cost on optimal policy should be investigated, and that operators or decision makers be allowed to observe optimum replacement policy for a range of penalty costs.

2.7. Resale values and purchase costs

Resale value is the second hand value of the plant. It is an age and time dependent cost factor which generally decreases in a very fast manner except in some situations where an unsteady and unstable economy prevails. In the absence of a second hand market, the resale value for some equipment is set to zero (scrap value) or the equipment is kept as a spare if it is not technically obsolete. Data for resale values are often obtained from some specified guide or directly from the second hand market. For example in the UK vehicle market, Glass's guide and the CAP Red Book (Kobbacy and Nicol, 1994) for private and commercial vehicles are available. From the prices given in these guides one can easily model the resale value function using regression techniques. The prices of old plant are also influenced by the introduction of new models in the market (Scarf, 1994).

In Christer and Waller (1987a) and in Walker (1994) the depreciation cost or resale value was modelled as

$$S(t) = R\gamma\delta^{t},\ 0 \le \gamma \le 1, 0 \le \delta \le 1, \qquad (2.9)$$

where R represents the purchase cost new of the equipment; t its current age; γ is the anti-log of the intercept of regression and represents the very early depreciation after purchase and finally δ is the anti-log of the slope of the regression and represents the long term depreciation. It is however, not the only formulation for the resale cost. Lake and Muhleman (1979) considered a replacement problem for a wrapping machine for biscuits. They used different models for resale values such as constant value, a linear decline of the resale value with the age of the machine and a monthly depreciation at constant rate.

The resale value of an item of equipment depends on its age, usage and condition and of course the state of the market of supply and demand. In certain instances, particularly in developing countries resale value is sometimes higher than the cost new. In the absence of historical data on resale values we choose the model of Christer and Waller (1987a) (equation 2.9), for resale values. As mentioned before, it is not the only representation of the resale value model. The other models might be considered, such as, a linear or an exponential model. A simple example is the resale values for Ford Escort cars data obtained from Glass's Guide Car Values. The cost new of these types of cars is £9915 in 1993. The resale values and the ages of the cars at resale are as in the Table 2.1. The model was fitted to the data, equation (2.9), giving estimates for γ and δ of 0.912 and 0.828 respectively. An illustration of the depreciation cost is shown in Figure 2.2. The purchase price corresponds to age zero (buying new) although buying old may be considered in replacement modelling. The purchase price is the cost of a replacement equipment and may increase in the future as a result of inflation, technological improvement or many other economic factors.

Table 2.1. The resale values and ages at resale for Ford Escort model.

Resale price	Age in years
2325	7.5
2750	6.5
3225	5.5
3500	5.0
3750	4.5
4100	4.0
4400	3.5
4800	3.0
2725	2.5
6275	2.0
6700	1.5
7425	1.0
8275	0.5
9915	0.0

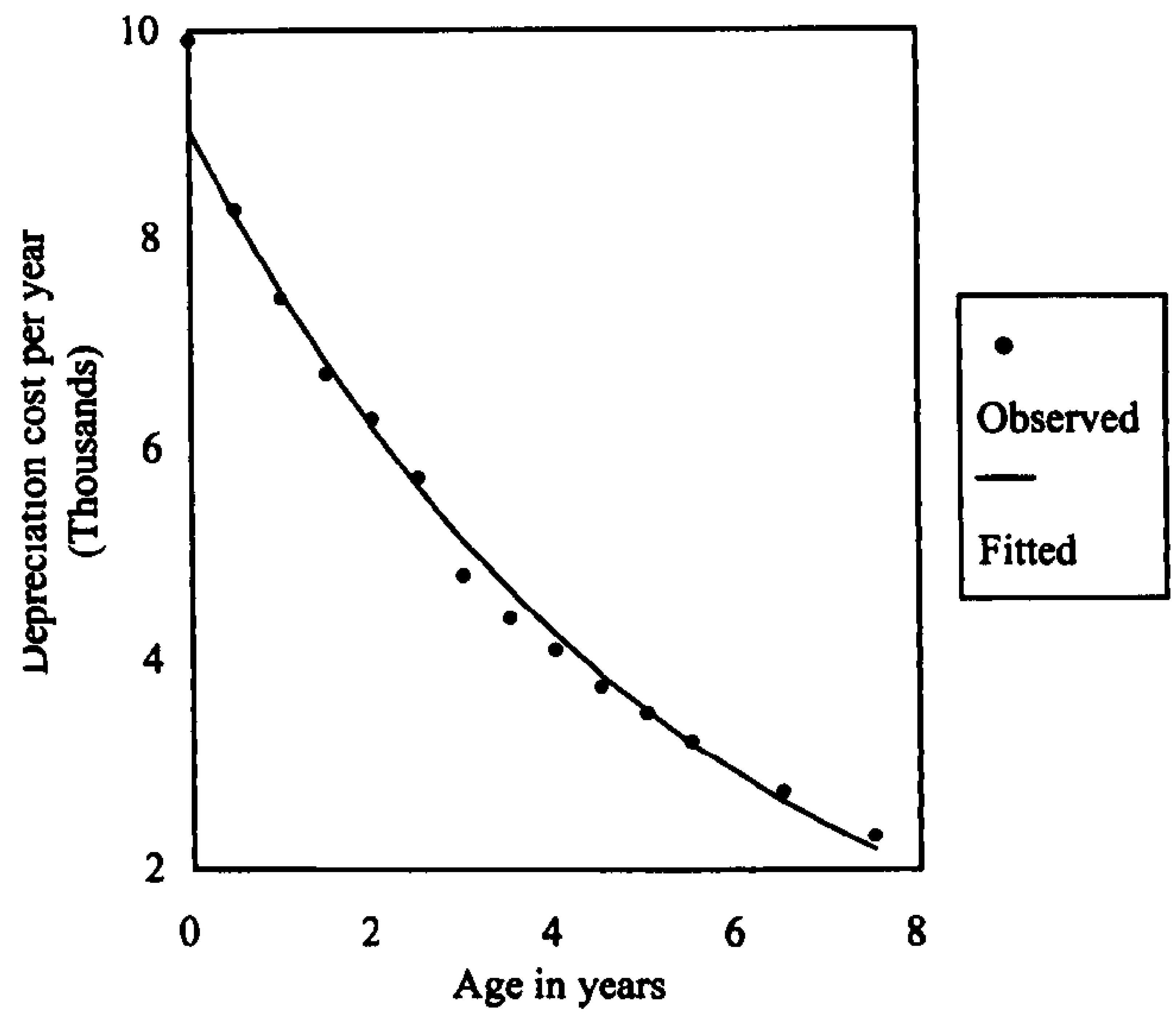


Figure 2.2. The model fitted along with the data observed for the resale values and cost new for Ford Escort model.

CHAPTER 3

CHAPTER 3

Capital Replacement Models

3.1. Introduction

Much research has been done in the replacement field so far, so before going directly to our work, we review this research. Many of the authors concentrate their studies on determining the economic life of an equipment (age at replacement). The costs mainly considered in economic life modelling are the maintenance cost (including the operating cost) and the replacement cost and the resale value. Need for replacement may also be driven by a lack of operating efficiency (failure) or by technical obsolescence.

In this chapter we describe the quantitative issues in replacement and repair cost limit models. Also in this chapter we describe replacement models of economic life type, from the early like-with-like infinite cycle models of Eilon et al. (1966) to the sub-fleet replacement fixed horizon models of Scarf and Hashem (1997). The models are discussed in the context of various criteria, and examples are given where appropriate.

3.2. Qualitative issues in capital replacement

For modelling the replacement problem, Christer (1988) presented a paper discussing the problem of when to replace an existing plant/item with an improved new type of a plant, and how the cost estimation and prediction should be undertaken. It was concluded that the replaced plant would be expected to have at least one improved attribute such: as lower running costs; higher unit output; improved quality of production; greater product variation; enhanced customer appeal; enhancement of the technical image of the company. The main task of the paper was to formulate a replacement criterion for technically improved equipment with operational measures reducible to a scalar measure called cash. In this way, the differences between the old (current) and the new plant would lie in purchasing costs and maintenance costs per unit time. Christer and Waller (1987b) discussed a descriptive model of the equipment-replacement decision process. The model was based on the results of a survey undertaken between February 1983 and May 1984 and the survey itself was based on a questionnaire. The survey observed that there is no particular decision technique or procedure peculiar to any one type of organisation and that a reasonably homogeneous mix is to be expected. The conclusion of this paper was that the descriptive model developed is appropriate to the decision process studied within the collaborating companies that provided the data. Care must be taken in extrapolating the results to other plant and companies.

Issues related to replacement of existing plant and equipment were discussed by Campbell (1994). The idea for this study rose when hospital executives found that they must make major strategic and replacement capital investment decisions for an environment that is likely to be quite different from the one that currently exists. Concerning replacement of different types of assets little was known about which factors are more important in the replacement of plant such as buildings and infrastructure relative to the replacement of medical equipment. For example, buildings and infrastructure improvements are likely to be less important than equipment replacement to hospital physicians. The study addressed the issues by examining plant and equipment replacement priorities in 116 large US hospitals. It was found that main factors affecting hospital capital replacement were financial factors (funding resource) and industry factors such as the variability in the demand for acute care hospital services with the expansion of man aged care and changes in government reimbursement. These factors dominate the timing of planned replacements. Other factors governing the capital replacement decision such as the degree of competition between the hospital and its peers, the demand for replacement by the hospital's doctor, equipment maintenance that may be more important than plant renovation to physicians, and patients' comments and concerns. A survey was done by randomly selecting a sample of financial managers from the 1992 Health Care Financial Management Association. The study results were intended to apply only to larger hospitals. The results revealed that the most important factor affecting plant replacement was meeting regular or accreditation requirements, and the leading factor in equipment replacement was technological obsolescence. The financial condition of the hospital was second in order of importance for plant replacement decisions; however, the financial condition was fourth in order of importance for equipment replacement. Plant improvements are likely to be less important than equipment replacement to a hospital physician and also equipment maintenance may be more important than plant renovation to a physician. The hospitals were highly influenced by physicians' requests in the decision to replace equipment; this was less so for plant. The plant in this context is the hospital buildings and their equipment needed for medical purposes. The priority given to patient complaints is consistent with the view that the hospital's primary customers are physicians and not patients. The major factor causing delays in replacements was lack of internal funds. On the basis of the responses in the survey, priorities need to be altered to reflect the importance of costs and utilisation in replacement decisions. The study considered that hospital executives must direct greater attention to: consumer concerns in the capital investments that they make; access to funds for replacement is directly dependent of profitability, financial condition, and perceived community benefit, as cost-based reimbursement is eliminated; and finally, hospital executives in the nation's large acute care hospitals will need to take a leadership role in altering their replacement priorities to reflect this new environment.

In his paper Russell (1982) presented a description of an operational research project team's assessment of the environment within which replacement decisions are made and the team's attempt to take these decisions into account to ensure effective implementation of assessment work. The work was done in the context of one vehicle replacement policy and the following were considered.

1-Developing a vehicle replacement model.

2-The collection of data for modelling.

3-Analysis of data on repair and maintenance costs of vehicles.

4-The development of a suggested approach to vehicle replacement.

5-Development of the ideas on aiding the selection of individual vehicles via the use of annual expenditure limits.

6-Consideration of transport management systems information.

3.3. Repair cost limit models

Sometimes, it may be economical to repair or replace before failure happens. As we mentioned above, most of the authors concentrate their study on the economic life models. Other authors consider repair cost limit models. Hastings (1967) described the replacement problem when an item requires repair. The item should be inspected to determine whether the estimated cost of repair is less or more than a certain level (the repair cost limit). The author used dynamic programming to determine optimum repair cost limits. Two main problems were analysed; the first one concerning condition related to age and the second where condition is related to the number of overhauls.

Jardine, Goldrick and Stender (1976) suggested the concept of annual maintenance cost limit (AMCL) as an approach to be used in making replacement decisions. The decision to replace is taken if the estimated maintenance bill for the next year exceeds the AMCL appropriate to the vehicle. A replacement model was constructed assuming that the equipment will be required over a fixed planning horizon. The objective was to minimise the total cost by the selection of optimal AMCL, and this was obtained by solving a recurrence relation considering the following.

1-The time required for running the vehicle until the end of the planning horizon.

2-The age of the vehicle at the beginning of the year.

3-The probability density function for the maintenance cost, for a vehicle of that age.

4-The maintenance cost limit for a vehicle of that age.

5-The minimal expected cost of replacing and maintaining a vehicle during the running time starting with age in (2) above.

The model was modified to deal with fleet replacement taking account of tax allowances, prices of new vehicles, resale values and discounting. Finally, the maintenance cost limit replacement indicated that premature replacement incurs heavy penalties and that delayed replacement results in far lower penalties. Replacement models are either related to components or related to a single complex system.

A recent study on vehicle replacement was done by Hensher and Zhu (1994); the authors suggested an approach to overcome some shortcomings of traditional methods related to the vehicle replacement decision process for different types of vehicles in a vehicle fleet subjected to budget and average fleet age constraints. This paper resolved the limitation of the traditional vehicle replacement methods by introducing the concept of a residual value function of which the positive portion is a linear function of used years and completed kilometres. This was used to calculate annualised costs of a vehicle. With the additional kilometres and years, the annualised equivalent cost (AEC) can be derived by the sum of capital AEC (AEC_c), operating and maintenance AEC (AEC_{om}), and major rebuild (engine rebuild, transmission rebuild) AEC (AEC_r), then the total annualised cost of the vehicle is

$$\mathrm{AEC}_{\mathrm{total}} = \mathrm{AEC}_{\mathrm{c}} + \mathrm{AEC}_{\mathrm{om}} + \mathrm{AEC}_{\mathrm{r}}. \tag{3.1}$$

The above AEC of the vehicle is the annualised equivalent cost without considering replacement. Therefore, there are three similar components of the total AEC upon replacement (the vehicle's useful life), and the total is expressed as

$$\mathrm{AEC}'_{\mathrm{total}} = \mathrm{AEC}'_{\mathrm{c}} + \mathrm{AEC}'_{\mathrm{om}} + \mathrm{AEC}'_{\mathrm{r}}. \tag{3.2}$$

Ideally, a vehicle with higher total AEC associated with rebuild than the total AEC upon replacement should be replaced. The paper proposed a 0-1 integer programming model taking into account the age and budget constraints. This model yielded an optimal solution which determines the particular decision for each vehicle in the fleet. A case study with a fleet of 21 vehicles was discussed, along with a software development called the Vehicle Replacement System (VRS). For simplicity, all vehicles were assumed to be of the same type.

3.4. Early replacement models.

Early replacement models were studied by Eilon et al. (1966). The authors developed two replacement models as an attempt to determine the optimum economic life of the equipment. For the first model attention was paid to the total cost per annum; for the second model the total cost discounted to present value was considered. The authors introduced the concept of the discount factor defined in terms of the inflation rate. This factor is used to calculate the present value of capital that will be spent in the future. The first model was formulated as

$$T = \frac{A - S - C\gamma}{n} + \frac{1}{n}\int_0^n f(t)dt, \tag{3.3}$$

where T = the total cost per annum

A = acquisition cost of new truck;

S = resale value of the existing truck (n years after it was purchased);

$f(t)$ = maintenance cost per unit time of a truck t years after acquisition;

n = age (in years) of truck when replaced;

C = capital allowances;

γ = rate of taxation.

Notice that the first term in equation (3.3) represents the average capital costs involved in the acquisition of the existing truck when the resale value and capital allowances are accounted for; the second term expresses the total maintenance costs of the existing truck averaged over the n year replacement cycle.

The second model was formulated as

$$P = \frac{A - Sr^n - C_p\gamma}{n} + \frac{1}{n}\int_0^n f(t)r^t dt, \qquad (3.4)$$

and for an infinite number of successive machine replacements the total of all future costs V discounted to present value was given by:

$$V = P(1 + r^n + r^{2n} + \dots\dots)$$

$$= \frac{P}{1 - r^n}, \qquad (3.5)$$

where r = discount rate;

C_p = present value of capital allowances;

P = total average cost per year discounted to present value.

The results obtained from applying model one (equation 3.3) were based on an imaginary average truck, with maintenance cost per unit time model according to that fitted to data for a sample of 10 trucks. Furthermore, it was found that the results for optimal replacement lives for the trucks considered varied from 5 to 12 years. The results from model two (equations 3.4 and 3.5) were affected by capital allowances, so the optimal replacement period was reduced by about 4-5 years. The study demonstrated the importance of considering capital allowances for tax purposes. The two models yielded a flat objective function near their optimal points. Model one suggests replacing the equipment more frequently than model 2, and for that reason it has the advantage that it provides an opportunity to assess technological innovation and new designs of equipment over shorter time intervals.

Elton and Gruber (1976) presented a paper for studying the equipment replacement model with equal replacement intervals. They presented proof for the optimality of the equal life policy incorporating technological change when new equipment is identical to old (current) equipment. They showed that; a policy of replacing at equal intervals does not have to be assumed but rather can be proved to be optimum. They found that over an infinite horizon if two replacement intervals (two economic lives) are different, future lives must decrease or increase with limit. Since such an increase or decrease is impossible, equal lives must be the only solution.

Another problem of equipment replacement was studied by Lake and Muhleman (1979). They developed a simulation model for the replacement of a particular type of machine in order to predict the effects of production stoppages that can sometimes result from the breakdown of the machine. They were concerned simply with determining the age at which a machine has to be replaced (economic life policy). Technological progress is relatively slow in this area so its effects were excluded from the model.

It was apparent that, one of the weaknesses of the pure economic life model is that it ignores the situation when a machine requires an expensive repair before the end of its economic life.

A recent study was performed by Scarf (1994), who considered a modelling approach aimed at answering the questions, " how old to buy a vehicle?" and "how long to run it before resale?". These questions were considered in the context of the replacement problem of a private motorist. The author proposed an equivalent rent criterion which assumed an infinite series of identical buy, operate and sell cycles, typical of models found in Operational Research literature. It was found that the model (infinite horizon model) is appropriate for this problem.

In their paper Hawkins and Nasoni (1977) presented a theoretical model for dealing with uncertainty in capital investment. They developed variable discounting as an approach to dealing with uncertainty. They considered a relatively simple prototype replacement model (stationary technology, no inflation and no salvage) assuming an instantaneous rate of cash flow per unit time, discounted net benefit to determine the economic life and constant discount factor over infinite horizon. The rate of change of cash flow with time was proportional to the variable discount factor. It was necessary to assume that if $k(t)$ and K are the variable and constant discounting factors respectively, then $0 \leq k(t) \leq K_{max}$.

They also used a linear approximation to the variable discounting function, which represents the next level of generality over that of constant. The conclusion of the study relating to a real example stated that there can be a substantial difference in present value results using a variable versus a constant discount rate.

The earliest infinite horizon model for like-with-like replacement was provided by Kaufman (1963), Churchman, Ackoff & Arnoff (1966) and Eilon et al. (1966). The earliest economic life models over one replacement cycle were basically all expressed as

$$C(t) = \frac{1}{t}\left[R + \int_0^t f(\tau)d\tau - S(t)\right], \tag{3.6}$$

where $C(t)$ the total cost per unit time, $f(\tau)$ is the maintenance cost per unit time of equipment aged τ, $S(t)$ is the resale value of equipment aged t, R is the purchase (capital cost). Equation (3.6) represents the cost per unit time over the cycle of length t. That value of t which minimises C is called the economic life. Without discounting the one cycle model and infinite cycle model are identical (assuming like-with-like replacement).

Here for our study, we assume that the individual requires an equipment over an indefinite period, so that it is necessary to consider a series of buy-run-sell-cycles. The decision variable is the optimum value of the age of the equipment at replacement, the so called "economic life" for equipment bought new.

3.4.1. The total discounted cost criterion

The total discounted cost criterion is an extension of the model (3.6). It is one possible criterion for the determination of the economic life. It considers an infinite planning horizon and a discount factor υ in order to consider costs at net present value. For a single purchase and resale cycle we have that the total cost discounted to present value is given by

$$\hat{C}(t) = \left[R + \int_0^t f(\tau)\upsilon^\tau d\tau - S(t)\upsilon^t \right].$$

Thus for an infinite series of such cycle, the total of all future costs discounted to present value is

$$C_1(t) = \hat{C}(t)(1 + \upsilon^t + \upsilon^{2t} + ...),$$

$$C_1(t) = \frac{1}{(1-\upsilon^t)} \hat{C}(t),$$

from which

$$C_1(t) = \frac{1}{(1-\upsilon^t)} \left[R + \int_0^t f(\tau)\upsilon^\tau d\tau - S(t)\upsilon^t \right]. \quad (3.7)$$

Notice that as $\upsilon \to 1, C_1(t) \to \infty$ and for this reason the cost C_1 does not have a straightforward intuitive interpretation. A computation approach for the determination of economic life using the total discounted cost criterion is expressed by the equation

$$C_2(n) = \frac{1}{(1-\upsilon^n)} \left[R + \sum_{j=1}^{n} M_j \upsilon^{j-1/2} - S(n)\upsilon^n \right], \ \upsilon < 1, \quad (3.8)$$

where $R, M_j, S(n)$ and υ are respectively the purchase cost new; the maintenance cost per unit time of equipment in its *jth* year; the resale value of equipment in its *nth* year and the discount factor. Here the cost functions may be considered with a unit time interval the month, year or whatever is appropriate. The maintenance costs are assumed to occur in the middle of each interval and are discounted accordingly.

3.4.2. The equivalent rent criterion

Consider now an infinite series of identical buy-run-sell cycles (see Figure 3.1) with total discounted cost given by $C_2(n)$, equation (3.8).

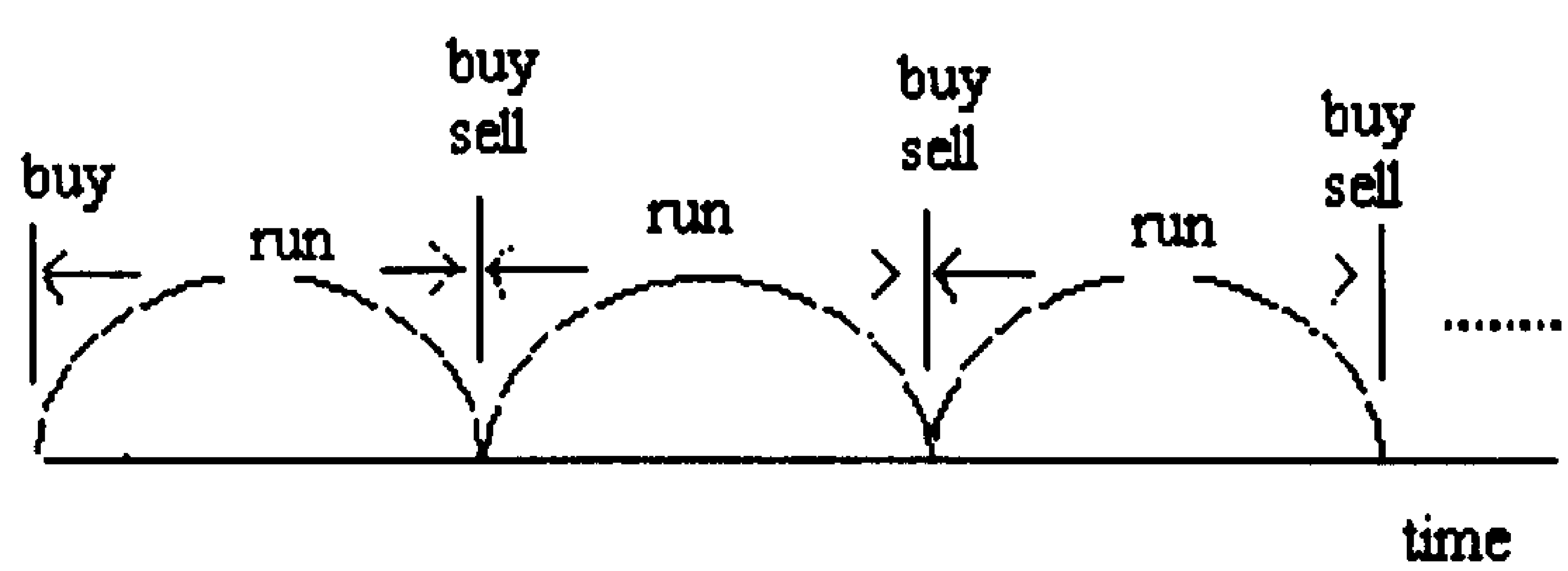


Figure 3.1. An infinite series of buy-run-sell-cycles, with values of n the same for each cycle.

Another approach to determine the economic life considers the rent payable per unit time period and payable over an indefinite period, which would be necessary to meet the total discounted cost $C_2(n)$. This periodic payment is called the equivalent rent and this criterion can be used to determine optimal policy. Discounting this rent payable in each time period in future to present value, it follows that

$$C_3(n)(1+\upsilon+\upsilon^2+.......)=C_2(n),$$

from which

$$C_3(n)\frac{1}{1-\upsilon} = C_2(n),$$

consequently we obtain

$$C_3(n) = (1-\upsilon)C_2(n), \tag{3.9}$$

where $C_3(n)$ is the equivalent rent. Notice that to obtain the equivalent rent $C_3(n)$ as $\upsilon \to 1$ we substitute $C_2(n)$ from equation (3.8) and consequently

$$C_3(n) \to \lim_{\upsilon \to 1}\left(\frac{1-\upsilon}{1-\upsilon^n}\right)\left[R + \sum_{j=1}^{n} M_j - S(n)\right],$$

where

$$\lim_{\upsilon \to 1}\left(\frac{1-\upsilon}{1-\upsilon^n}\right),$$

can be obtained by using L'Hopital's rule as

$$\lim_{\upsilon \to 1}\left(\frac{1-\upsilon}{1-\upsilon^n}\right) = \lim_{\upsilon \to 1}\frac{-1}{-n\upsilon^{n-1}} = \frac{1}{n},$$

from which

$$C_3(n) \to \frac{1}{n}\left[R + \sum_{j=1}^{n} M_j - S(n)\right]. \tag{3.10}$$

Thus when there is no discounting the equivalent rent is equivalent to the average cost per unit time.

3.4.3. Numerical example

Using the two computational approaches in equation (3.8) and equation (3.9), we can determine the economic life for an equipment. For example, we consider this for five models of Malaysian buses such as Mercedes, Isuzu CSA, Mitsubishi, Isuzu CJR and Cummins. The purchase cost new for the five models are M$500K, M$230K, M$750K, M$300K and M$800K, respectively. Maintenance cost per unit time functions were obtained from maintenance records (full details are given in chapter 6). Using a typical discount factor of 0.98, the results are as given in Table 3.1. Here the results represent the equivalent rent per bus per year and the total discounted cost per unit time for each model type. It is obvious that the two models give the same n^* (economic life), but of course different minima of both the equivalent rent and the total cost discounted respectively. The equivalent rent has an easier interpretation than the total discounted cost. Figure 3.2 and Figure 3.3 illustrate these results. Also it should be noted that Cummins is the most expensive one among all models and this is because of the high maintenance cost per unit time of Cummins.

Tale 3.1. Results for total discounted cost and equivalent rent criteria; discount factor=0.98

	Rent criterion		**Total discounted cost criterion**	
Bus model	n^*	Min. rent	n^*	Min. cost
Mercedes	18	42132*	18	5408988
Isuzu CSA	20	98575	20	428907*
Mitsubishi	18	108180	18	2106609
Isuzu CJR	5	83975	5	4197855
Cummins	6	126433	6	6321632

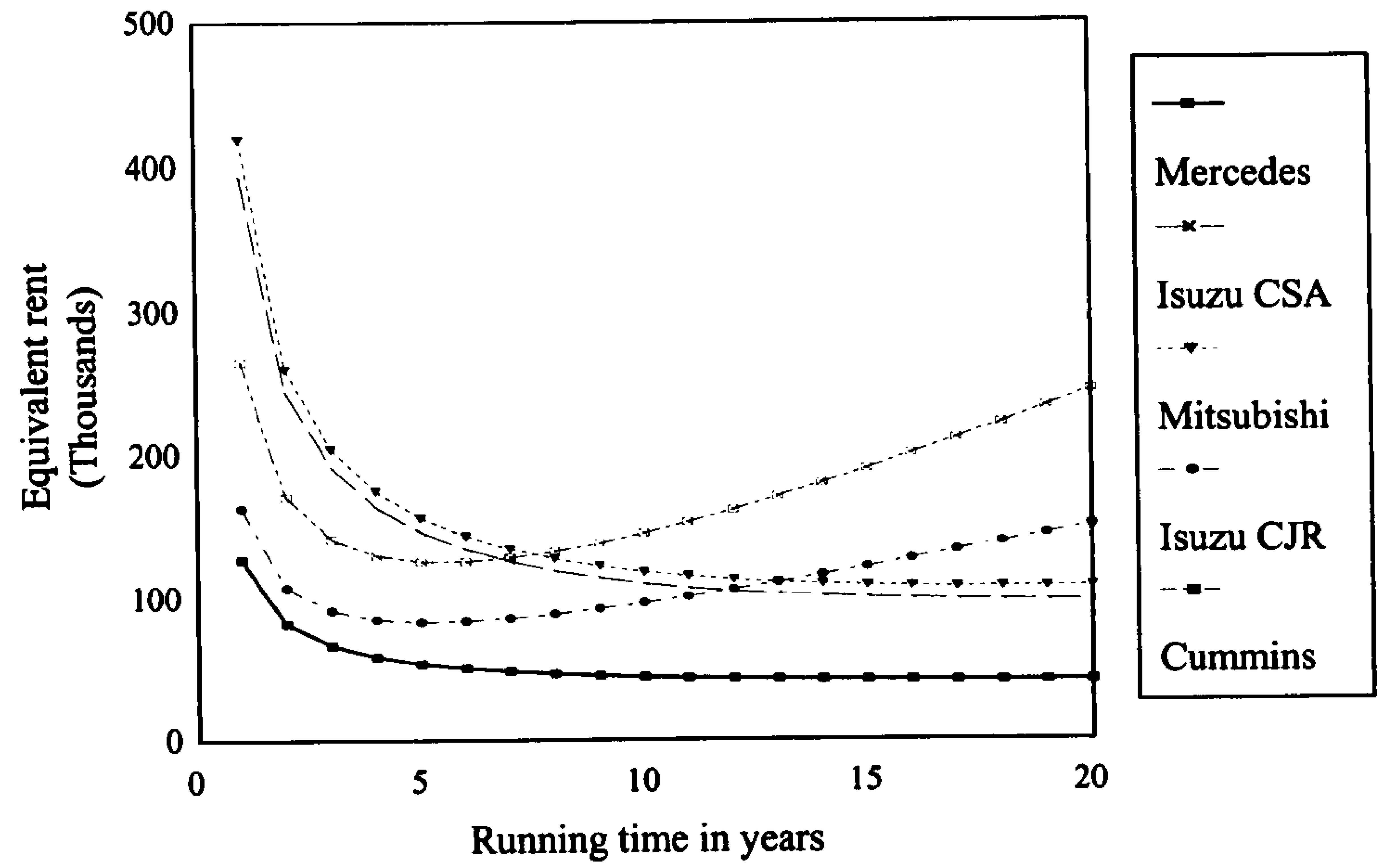


Figure 3.2. Rent versus age at replacement / running time.

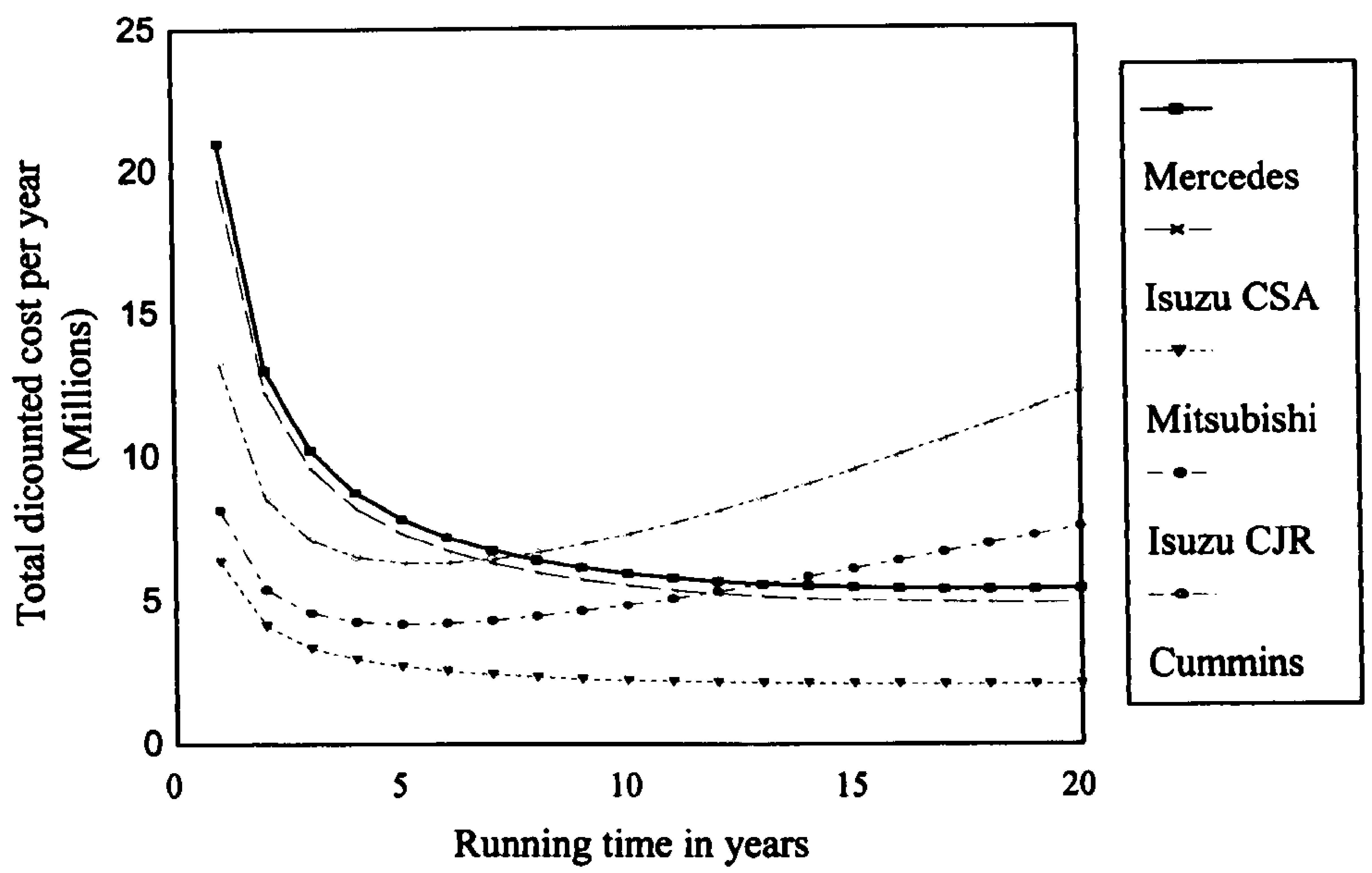


Figure 3.3. Total discounted cost versus age at replacement / running time.

3.4.4. Model extension

The infinite horizon model for like-with-like replacement can be extended to the case of buying old equipment aged m years at purchase. In this case it is necessary to consider a series of buy-run-sell cycles (Scarf, 1994) with decision variables: the age of the equipment at purchase; and the running time, a time to replacement, of the equipment.

The approach, equation (3.8), can be extended to this case and the total discounted cost is expressed as

$$C_4(m,n) = \frac{1}{(1-\upsilon^n)}\left[R_m + \sum_{j=1}^{n} M_{m+j}\upsilon^{j-1/2} - S_{m+n}\upsilon^n\right], \quad \upsilon < 1. \tag{3.11}$$

Here m is the age of the equipment at purchase (buying old). R_m, M_m, S_m and υ are respectively the purchase cost of equipment aged m years; the maintenance cost per unit time for equipment aged m; the resale value of equipment aged m, and the discount factor.

The equivalent rent is expressed as

$$C_5(m,n) = (1-\upsilon)C_4(m,n). \tag{3.12}$$

m and n are the decision variables in this model.

3.5. Finite horizon models

Christer and Goodbody (1980) considered the decision problem concerning the replacement of members of a fleet of fork lift trucks during a period of inflation and economic uncertainty. The authors used an alternative model based upon relatively short term estimates of costs. The replacement decision was determined for both constant and variable discount rates. They developed a model of the maintenance cost per unit time for fork lift trucks with a view to determining the optimal replacement age. The criterion function was selected to be related to the average discounted cost over two replacement cycles. The process can be associated with the short term decision problem of when to replace currently operating plant of a certain age. Factors such as tax allowances, regional development grants and technological improvements were readily encompassed within the proposed models.

Christer and Waller (1987a) extended the one and two-cycle rent criteria for capital equipment replacement and the infinite-cycle discounted-cost criterion to incorporate new tax features of the 1984 finance act. A basic discrete rent criterion was described prior to taking account of any adjustment for tax allowances. Results from the one cycle model stated that:

1-the optimum age is directly proportional to the tax rate, but inversely proportional to the writing down allowance ,

2- the delay in tax payment has very little effect on the results.

Results from the two-cycle model showed that:

1-the values of maintenance costs, initial age of the first vehicle and replacement age of the second vehicle are major influences on the model's results;

2-interest rates have very little influence on the results;

3- variation in the tax parameters has virtually no effect on the results.

The results of the infinite-cycle model showed that the effect of tax parameters remained very small. The authors found that modellers may, with some justification, consider simplifying matters by omitting tax parameters from replacement models until such time as the legislation changes.

Recently, Bean, Lohman and Smith (1994) presented a paper on equipment replacement under technological change. They developed bounds on the error due to truncation involved by truncating infinite-horizon replacement economy problems at some finite horizon. These bounds were illustrated through a numerical example from a real case in vehicle replacement. The analysis allowed for the existence of both revenues and costs. The results were of value because of the high cost of gathering information for decision problems of this type. It was apparent that if a small number of years of data is available, and no forecast horizon (period over which one is concerned with making replacement decisions) is found within that time frame, the results obtained give the decision maker the power to evaluate the cost of not planning further in time.

A robust replacement model with applications to medical equipment was formulated by Christer and Scarf (1994) who perceived shortcomings in the applicability of capital equipment replacement modelling. These shortcomings were identified in a 1987 survey within the UK by Hsu (1988). A comparison between a 1987 survey and a similar 1988 survey undertaken in the USA was made with the explanations for apparent differences. The replacement model was developed, however, taking into account some factors such as service and risk, and introducing the concept of a penalty factor. A prototype replacement model of medical equipment was contemplated in an appropriate format. The prime aim of the model was to aid replacement decision making by identifying a "good" replacement decision and the consequences of alternative decisions. The marginal costs associated with delayed replacement were also considered. Finally the model was found to be straightforward, responsive to parameter changes, allows technological development and can be extended to incorporate variable discount or tax considerations.

Scarf (1994) considered a two-cycle replacement model which involved three decision variables for the currently running vehicle. These variables were the time (the length of the first cycle) of operating a vehicle before replacement, the age at purchase of the replacement vehicle and the ongoing requirement for a vehicle (the length of the second cycle). Since the planning horizon is finite and variable, circumstances can arise in which it is optimal to resale the current asset.

In real world applications, finite horizon models are desired and accepted, especially for cost prediction as well as for consideration of economic factors such as inflation rate or discount factor. Also with a finite planning horizon, prediction of the costs of the model of equipment is relatively easy to consider. In this case the length of the planning horizon is either variable or fixed.

3.5.1. Variable length finite planning horizon models

Christer and Goodbody (1980) developed a two-cycle model with variable length for the planning horizon and the function to be minimised was expressed as

$$C(\tau;K,L)=\frac{1}{K+L}\left\{\int_0^K f_1(\tau+t)\upsilon^t dt+\upsilon^K\left[R+\int_0^L f_2(t)\upsilon^t dt+\upsilon^L R\right]\right\},\quad \upsilon>0. \quad (3.13)$$

This function represents the total discounted cost per unit time of operating a plant, currently τ years old for a further K years, replacing it with a possibly different model and then operating for a further L years before replacing again with an equipment model of the same type. Thus K and L are the decision variables in this model.

Notice that the resale value of the plant was not considered but could be incorporated without any difficulty. This model was later refined (Christer and Waller, 1887a; Christer, 1988; Christer and Scarf, 1994). The outline of the model is shown in Figure 3.4. Here, $f(\tau+t)$, R and υ are respectively the maintenance cost per unit time of an equipment currently aged τ at age t, the purchase price and the discount factor.

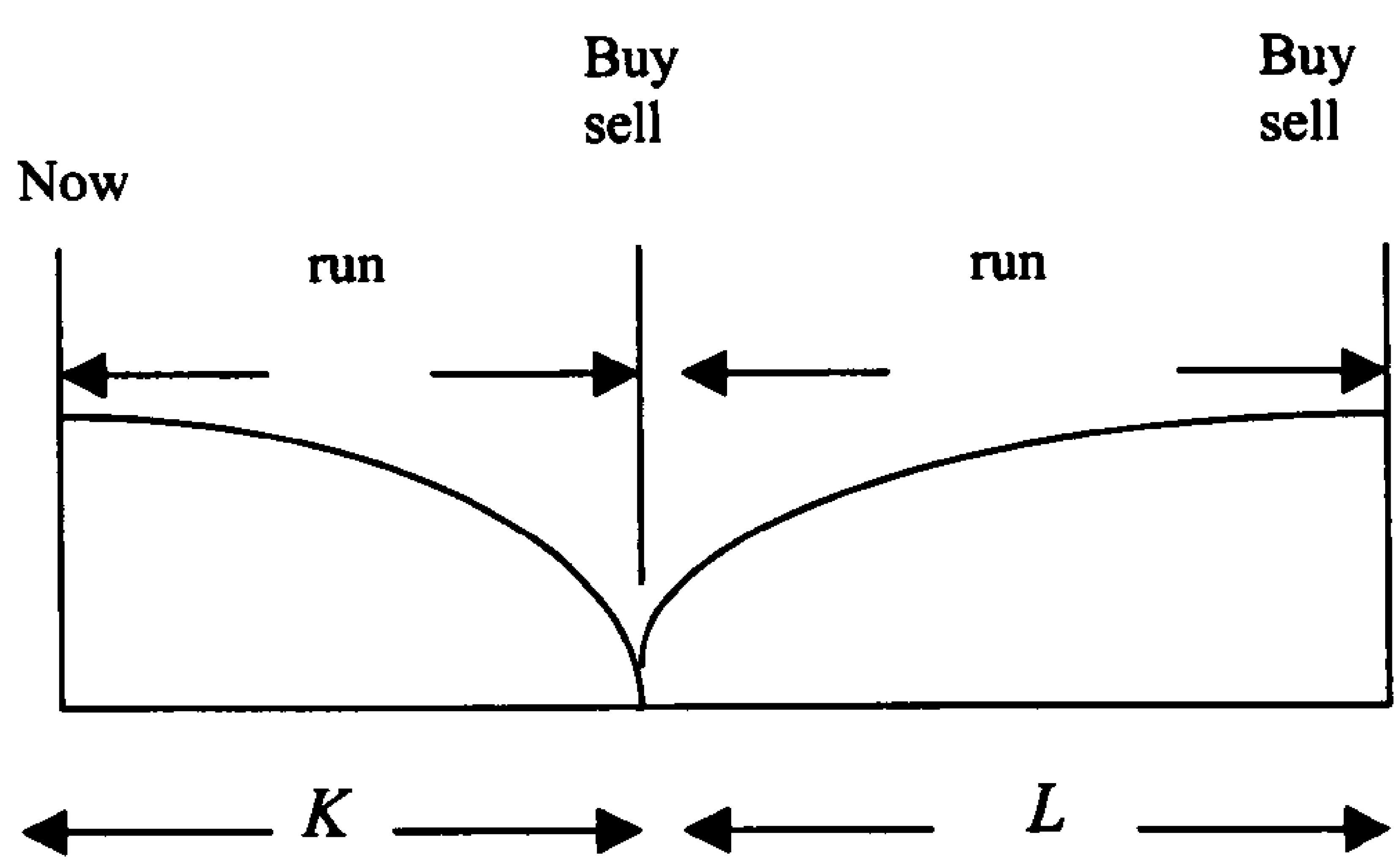


Figure 3.4. Two-cycle replacement model with variable planning horizon.

Technological change is recognised for the case in which replacement is not like-with-like and the second cycle addresses the cost consequences of such technological change. Technological development was discussed in detail in Christer (1988), and was applied to the problem of non-like-with-like replacement in which prediction for maintenance cost of new plant was made using historical data and a predictive ratio method.

The discrete form of equation (3.13) is the most practical approach to determine the replacement decision variables. Thus, for the model which considers resale values, the total discounted cost over two cycles is defined by

$$TDC(K,L) = \sum_{t=1}^{K} M_1(\tau + t)\upsilon^{t-1/2} + \upsilon^K \left[\begin{array}{l} R - S_1(\tau + K) + \sum_{t=1}^{L} M_2(t)\upsilon^{t-1/2} \\ \qquad + \upsilon^L \{R - S_2(L)\} \end{array} \right], \tag{3.14}$$

where $M_1(.)$ and $M_2(.)$ are the age related maintenance costs per unit time for current and new plant respectively; R is the cost of the new plant ; $S_1(.)$ and $S_2(.)$ are the age related resale values for current and new plant respectively, and υ is the discount rate.

All costs are discounted to present values. In practice the discrete formulation, (3.14), is a necessary requirement, and appropriate units for K and L should be used. Here all maintenance costs are assumed to occur in the middle of the respective time period. Thus, the average cost per year over the length of the two cycles, discounted to present value, is then

$$ATDC(K,L) = TDC(K,L)/(K+L). \tag{3.15}$$

The annual equivalent rent $R(K,L)$, which would be necessary to meet the total discounted cost over the two cycles, can be obtained by noting that

$$R(K,L)\left[\upsilon + \upsilon^2 + \ldots\ldots + \upsilon^{K+L}\right] = TDC(K,L),$$

whence

$$R(K,L)\sum_{i=1}^{K+L} \upsilon^i = TDC(K,L).$$

Therefore

$$R(K,L) = TDC(K,L) \Big/ \sum_{i=1}^{K+L} \upsilon^{i} . \qquad (3.16)$$

3.5.1.1. Numerical example

This example considers replacing a Cummins bus with an Isuzu CJR bus, and also a Mercedes with an Isuzu CJR. The data for these buses operated by a Malaysian bus company is considered in detail in chapter 6. The prices new are of M$500K, M$300K and M$800K for Mercedes, Isuzu CJR and Cummins respectively. The maintenance costs per year for Mercedes, Isuzu CJR and Cummins are $9680t^{0.50}, 9680t^{1.14}$ and $9680t^{1.33}$ respectively (in Malaysian dollars, M$) where t is the running time in years for the bus. This maintenance cost per unit time form considered is discussed in detail in chapter 2. The resale values are given by $S(t) = R\gamma\delta^{t}$ where R is the cost new for the bus (in Malaysian dollars, M$), γ and δ are of 0.613 and 0.811 respectively and t is the age in years at resale for the bus. Using a discount factor of 0.98 and various lengths of the second cycle L, the results are presented in Tables 3.2 and 3.3.

Table 3.2. Replacement decision for replacing Cummins with Isuzu CJR for various lengths of the second cycle L ; discount factor =0.98.

Rent			Discounted cost		
K^*	L	**Min. rent**	K^*	L	**Min. cost**
3	3	114679	4	3	106387
3	6	101312	3	6	91703
3	9	102714	3	9	90293
3	12	110889	4	12	94259

Table 3.3. Replacement decision for replacing Mercedes with Isuzu CJR for various lengths of the second cycle L ; discount factor =0.98.

Rent			Discounted cost		
K^*	L	**Min. rent**	K^*	L	**Min. cost**
18	3	59251	20	3	47004
19	6	60482	20	6	46596
20	9	64192	20	9	48091
20	12	70025	20	12	51052

Table 3.2 shows that Cummins is very expensive because its maintenance cost increases rapidly with its early life. The decision is replace Cummins as soon as possible after at most 4 years usage. Table 3.3 presents the minimum rent and minimum discounted cost for replacing Mercedes with Isuzu CJR. The table illustrates that Mercedes has a very low maintenance cost and that explains why the decision is to keep Mercedes as long as possible. From the two tables it should

be noted that the length of the planning horizon ($K + L$) depends on the choice of bus to be replaced. It is noted that the values of K^* produced in the case of rent criterion are always smaller than that in the case of discounted cost criterion. Figures 3.5,3.6, 3.7 and 3.8 illustrate these results.

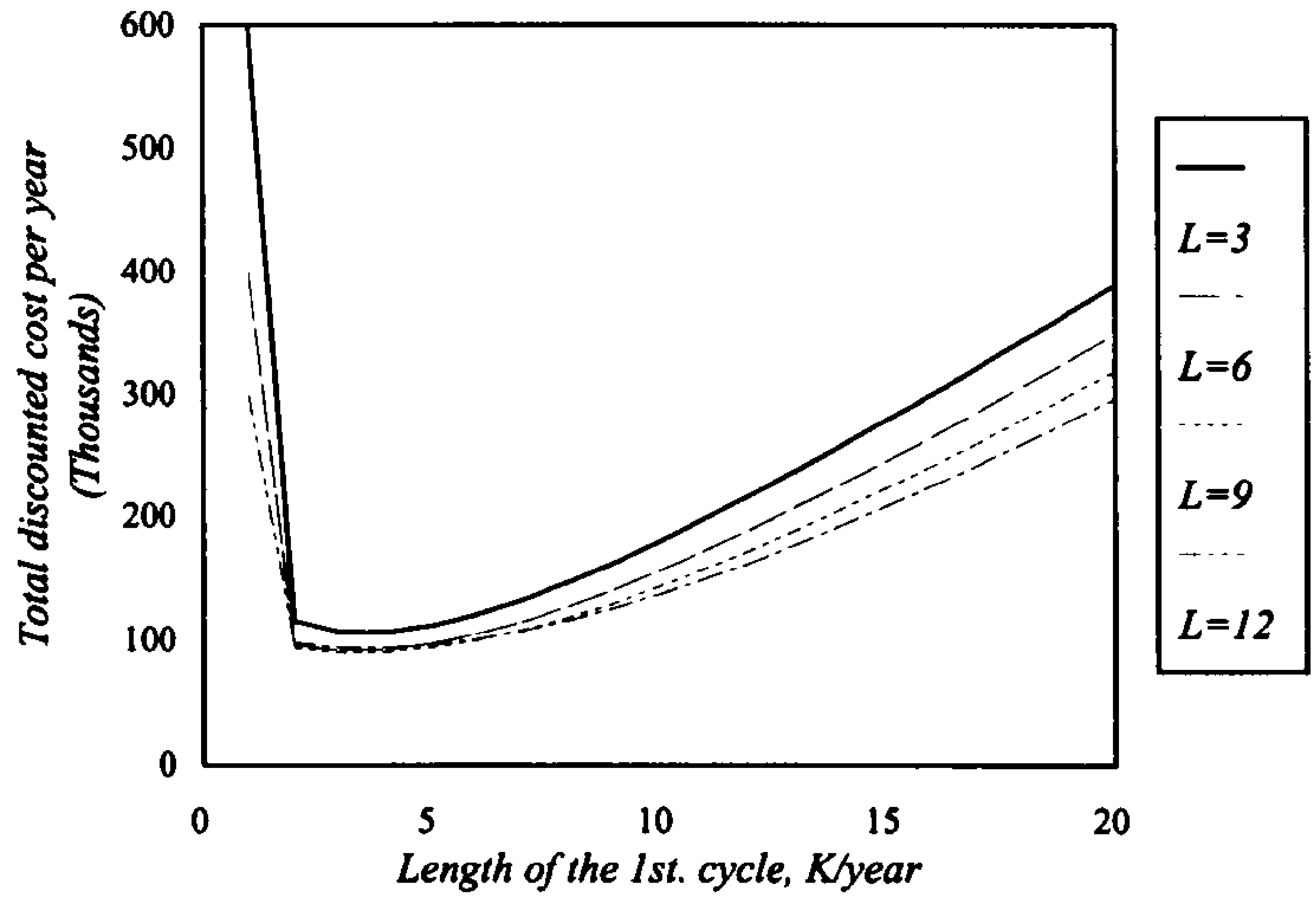


Figure 3.5. Total discounted cost per year versus length of first cycle K, for various lengths of second cycle L, Cummins replaced with Isuzu CJR. Two-cycle model with variable planning horizon.

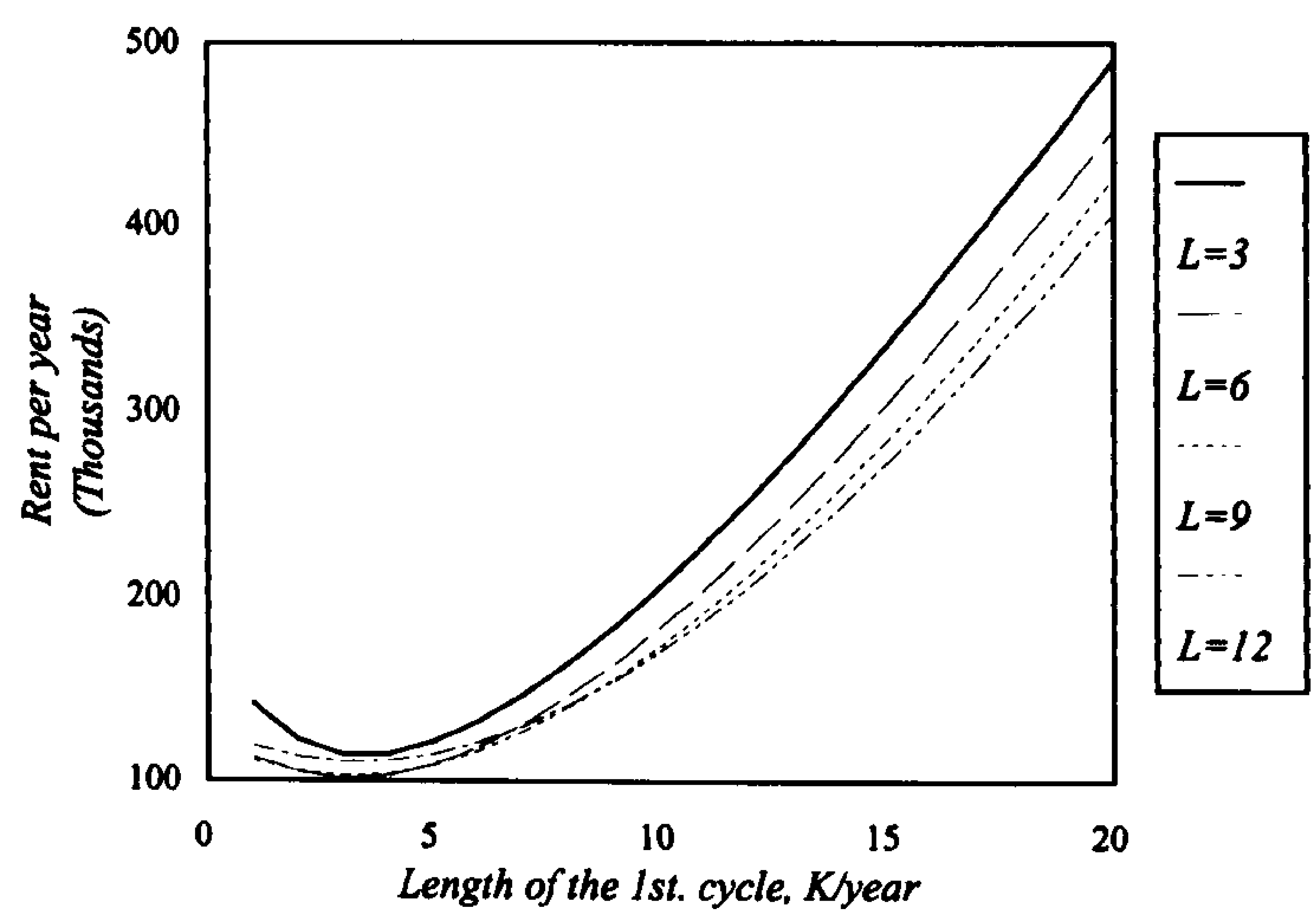


Figure 3.6. Rent per year versus length of first cycle K, for various lengths of second cycle L, Cummins replaced with Isuzu CJR. Two-cycle model with variable planning horizon.

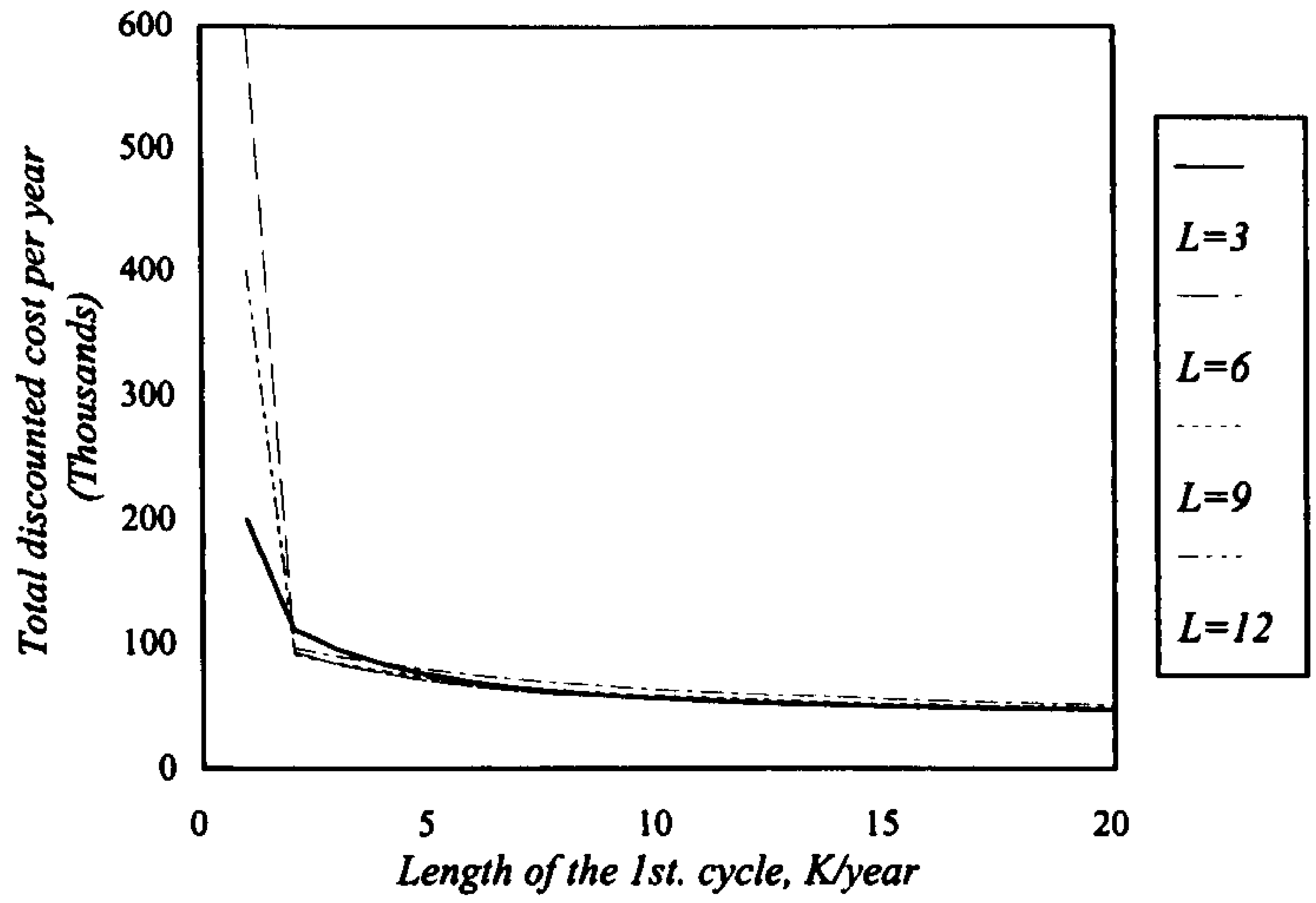


Figure 3.7. Total discounted cost per year versus length of first cycle K, for various lengths of second cycle L, Mercedes replaced with Isuzu CJR. Two-cycle model with variable planning horizon.

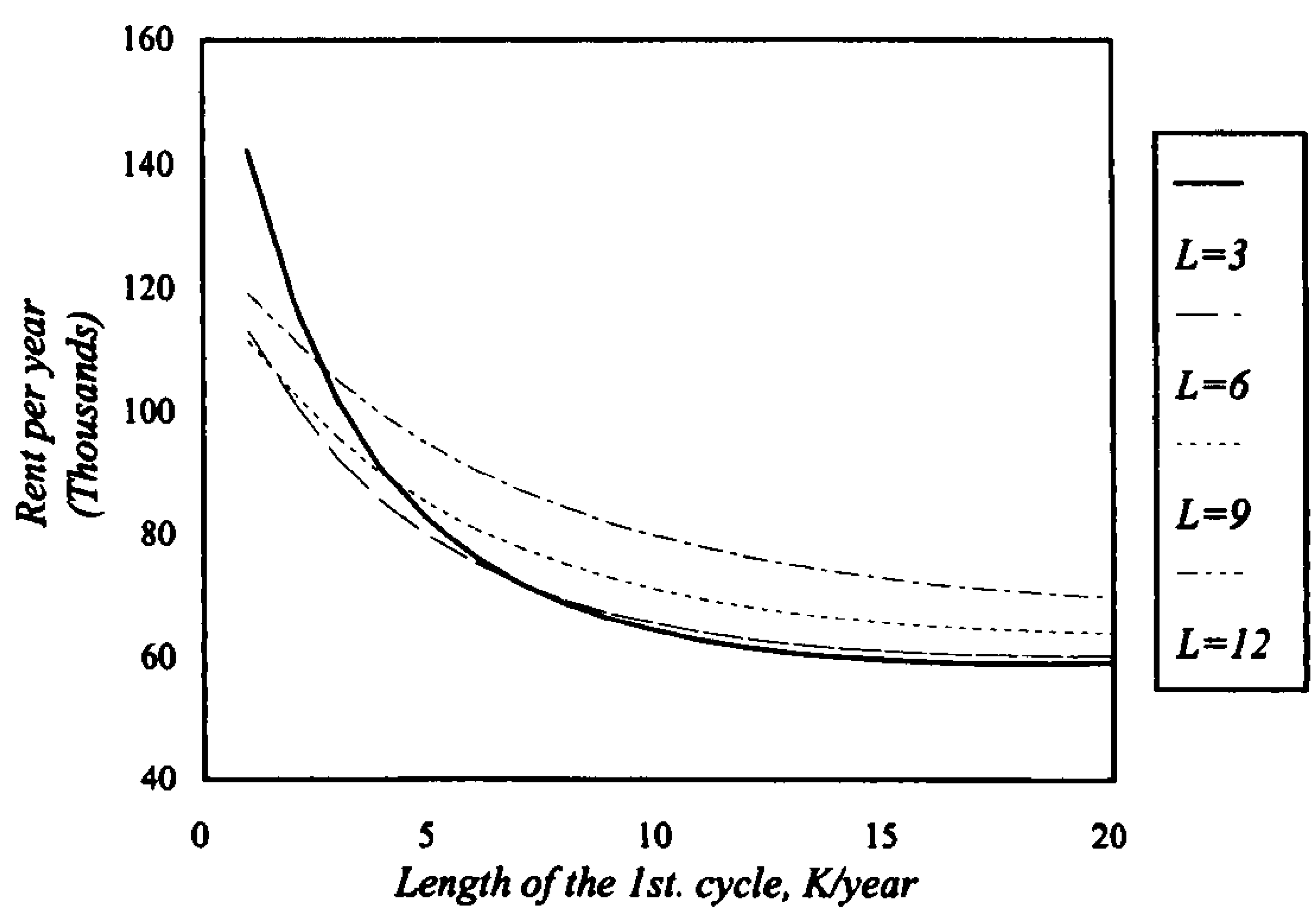


Figure 3.8. Rent per year versus length of first cycle K, for various lengths of second cycle L, Mercedes replaced with Isuzu CJR. Two-cycle model with variable planning horizon.

3.5.1.2. Model extensions

The use of two cycles in the model can be considered as a compromise between the need to model the on-going requirement for the plant, and the requirement for reasonable forecasts for future costs. The model can be adapted to the case of variable length planning horizon models with one cycle or three cycles. Here the lengths of the cycles are decision variables and the number of cycles chosen is a control variable for the replacement decision.

The model also can be extended to the case of sub-fleet replacement. This is when the fleet is inhomogeneous, so that plant differs in model specification, age and/or condition, say. Here, replacement models which consider a typical plant are inappropriate. When the fleet is to be replaced as an entire fleet, then model, equation (3.8), may be extended. This can be done by summing age (and plant specification) related maintenance costs per unit time and resale values over the entire fleet. However, it is unlikely that an operator would consider replacing an inhomogeneous fleet in its entirety. When plant are to be replaced singly, then, given individual maintenance cost history, repair cost limit replacement policies (Hasting, 1967; Jardine et al., 1976) may be considered. Of course, the replacement of plant singly may be impractical in certain cases. The replacement criteria, equations (3.7) and (3.8), are appropriate for a fleet which is homogeneous.

It may be often natural to subdivide the fleet, with interest focusing on which single sub-fleet is to be replaced and when to replace it. The choice of the equipment model type for purchase will usually be defined in advance by the operator/manager of the fleet. Optimal policy for the replacement of any particular sub-fleet would depend on the cost of the whole fleet. With new sub-fleet (s) likely to be different from the replaced sub-fleet (s), technological change is accepted and may be modelled.

Scarf and Bouamra (1995) presented a model for the case in which an inhomogeneous fleet is considered as comprising r sub-fleets of sizes

n_i $(i = 1,..,r)$. The approach was used to determine: which sub-fleet to replace first (on the basis of minimum cost over the horizon); when to replace it; and the increased cost of alternative (sub-optimal) choices for sub-fleet to be replaced first.

Some papers have been presented about the problem of mixed fleet and the size and composition of a fleet. An early attempt to tackle the problem of fleet size optimisation was made by Kirby (1959). He described the two-sided problem of both preventing a low utilisation of owned wagons in a small railway system and conversely preventing the frequent hire of costly extra wagons. He obtained an expression for the total expected cost per day and via this he could determine the number of owned and hired wagons that would minimise cost.

Gould (1969) discussed the simple fleet size problem and then presented an actual case study. Linear programming was used to find the optimum size and composition of the fleet. The study resulted in some principle recommendations.

1- Reduction in the size of the company fleet.

2- More emphasis on large vehicles.

3- More emphasis on stainless steel vehicles.

These recommendations were accepted by the company.

Mole (1975) extended previous work on the fleet size problem. A dynamic programming model was developed to determine the optimum fleet size that is time dependent. In order to allow for obsolescence of vehicles, a particular disposal policy was specified in advance, and it was assumed that the purchase price can be appropriately reduced to allow the scrap value in the period specified by the disposal policy. This paper took no account of the possibility of a mix of differing vehicle types and sizes, or the possibility of incompatibility between vehicles and loads. The model discussed was quite flexible enabling the cost implication of many practical policies to be estimated.

Woods and Harris (1979) used a simulation approach for investigating fleet composition for concrete distribution. Each vehicle handled only one order at a time, and vehicle trips were thus round trips between the depot and a single customer. They denoted that order was to some extent matched to vehicle size. Statistical analysis was used to determine the percentage of customers who would switch order sizes if the fleet mix was changed.

In his paper Parikh (1977) described an approximate but quick method for solving a fleet sizing and allocation problem. This paper presented a general queuing-theory-based approach to obtain a fast, approximate solution to the problem which arises when there is a lack of immediate availability of a transport vehicle unit. The lack of service was measured in terms of the factor of customer delayed orders and was assumed to be less than unity. Finally the fleet sizing problem corresponds to a multi-server queuing system with known mean arrival and service rates.

Etezali and Beasley (1983) considered the problem of determining the optimal composition of a vehicle fleet (the best fleet size and the best fleet mix). This paper was an attempt to tackle the problem of fleet composition only rather than the vehicle fleet size problems that had received more attention in the literature. The authors developed a mixed integer programming formulation of the problem; they were concerned with long-term decisions concerning the number and type of vehicles that the company should operate. They found that the solution to the programme, for any particular set of data, will give a vehicle fleet judged to be the best vehicle fleet that can be examined in more detail using the simulation approach.

A Recent study on the fleet size and fleet mix was presented by Scarf and Bouamra (1999). They tackled the capital replacement model for a fleet with variable size. They were concerned with the question : "how many items of plant are required to maintain a certain level of availability?"; this was associated with the question of "at what time should a currently operating plant or fleet of plant be replaced ?" . They developed a simple two-cycle model in which the size and the age at replacement of the fleet replacement are the principal decision variables. They also used a birth and death process to model the unavailability due to failure, and this unavailability was considered as a penalty cost. Optimal fleet replacement decisions were presented over a certain range of the penalty costs. The model allowed for the possibility of modelling varying demand with time. They applied the model developed to a homogeneous fleet using real data related to ventilators in the operating department of a large hospital. A difficulty of the model is that the optimum value of the horizon length depends on the choice of sub-fleet to be replaced (replacement schedule).

Vemuganti, Oblak & Aggarwal (1989) presented network models to determine the optimal replacement policy for a fleet of vehicles of various types and ages over a finite planning horizon. Although the models were formulated in the context of a fleet of vehicles, they are applicable to many-equipment replacement problems. An interesting feature of the network models was an allowance for fleet size variation during the planning horizon. The paper suggested further investigation for the composition of the vehicles of various ages in the fleet.

Another study recently done by Scarf and Bouamra (1995) considers a model for a mixed inhomogeneous fleet. They proposed a method to consider the optimum size of new sub-fleets replaced over a certain horizon. A two-cycle replacement model with a variable finite planning horizon was formulated with the time to first replacement and the size of the new sub-fleet at this time as principal decision variables. The model was applied to a mixed fleet of buses operated by a large Malaysian intercity bus company. Results for fixed fleet size and variable fleet size were obtained indicating the optimality of decision variables and the replacement order to the different sub-fleets.

In their paper Simms, Lamarre, Jardine and Boudreau (1984) described a policy for buying, operating and selling buses. The problem arose when a large urban transient authority with a yearly budget constraint operating a mixed fleet of buses which accumulated a number of kilometres per year, did not know whether to replace or not. A model was developed using linear and dynamic programming to select buy, sell and operating policies to minimise the total discounted cost over the finite horizon. Also a computer program was developed to optimise a flexible model to be used in a wide variety of replacement problems. The results could be equally useful to maintenance engineers, operations managers and budgeting officers.

Retirement of sub-fleet as spares occasionally (or even the case when sub-fleet is retained and fully used), can be modelled using the approach described above. The number of sub-fleets would simply increase by one at each replacement, with the cost associated with the retired sub-fleet added. Predicting the mean number of failures and maintenance costs for a retired sub-fleet would be difficult however, and it is likely that no data would be available for this, because such item would be used occasionally.

3.6. Fixed horizon Models

The previous models consider a variable planning horizon with a fixed number of cycles as a control variable, usually two in number. In the fixed horizon models the length of the planning horizon is fixed and is considered as a control variable. There will then be a variable number of cycles each of variable length. The number of cycles and their lengths are considered as decision variables.

A capital replacement model with a fixed planning horizon, h, say, may be formulated with number of cycles, $N(\geq 2)$, say, as a decision variable (Scarf and Christer, 1997). Other decision variables are the time, K, from now to replacement of the current plant (length of first cycle), and the lengths, $L_2,..,L_N$, of subsequent cycles. This scheme of the model behaviour is illustrated in Figure (3.9).

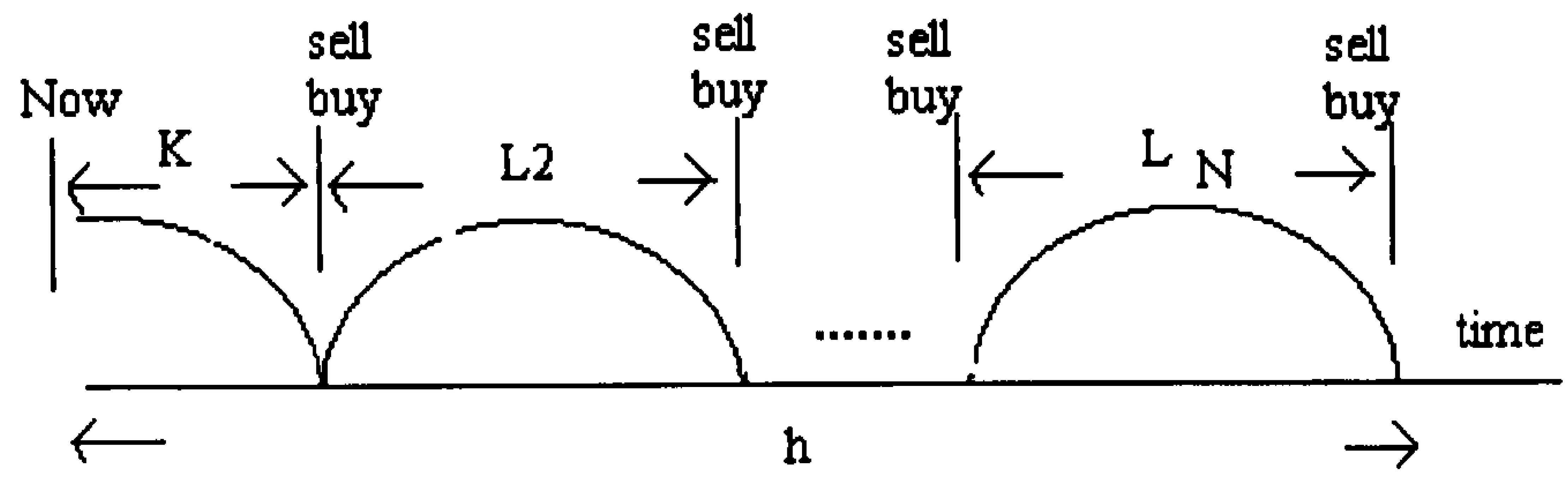


Figure 3.9. Replacement model with fixed planning horizon of length h, and variable number of cycles, N, of length $K, L_2,.., L_N$.

A special case of this model was first proposed by De Sousa and Guimaraes (1992). Bean et al. (1994), for example, have a fixed horizon for modelling technological change, but the approach is only applicable to a single plant or homogeneous fleet. Simms et al. (1984) sought an optimum age based mix for a fleet over a fixed horizon. Bouamra (1996) briefly describes a sub-fleet replacement model over a fixed planning horizon. Scarf and Hashem (1997) described sub-fleet replacement for a large inhomogeneous fleet of buses and also described the behaviour of a simple fixed planning horizon model in general context.

The approach which uses a fixed planning horizon replacement model resembles more closely the real world situation. Here an operator fixes a horizon of a certain length and is then interested in how many replacement cycles are needed to span the horizon, and in the length of each replacement cycle. This approach can also reduce the "end of horizon effect" that variable length planning horizon model may produce. In their paper Scarf and Christer (1997) presented a capital replacement model over a fixed planning horizon. The model is for the replacement of a single plant or homogeneous fleet, and the total discounted cost is expressed in the form

$$TDC(N,K,L_2,..,L_N;h) = \upsilon^K [\sum_{t=1}^{K} C_1(\tau+t)\upsilon^{t-K-1/2} + R_2 - S_1(\tau+K)$$
$$+ \sum_{i=2}^{N} \upsilon^{m(i)} \{\sum_{t=1}^{L_i} C_i(t)\upsilon^{t-L_i-1/2} + R_{i+1} - S_i(L_i)\}], \qquad (3.17)$$

where

$$m(i) = \sum_{j=2}^{i} L_j \, ;$$

τ is the age now of existing plant; $C_i(.)$ and $S_i(.)$ are the age related maintenance cost per unit time and resale value of the plant in cycle $i\,(i=1,..,N)$; R_i is the replacement cost of plant operated in cycle $i\,(i=2,..,N)$, and purchased at the end of the planning horizon $(i=N+1)$; and υ is the discount rate. The objective function may be considered as either the total discounted cost per unit time as $TDC(N,K,L_2,..,L_N;h)/h$, or the equivalent rent as $TDC(N,K,L_2,..,L_N;h)/\sum_{i=1}^{h}\upsilon^i$, and optimised subject to the constraint

$$k + \sum_{i=2}^{N} L_i = h .$$

It is recommended that optimum policy be determined for a range of values of h, and then h chosen not too large, but large enough in order not to increase costs by imposing a poorly scheduled replacement. If replacement is like-with-like, then as $h \to \infty$, optimum policy reduces to that based on an infinite cycle model. Fixed planning horizon models can be used to illustrate the effect of length of the horizon on the resulting optimum policy. Looking at the resulting optimum policy as a function of the horizon length (control variable) then allows the decision maker to choose a "robust" optimum policy.

3.6.1. A simple fixed planning horizon model

In their paper Scarf and Christer (1997) reviewed models with finite horizons that may be classified according to two types: variable planning horizon models and fixed planning horizon models. The models were also classified according to their use for modelling replacement of single plant, an entire fleet, or sub-fleet of a large inhomogeneous fleet. They pointed out that the models discussed were also appropriate for equipment improvement and refurbishment decisions such as major redesigns costing substantial sums of money and therefore requiring both justification and strategic planning.

A fixed planning horizon model with at most two replacements performed over the horizon is presented in its simplest form as

$$C(x)=\begin{cases}\int_0^x \alpha(t+\tau)^\beta dt+\int_0^{h-x}\alpha t^\beta dt+2R, & 0\le x<h,\\ \int_0^h \alpha(t+\tau)^\beta dt+R, & x=h,\end{cases} \tag{3.18}$$

where $\alpha(t+\tau)^\beta, \alpha t^\beta$ are respectively the age-related maintenance cost per unit time for the current equipment and the new equipment; x is the time until first replacement; τ is the age of the current equipment; R is the price of the new equipment and h the length of the horizon. Here h is a control variable and x is the decision variable. The model outline is illustrated in Figure 3.10. We impose a replacement at the end of the horizon in order to allow a comparison with the Christer & Goodbody model.

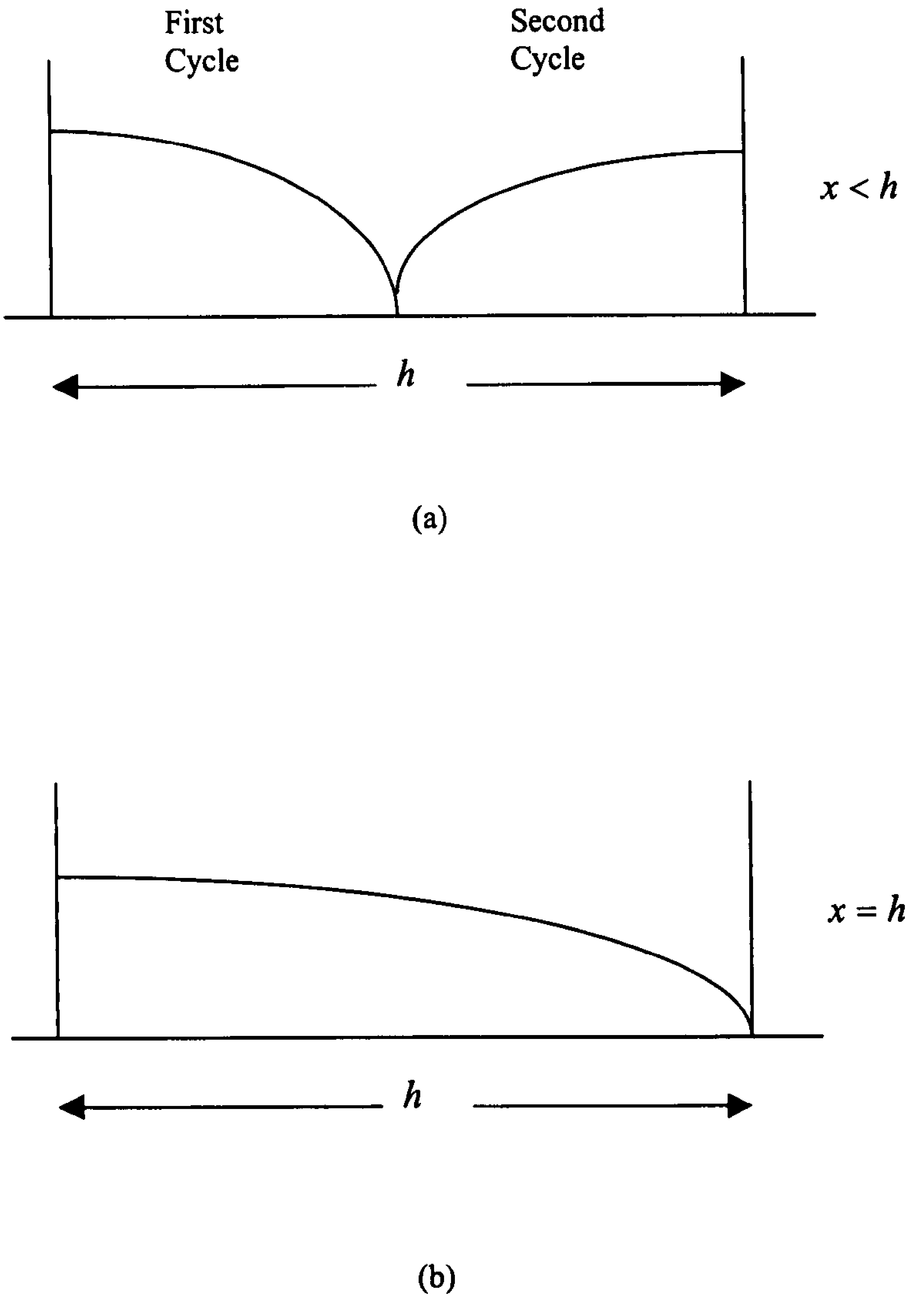


Figure 3.10. A simple fixed horizon model with at most two replacements: (a) two replacements at $x < h$ and at $x = h$; (b) single replacement at the end of the horizon $x = h$.

The cost function $C(.)$ for this model has the form illustrated in Figure 3.11.

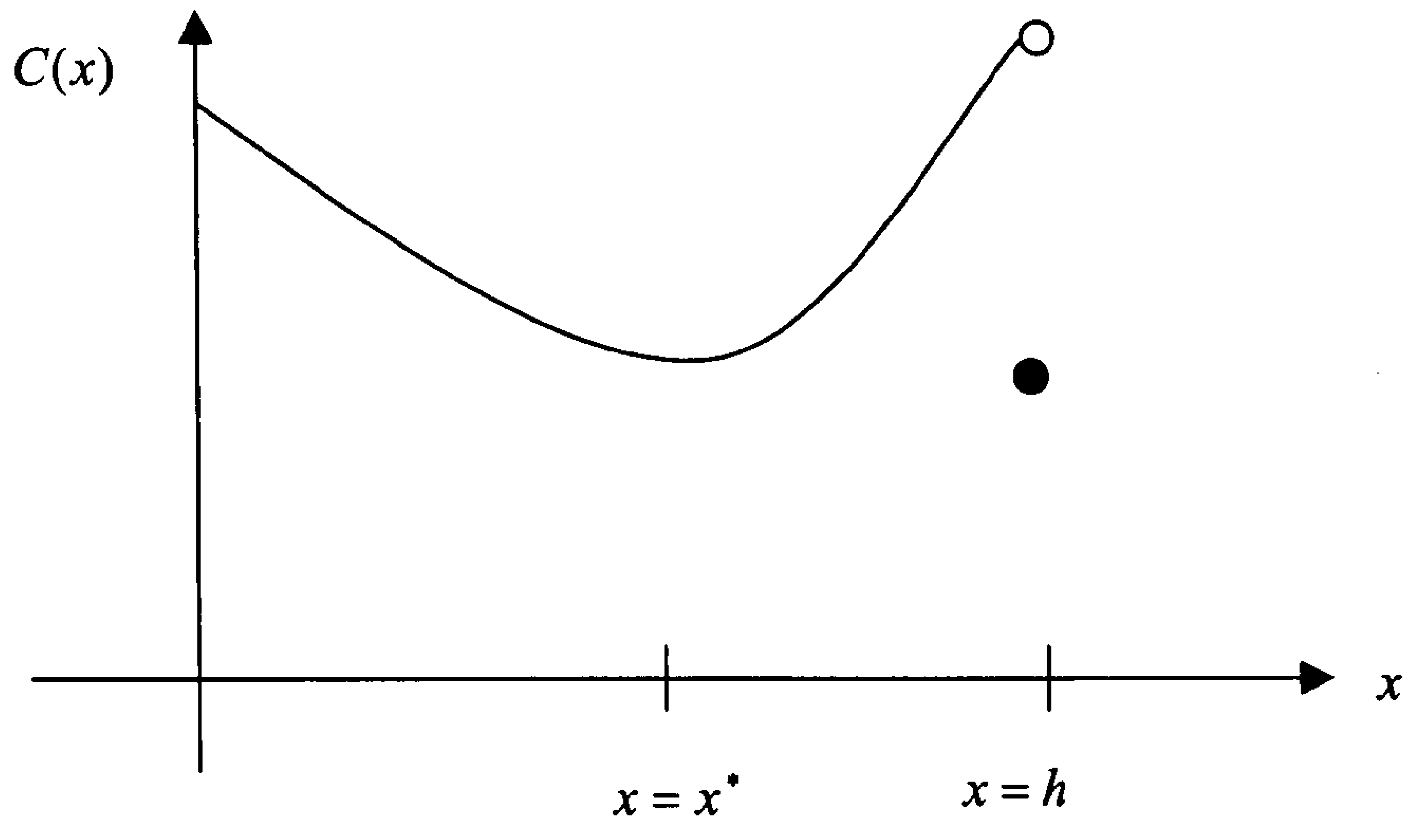


Figure 3.11. Cost function for a simple fixed horizon model with two local minima

Note that, for simplicity, the resale value is ignored and the discount factor is considered as unity.

This model is useful in practice, because often the life of the equipment is similar in duration to the length of the planning horizon. We consider this model in particular in order to study the behaviour of the optimal policy. This we do in chapter 4.

CHAPTER 4

CHAPTER 4

The Behaviour of Optimal Policy for Simple Capital Replacement Models

4.1. Introduction

In this chapter we consider the behaviour of optimal policy for a number of different capital replacement models. We begin this by looking at the behaviour of a simple economic life model of Eilon et al. (1966). Then we consider the behaviour of replacement models with a fixed planning horizon of length h (Scarf & Hashem, 1997).

We contrast these models with variable planning horizon models proposed by Christer & Goodbody (1980), which were motivated by the need to consider both technological change and the fact that decision-makers need to take account of the age of the equipment or fleet currently in operation.

4.2. Infinite horizon model of like-with-like replacement

In this simple economic life model we suppose that the equipment is used until aged y and then replaced. The model is formulated such that the average cost per unit time is

$$C(y) = \frac{1}{y}\left[\int_0^y f(t)dt + R\right], \tag{4.1}$$

where $f(t)$ is the maintenance cost per unit time for equipment aged t, y is the age at replacement and R is the purchase cost of the new equipment. This model is presented in section 3.4. We assume that the discount rate $\upsilon = 1$. We seek the value of y which minimises the cost $C(y)$. With the maintenance cost per unit time $f(t)$ in the form αt^{β}, which was discussed in section 2.1, equation (4.1) takes the form

$$C(y) = \frac{1}{y}\left[\int_0^y \alpha t^{\beta} dt + R\right].$$

The minimisation procedure is carried out as follows. Differentiate with respect to y to get

$$\frac{dC}{dy} = \frac{1}{y^2}\left[\alpha y^{\beta+1} - \frac{\alpha y^{\beta+1}}{\beta+1} - R\right].$$

Setting $dC/dy = 0$, we get

$$y^* = [R(\beta+1)/\alpha\beta]^{1/(\beta+1)}. \tag{4.2}$$

y^* is called the economic life. The value of d^2C/dy^2 can be calculated from the following formula:

$$\frac{d^2C}{dy^2} = \frac{1}{y^3}\left[\alpha\beta y^{\beta+1} + \frac{2\alpha}{\beta+1} y^{\beta+1} + 2R\right]. \tag{4.3}$$

Equation (4.3) indicates that $d^2C/dy^2 > 0$, since the quantities α, β and R are all positive and so $C(y)$ the average cost per unit time, is minimised at $y = y^*$. Equation (4.2) describes the behaviour of optimal policy in terms of the replacement cost, R and the maintenance cost per unit time function $f(t) = \alpha t^\beta$. If the replacement has to be made on the basis of non-like-with-like, the infinite planning horizon implies that the new model of equipment, as well economic factors and failure costs, need to be predicted in an objective fashion.

The cost function in the case of non-zero resale value ($S \neq 0$) is

$$C(y) = \frac{1}{y}\left[\int_0^y f(t)dt + R - S(t)\right].$$

Again to minimise $C(y)$ we set $dC/dy = 0$ where, with $f(t) = \alpha t^\beta$,

$$\frac{dC}{dy} = \frac{1}{y^2}\left[\alpha y^{\beta+1} - \frac{\alpha y^{\beta+1}}{\beta+1} - R - yS' + S\right],$$

where S' represents dS/dt.

Setting $dC/dy = 0$ yields

$$\alpha y^{\beta+1}(\beta/\beta+1) - R - yS' + S = 0, \tag{4.4}$$

from which

$$R[(y/y^*)^{\beta+1} - 1] = yS' - S,$$

and so

$$(y/y^*)^{\beta+1} = \frac{yS' - S}{R} + 1.$$

Hence

$$y/y^* = \left\{\frac{yS' - S}{R} + 1\right\}^{\frac{1}{\beta+1}}.$$

Suppose $S = ae^{-bt}$, $a, b > 0$, then,

$$S' = -abe^{-bt}.$$

Hence

$$0 < \frac{yS' - S}{R} + 1 = \frac{-a}{R}[yb + 1]e^{-bt} + 1 < 1.$$

In fact, provided S decreases with t then $yS' - S < 0$, so that $y / y^* < 1$, that is $y^* > y$. Thus if S decreases with t then y^* will always be an upper bound for the economic life. Given the numerous choices for S that could be adopted in practice and the non-linear nature of equation (4.4) we do not consider non-zero resale value further in this chapter.

4.3. Fixed planning horizon models

In order to gain insight into the behaviour of models with a fixed planning horizon, we study the behaviour of optimal policy for a simple model. This is done using certain restrictive assumptions regarding the number of replacements, the form of the maintenance cost per unit time for current and future equipment, and the replacement costs and resale values.

4.3.1. Notation and assumptions

Consider a fixed planning horizon, h. In line with the earlier models (Christer & Goodbody 1980, Christer & Scarf 1994, Scarf & Bouamra, 1995) we impose a resale and replacement at the end of the horizon in order to make a comparison with Christer and Goodbody model. For simplicity we also assume that:

(i) at most two replacements will take place over the horizon

(ii) a typical equipment of age τ is currently being operated and on replacement the new equipment is of age zero (new equipment).

(iii) the purchase cost of the new equipment is R;

(iv) the maintenance cost per unit time of the equipment age t is αt^{β}, where $\alpha > 0$; $0 \le \beta \le 1$; (see Figure 4.4).

(v) the resale value of equipment is zero;

(vi) the discount rate is unity.

4.3.2. Like-with-like replacement

In this section we suppose that the new plant is an identical model to the current plant. So, if the time to the first replacement is x, the total cost over the horizon h takes the following form (as described in equation 3.18):

$$C(x) = \begin{cases} \int_0^x \alpha(t+\tau)^{\beta} dt + \int_0^{h-x} \alpha t^{\beta} dt + 2R, & 0 \le x < h, \\ \int_0^h \alpha(t+\tau)^{\beta} dt + R, & x = h, \end{cases} \tag{4.5}$$

where αt^{β} is the maintenance cost per unit time of an equipment at age t; R is the purchase cost new for an equipment; x is the time of the first replacement and h is the length of the fixed planning horizon. Notice that h is fixed; strictly we consider it as a control variable which may be varied by the modeller and/or decision-maker. Notice that if $x = h$, then there is only one replacement over the horizon (see Figure 4.1) and for this reason $C(x)$ is discontinuous at $x = h$.

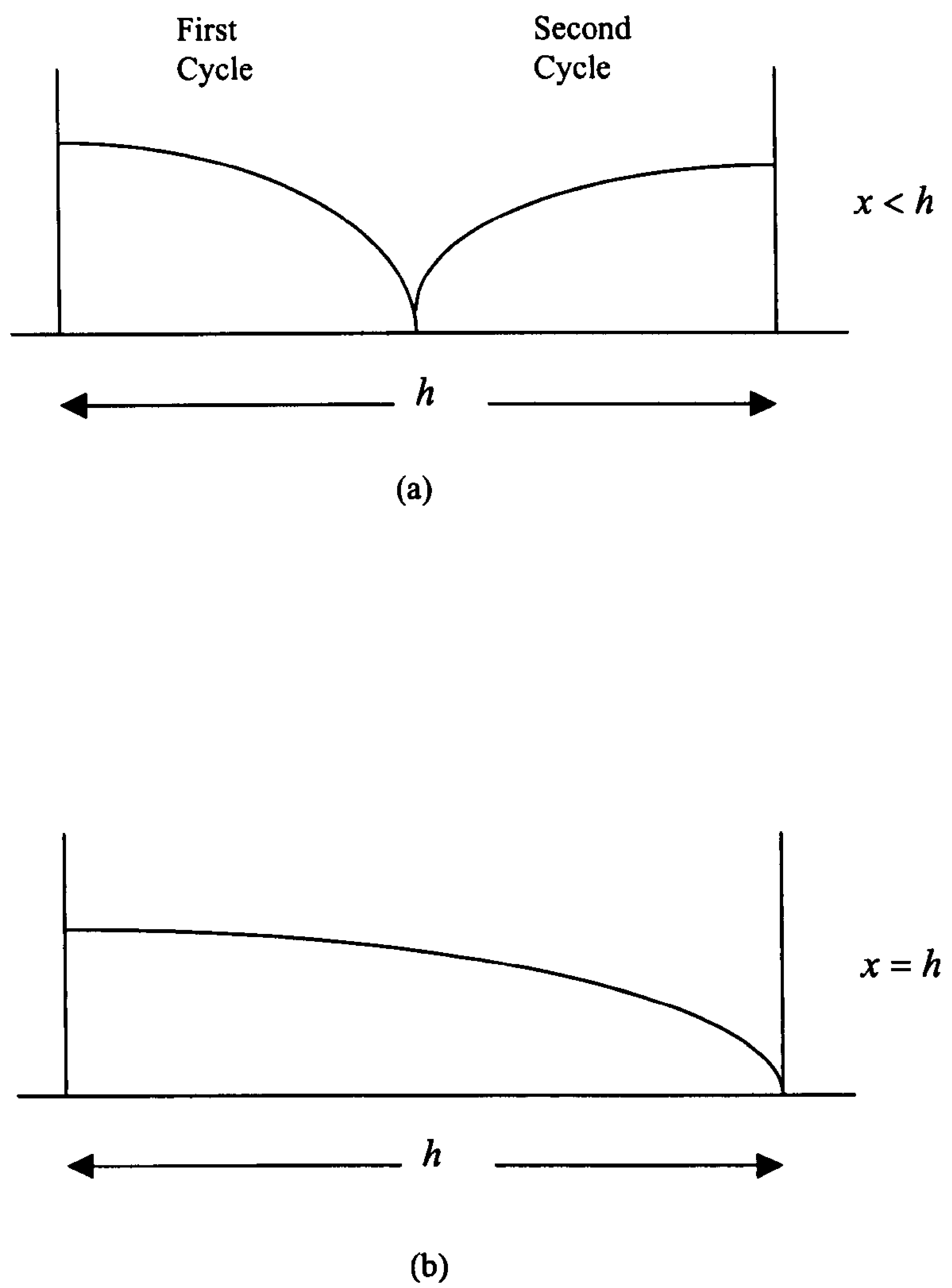


Figure 4.1. A simple fixed horizon model with at most two replacements:
(a) two replacements; (b) single replacement at the end of the horizon.

We have to minimise the cost function (4.5). First note that

$$C(x) = \frac{\alpha(x+\tau)^{\beta+1}}{\beta+1} - \frac{\alpha\tau^{\beta+1}}{\beta+1} + \frac{\alpha(h-x)^{\beta+1}}{\beta+1} + 2R, \qquad 0 \le x < h.$$

Differentiating with respect to x we get

$$dC/dx = \alpha(x+\tau)^{\beta} - \alpha(h-x)^{\beta}.$$

Setting $dC/dx = 0$, we obtain $x = (h-\tau)/2$, $(\tau \le h)$, as a solution. Also

$$d^2C/dx^2 = \alpha\beta\{(\tau+x)^{\beta-1} + (h-x)^{\beta-1}\} > 0 \qquad \text{for } 0 \le x < h.$$

Thus $C(x)$ has local minima at $x = (h-\tau)/2$ and at $x = h$. The global minimum can be obtained by comparing the two local minima $C((h-\tau)/2)$ and $C(h)$. Thus, $C(x)$ has a global minimum at $x = (h-\tau)/2$, and it is optimal to perform two replacements, if

$$[h+\tau]^{\beta+1} - 2[(h+\tau)/2]^{\beta+1} > R(\beta+1)/\alpha. \qquad (4.6)$$

Otherwise it is optimal to make only one replacement at the end of the planning horizon (by default).

Equation (4.6) is most easily interpreted when $\tau = 0$, whence $C(x)$ has a minimum at $x = h/2$ if

$$h^{\beta+1} - 2^{-\beta}h^{\beta+1} > R(\beta+1)/\alpha. \qquad (4.7)$$

Inequality (4.7) leads to the conclusion that it is optimal to perform two replacements if

$$h > \left[R(\beta+1)/\alpha(1-2^{-\beta})\right]^{1/(\beta+1)}.$$

Note that the replacement is carried out at the midpoint of the planning horizon.

For $\beta = 0.7$, $\alpha = 25$, $R = 450$, say, and t measured in years, we obtain that it is optimal to perform two replacements, one at $h/2$ and the other at h, when $h > 13.1$ otherwise we perform only one replacement at h.

Also from equation (4.2) and inequality (4.7), we have that for the fixed horizon model it is optimal to replace at $x = h/2$ if:

$$h/y^* > \left[\beta/(1-2^{-\beta})\right]^{1/(\beta+1)} \approx \sqrt{2}, \qquad (\tfrac{1}{2} \le \beta \le 1).$$

The conclusion of this result is that if the economic life of equipment is y^*, then the fixed horizon model will always replace once only at the end of the planning horizon if $h < \sqrt{2}y^*$ approximately (taking into account the typical values of β occurring in practice). Otherwise for $\tau = 0$, replacements should be performed at the mid point of the planning horizon and at the end. Consequently, the optimum policy is sensitive to the length of the planning horizon h. This emphasizes that h should be chosen with care.

When $\tau \neq 0$ we have that it is optimal to perform two replacements, one at $(h - \tau)/2$ and the other at h when (from inequality 4.7)

$$h > \{R(\beta + 1)/\alpha(1 - 2^{-\beta})\}^{1/(\beta+1)} - \tau. \qquad (4.8)$$

Alternatively we can view this inequality (4.8) as a condition on τ, that is

$$\tau > \{R(\beta + 1)/\alpha(1 - 2^{-\beta})\}^{1/(\beta+1)} - h = \tau_c.$$

If $\tau > \tau_c$ (some critical value) then two replacements, one within the horizon, and one at the end (by default), is optimal.

For $\beta = 0.7$, $\alpha = 25$, $R = 450$, $\tau = 5(10)$ and t measured in years, we have that it is optimal to replace at $x = (h - \tau)/2$ provided that $h > 8.1$ for $\tau = 5$ but for $\tau = 10$ the equipment is currently old so that $\tau > h$, leading to replacement immediately. The optimum policy changes with τ in a simple manner.

Also in the case of $\tau \neq 0$ and from equation (4.2) and inequality (4.8) we have that for the fixed horizon model it is optimal to replace at $x = (h - \tau)/2$ if

$$(h + \tau)/y^* > [\beta/(1 - 2^{-\beta})]^{1/(\beta+1)}.$$

For $1/2 \le \beta \le 1$, the typical values of β that occur in practice, $[\beta/(1-2^{-\beta})]^{1/\beta+1}$ has approximate minimum value $\sqrt{2}$. Thus if the economic life of an equipment is y^*, then the fixed horizon model will always replace once only at the end of the planning horizon if $h < \sqrt{2}y^* - \tau$ approximately. Otherwise replacements should be made at $(h-\tau)/2$ $(\tau \le h)$ and at the end of the planning horizon.

Now if $\tau > h$, it follows that $dC/dx \ne 0$ for $0 \le x \le h$. Therefore $C(x)$ has local minima at $x = 0$ and at $x = h$, and it is optimal to replace immediately (at the beginning of the horizon) if $C(0) < C(h)$, that is if

$$[h+\tau]^{\beta+1} - [h^{\beta+1} + \tau^{\beta+1}] > R(\beta+1)/\alpha.$$

There will also be a replacement at the end of the horizon (by default). For example, if $\beta = 1$, then it is optimal to replace immediately if $\tau > R\alpha^{-1}/h$.

We should briefly note that if $\beta = 0$, then the maintenance costs are not age-related and so for like-with-like replacement, replacement of existing plant will not be optimal. This is an obvious point here but will become more relevant in the following section when we consider non-like-with-like replacement with $\beta = 0$.

4.4. Fixed planning horizon models and non-like-with-like replacement

Suppose now that the new plant differs from the current plant, and that this difference is only in economic terms (purchase cost, maintenance cost per unit time); the new equipment performs the same function as the current equipment. The maintenance cost per unit time of the new equipment is assumed to take the form $\alpha_2 t^{\beta_2}$ while that of the existing equipment takes the form $\alpha_1 t^{\beta_1}$.

4.4.1. Case $\beta_1 = \beta_2$

For the simple case $\beta_1 = \beta_2 = \beta$, we first let $\alpha_1 = \alpha$ and $\alpha_2 = \rho\alpha$. If the time to first replacement is x, the total cost over the planning horizon $[0,h]$ is then

$$C(x) = \begin{cases} \int_0^x \alpha(t+\tau)^\beta dt + \int_0^{h-x} \rho\alpha t^\beta dt + 2R, & 0 \le x < h, \\ \int_0^h \alpha(t+\tau)^\beta dt + R, & x = h, \end{cases} \tag{4.9}$$

where ρ is a factor representing the ratio of maintenance cost per unit time of the new equipment to that of the old (current) equipment and the other parameters are as defined earlier (section 4.2.1). Again if $x = h$ then there is only one replacement over the horizon and $C(x)$ is discontinuous at $x = h$. Differentiating with respect to x, we have

$$dC/dx = \alpha(x+\tau)^\beta - \rho\alpha(h-x)^\beta.$$

Setting $dC/dx = 0$ we obtain:

$$\alpha(x+\tau)^\beta = \rho\alpha(h-x)^\beta, \qquad 0 \le x < h. \tag{4.10}$$

Also

$$d^2C/dx^2 = \alpha\beta[(x+\tau)^{\beta-1} + \rho(h-x)^{\beta-1}] > 0, \qquad 0 \le x < h,$$

so $C(x)$ has local minima at:

$$x = x' = (h\rho^{1/\beta} - \tau)/(\rho^{1/\beta} + 1),$$

and at:

$$x = h, \qquad h \ge \tau\rho^{-1/\beta}.$$

The local minimum at $x = x' = (h\rho^{1/\beta} - \tau)/(\rho^{1/\beta} + 1)$, is the global minimum if $C(x') < C(h)$, that is if

$$\left\{\frac{(h\rho^{1/\beta}-\tau)}{(\rho^{1/\beta}+1)}+\tau\right\}^{\beta+1}+\rho\left\{h-\frac{(h\rho^{1/\beta}-\tau)}{(\rho^{1/\beta}+1)}\right\}^{\beta+1}-(h+\tau)^{\beta+1}+R(\beta+1)/\alpha<0. \tag{4.11}$$

Then there are two replacements over the horizon. Otherwise there is only one replacement at the end of the horizon. Inequality (4.11) explicitly provides the value of h for which there are two replacements, that is if

$$h>\left[R(\beta+1)/\alpha\{1-\rho(1+\rho^{1/\beta})^{-\beta}\}\right]^{1/\beta+1}-\tau. \tag{4.12}$$

Alternatively we can view this inequality (4.12) as a condition on τ, that is

$$\tau>\left[R(\beta+1)/\alpha\{1-\rho(1+\rho^{1/\beta})^{-\beta}\}\right]^{1/\beta+1}-h=\tau_c.$$

If $\tau>\tau_c$ (some critical value) then two replacements, one within the horizon, and one at the end (by default), is optimal. Inequality (4.12) may also be considered as a condition on ρ, the ratio of the instantaneous age related maintenance costs per unit time of the new to current equipment, so that it is optimal to replace at time $x=x'=(h\rho^{1/\beta}-\tau)/(\rho^{1/\beta}+1)$, (and at h by default), if

$$\rho(1+\rho^{1/\beta})^{-\beta}<1-\frac{R\alpha^{-1}(\beta+1)}{(h+\tau)^{\beta+1}}.$$

When $\tau=0$, it is optimal to replace at $x=h\rho^{1/\beta}/(\rho^{1/\beta}+1)$ if

$$h>\left[R(\beta+1)/\alpha\{1-\rho(1+\rho^{1/\beta})^{-\beta}\}\right]^{1/\beta+1}.$$

Figures 4.2 and 4.3 illustrate the cost function.

For $\tau=0$, $\beta=0.7$, $\alpha=25$, $R=450$, $\rho=2$ (the maintenance cost per unit time of the new equipment is twice as expensive as the maintenance cost per unit time of the current one) and t measured in years - we have that it is optimal to replace at $0.7h$ provided that $h>19.5$ years. If the maintenance cost per unit time of the new equipment is less expensive than that of the old (current) one, thus, for the same values mentioned above, and $\rho = 1/2\,(2/3)$, the optimal replacement occurs at $0.3h\,(0.4h)$ provided $h > 10.1\,(11.1)$ years. We consider other values for ρ and the results are shown in Table 4.1. Obviously, from Table 4.1, we notice that as the value of ρ increases the first replacement time gets close to h and that the planning horizon length for which two replacements would be made increases. Also from the table it is noticed that there is a symmetrical behaviour between the values of ρ and the values of x (for example $\rho = 1/2$ gives $x = 0.3h$ and $\rho = 2$ gives $x = 0.7h$). Note that the value of h, as we mentioned in many situations above, is a control variable and that the optimum policy would be, in practice, determined for a range of values of h.

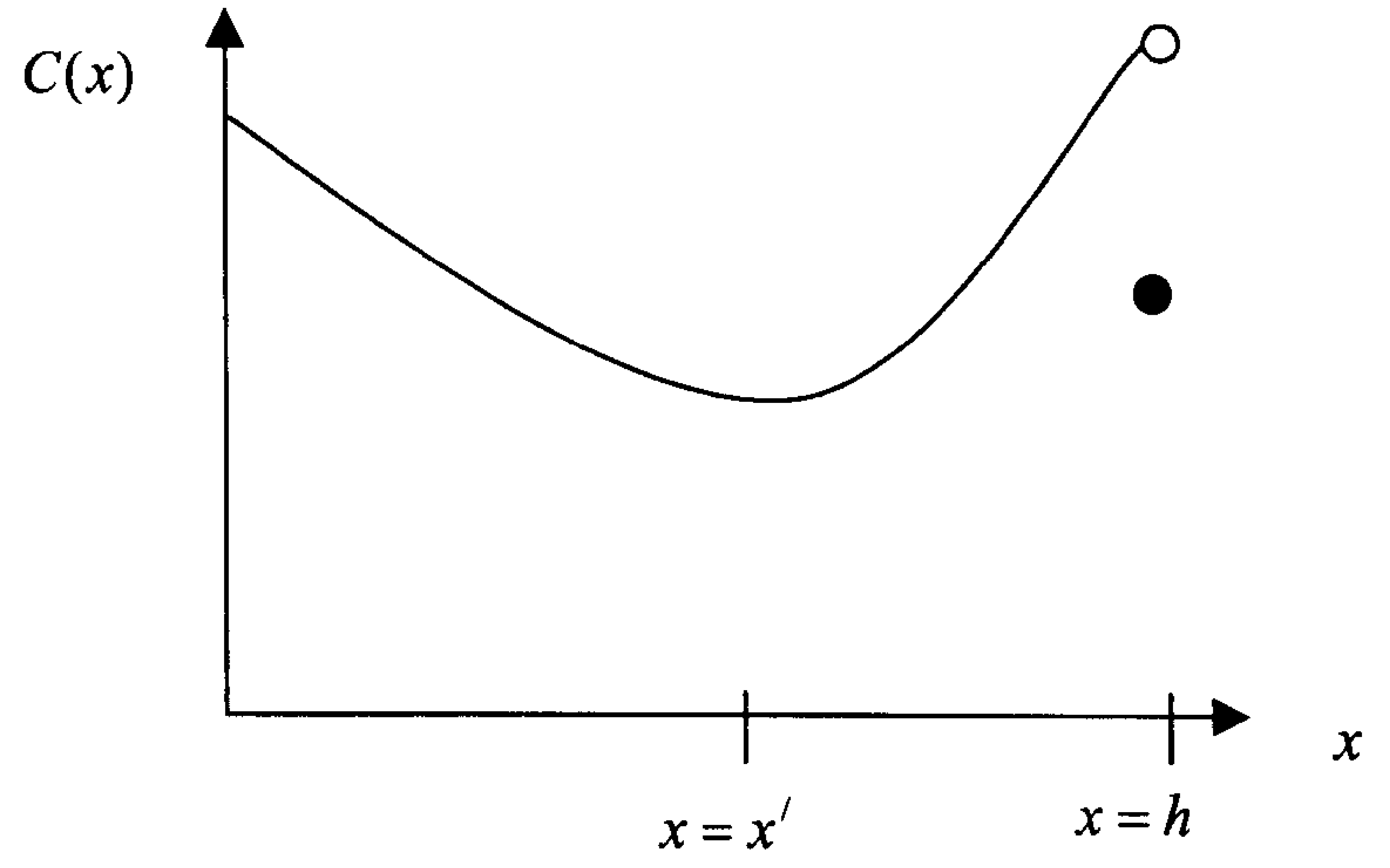


Figure 4.2. Cost function for a simple fixed planning horizon with two local minima at x' and h for non-like-with-like replacement model $(\tau \leq h\rho^{1/\beta})$.

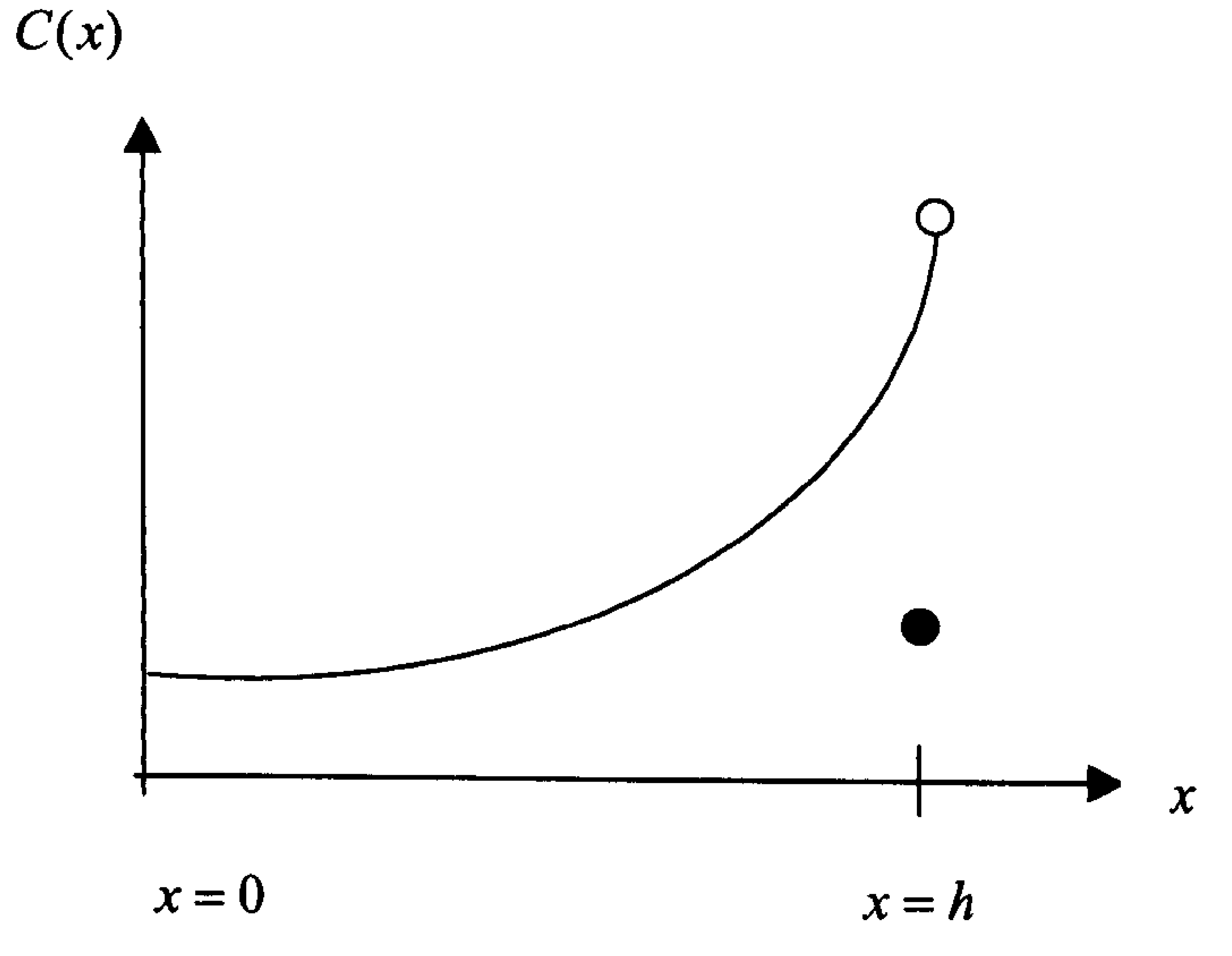


Figure 4.3. Cost function for a simple fixed planning horizon with two local minima at 0 and h for non-like-with-like replacement model $(\tau > h\rho^{1/\beta})$.

Table 4.1. The times of first replacement and lengths of horizon at which two replacements would be made for different maintenance cost per unit time factors, ρ.

ρ	x (time to first replacement)	h (horizon length)
1/4	$0.1h$	8.7
1/3	$0.2h$	9.2
1/2	$0.3h$	10.1
2/3	$0.4h$	11.1
2	$0.7h$	19.5
3	$0.8h$	25.5
4	$0.9h$	31.6

If $\tau > h\rho^{1/\beta}$, it follows that $dC/dx \neq 0$ for $0 \leq x \leq h$. Therefore $C(x)$ has local minima at $x=0$ and at $x=h$, and it is optimal to replace immediately (at the beginning of the horizon) if $C(0) < C(h)$, that is if

$$[h+\tau]^{\beta+1} - [\rho h^{\beta+1} + \tau^{\beta+1}] > R(\beta+1)/\alpha. \qquad (4.13)$$

There will also be a replacement at the end of the horizon (by default). Thus, immediate replacement is optimal if

$$\rho < h^{-(\beta+1)}\{(h+\tau)^{\beta+1} - \tau^{\beta+1} - R\alpha^{-1}(\beta+1)\}.$$

If $\beta = 1$ this simplifies to

$$\rho < 1 - \frac{2R}{\alpha h^2} - \frac{2\tau}{h}.$$

We can consider inequality (4.13) as providing a limit for the age of existing equipment beyond which immediate replacement should be made, or as a condition on the horizon length given the age and cost parameters. For example, if $\beta = 1$, then it is optimal to replace immediately if

$$\tau > \frac{R\alpha^{-1}}{h} - \frac{h}{2},$$

or if

$$h > \{-\tau + \sqrt{\tau^2 + 2R\alpha^{-1}(1-\rho)}\}/(1-\rho), \qquad (\rho < 1).$$

4.4.2. Case $\beta_1 = \beta_2 = 0$

This case in which the age-related maintenance costs per unit time are constant may at first sight appear trivial, and in fact the behaviour of optimal policy is straightforward. However there are practical situations in which the assumption of non-age-related maintenance costs will be appropriate. Furthermore, in this case, the variable planning horizon model of Christer and Goodbody (1980) will not give sensible results; this is because the optimum value of the length of the current replacement cycle will be non-finite; this is discussed later in this chapter.

Now the cost function is

$$C(x) = \begin{cases} \int_0^x \alpha\,dt + \int_0^{h-x} \rho\alpha\,dt + 2R, & 0 \le x < h, \\ \int_0^h \alpha\,dt + R, & x = h, \end{cases}$$

from which

$$C(x) = \begin{cases} \alpha x + \rho\alpha(h-x) + 2R & 0 \le x < h, \\ \alpha h + R, & x = h, \end{cases}$$

giving that

$$C(x) = \begin{cases} \alpha(1-\rho)x + \rho\alpha h + 2R & 0 \le x < h, \\ \alpha h + R, & x = h. \end{cases}$$

The minimisation technique in this case implies that $C(x)$ has a minimum when $x = 0$. Therefore, if $\rho < 1$ then $C(x)$ is strictly increasing in $[0, h)$, so that $C(x)$ has local minima at $x = 0$ and at $x = h$, and it is optimal to replace immediately (at the beginning of the horizon) if

$$C(0) < C(h),$$

that is if

$$\rho < 1 - \frac{R}{\alpha h},$$

or equivalently if

$$h > h_L = R\alpha^{-1}(1-\rho).$$

Note that if $\rho > 1$ then $C(x)$ is strictly decreasing in $[0, h)$, and

$$\lim_{x \to h} C(x) > C(h),$$

so that it is optimal not to replace within the horizon (only at the end by default).

4.4.3. Case $\beta_1 \ne \beta_2$

Now consider the case in which $\alpha_1 \ne \alpha_2$ and $\beta_1 \ne \beta_2$. Thus $C(x)$ has the following form:

$$C(x) = \begin{cases} \int_0^x \alpha_1 (t+\tau)^{\beta_1} dt + \int_0^{h-x} \alpha_2 t^{\beta_2} dt + 2R, & 0 \le x < h, \\ \int_0^h \alpha_1 (t+\tau)^{\beta_1} dt + R, & x = h, \end{cases} \quad (4.14)$$

where $\alpha_1 t^{\beta_1}$ is the maintenance cost per unit time for the current equipment at age t; $\alpha_2 t^{\beta_2}$ is the maintenance cost per unit time of a new equipment at age t; R is the purchase cost new for an equipment; x is the time of the first replacement and h is the length of the fixed planning horizon. To determine the optimum value of x (the time of first replacement) we initially use the linear function at which is closest to the power law function αt^{β} over the range of values of the age of the equipment, that is $[0, h+\tau]$. This eases the calculations and makes the solution tractible. The mathematical expression for the distance (squared to ensure the positive value of it) function denoted by D is

$$D = \int_0^{h+\tau} \left[\alpha t^{\beta} - at\right]^2 dt, \tag{4.15}$$

where a is a constant for which D is minimum. Thus,

$$D = \alpha^2 \frac{(h+\tau)^{2\beta+1}}{\beta+1} - 2\alpha a \frac{(h+\tau)^{\beta+2}}{\beta+2} + a^2 \frac{(h+\tau)^3}{3}.$$

Differentiating with respect to a we obtain

$$\partial D / \partial a = 2a \frac{(h+\tau)^3}{3} - 2\alpha \frac{(h+\tau)^{\beta+2}}{\beta+2}.$$

Setting $\partial D / \partial a = 0$ we obtain

$$a = 3\alpha \frac{(h+\tau)^{\beta-1}}{\beta+2}. \tag{4.16}$$

Using the form at, the total cost is expressed as

$$C(x) = \begin{cases} \int_0^x a_1(t+\tau)dt + \int_0^{h-x} a_2 t dt + 2R, & 0 \le x < h, \\ \int_0^h a_1(t+\tau)dt + R, & x = h, \end{cases}$$

from which we obtain

$$C(x) = \begin{cases} \frac{a_1}{2}\left[(x+\tau)^2 - \tau^2\right] + a_2 \frac{(h-x)^2}{2} + 2R, & 0 \le x < h, \\ \frac{a_1}{2}\left[(h+\tau)^2 - \tau^2\right] + R & x = h. \end{cases} \tag{4.17}$$

Differentiating with respect to x we obtain

$$dC/dx = a_1(x+\tau) - a_2(h-x).$$

Setting $dC/dx = 0$ we obtain

$$x' = \frac{a_2 h - a_1 \tau}{a_1 + a_2}, \tag{4.18}$$

where $a_1 = 3\alpha_1 \dfrac{(h+\tau)^{\beta_1 - 1}}{\beta_1 + 2}$ and $a_2 = 3\alpha_2 \dfrac{(h+\tau)^{\beta_2 - 1}}{\beta_2 + 2}$.

From which it follows that

$$x' = \frac{\alpha_2 (h+\tau)^{\beta_2 - 1} h(\beta_1 + 2) - \alpha_1 (h+\tau)^{\beta_1 - 1} \tau(\beta_2 + 2)}{\alpha_1 (h+\tau)^{\beta_1 - 1}(\beta_2 + 2) + \alpha_2 (h+\tau)^{\beta_2 - 1}(\beta_1 + 2)}.$$

The local minimum at x' is the global minimum if the value of h satisfies the inequality

$$a_1\left\{\frac{a_2 h - a_1\tau}{a_1 + a_2} + \tau\right\}^2 + a2\left\{h - \frac{a_2 h - a_1 \tau}{a_1 + a_2}\right\}^2 - a_1(h+\tau)^2 + 2R < 0. \tag{4.19}$$

In this case there are two replacements over the horizon. Otherwise there is only one replacement at the end of the horizon.

We consider this case further in section 4.5 using a quadratic approximation to αt^β, and a numerical example in section 4.6.

4.4.4. Case $\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$

Also consider the case $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 \neq \beta_2$, proceeding as in the case $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$, writing down the total cost, $C(x)$, over the planning horizon h then the values of a_1 and a_2 are given as

$$a_1 = 3\alpha \frac{(h+\tau)^{\beta_1 - 1}}{\beta_1 + 2} \text{ and } a_2 = 3\alpha \frac{(h+\tau)^{\beta_2 - 1}}{\beta_2 + 2}.$$

The optimal policy will follow the same concept as in the case of $\alpha_1 \neq \alpha_2$ and $\beta_1 \neq \beta_2$.

(Note that if $\alpha_1 = \alpha_2$ then we have like-with-like replacement).

4.5. The behaviour of optimal policy with second order approximation

In this section we apply a second order approximation to the maintenance cost per unit time function αt^{β}. In fact we use a quadratic function of the form $at^2 + bt$, where a and b are constants determined by minimising the integral of the square distance between the function $at^2 + bt$ and the function αt^{β} over the range of values of the age of the equipment, that is $[0, h+\tau]$. This allows us to obtain an explicit value for the optimum time to replacement, x^*. The distance function takes the following form

$$D = \int_0^{h+\tau} \left[\alpha t^{\beta} - (at^2 + bt)\right]^2 dt, \tag{4.20}$$

from which we obtain that

$$D = \frac{\alpha^2 (h+\tau)^{2\beta+1}}{2\beta+1} - \frac{2\alpha a (h+\tau)^{\beta+3}}{\beta+3} - \frac{2\alpha b (h+\tau)^{\beta+2}}{\beta+2} + \frac{a^2 (h+\tau)^5}{5} + \frac{2ab(h+\tau)^4}{4} + \frac{b^2 (h+\tau)^3}{3}. \tag{4.21}$$

To obtain expressions for a and b, we minimise D with respect to a and b. The minimisation procedure is as follows

$$\frac{\partial D}{\partial a}=\frac{2a(h+\tau)^5}{5}+\frac{2b(h+\tau)^4}{4}-\frac{2\alpha(h+\tau)^{\beta+3}}{\beta+3}.$$

Setting $\partial D/\partial a=0$ we obtain

$$\frac{2a(h+\tau)^5}{5}+\frac{2b(h+\tau)^4}{4}=\frac{2\alpha(h+\tau)^{\beta+3}}{\beta+3}. \tag{4.22}$$

Differentiating D with respect to b and setting $\partial D/\partial b=0$ we obtain

$$\frac{2a(h+\tau)^4}{4}+\frac{2b(h+\tau)^3}{3}=\frac{2\alpha(h+\tau)^{\beta+2}}{\beta+2}. \tag{4.23}$$

Solving equation (4.22) and equation (4.23) simultaneously we obtain

$$a=\frac{20\alpha(h+\tau)^{\beta-2}(\beta-1)}{(\beta+3)(\beta+2)}, \tag{4.24}$$

and

$$b=\frac{12\alpha(h+\tau)^{\beta-1}(2-\beta)}{(\beta+3)(\beta+2)}. \tag{4.25}$$

The behaviour of the power law function αt^{β}, the first order approximation function and the second order approximation function is illustrated in Figure 4.4.

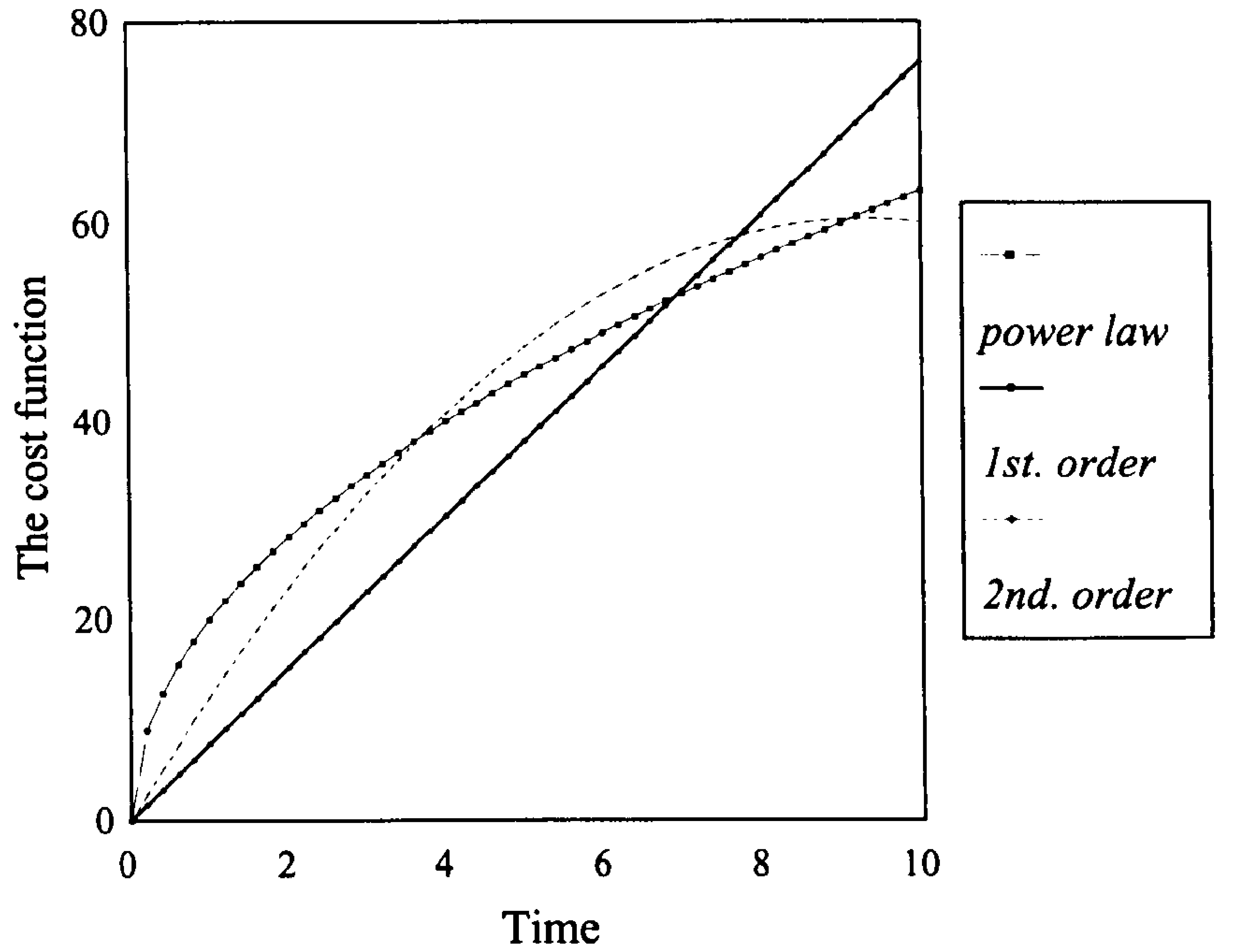


Figure 4.4. The behaviour of the two different approximate functions along with the behaviour of the function αt^{β} with α =20, β =0.5 and h =10, $\tau = 0$.

The total cost function in the case of non-like-with-like as presented in equation (4.14) is then

$$C(x)=\begin{cases}\int_0^x \{a_1(t+\tau)^2+b_1(t+\tau)\}dt+\int_0^{h-x}(a_2t^2+b_2t)dt+2R, & 0\le x<h,\\ \int_0^h \{a_1(t+\tau)^2+b_1(t+\tau)\}dt+R, & x=h,\end{cases} \tag{4.26}$$

where $a_1t^2+b_1t$ is the maintenance cost per unit time function of the current equipment at age t; $a_2t^2+b_2t$ is the maintenance cost per unit time function of a new equipment at age t; R is the purchase cost new for an equipment; x is the time of the first replacement; and h is the length of the fixed planning horizon. The cost function then becomes

$$C(x)=\begin{cases}\frac{a_1}{3}\left[(\tau+x)^3-\tau^3\right]+\frac{b_1}{2}\left[(\tau+x)^2-\tau^2\right]+\frac{a_2}{3}(h-x)^3+\frac{b_2}{2}(h-x)^2+2R, & 0\le x<h,\\ \frac{a_1}{3}\left[(\tau+h)^3-\tau^3\right]+\frac{b_1}{2}\left[(\tau+h)^2-\tau^2\right]+R, & x=h.\end{cases}$$

Differentiating with respect to x we obtain

$$\frac{dC}{dx}=(a_1-a_2)x^2+(2a_2h+2a_1\tau+b_1+b_2)x+a_1\tau^2-a_2h^2+b_1\tau-b_2h.$$

and so $C(x)$ has a local minimum at $x=h$ and at

$$x=x'=\frac{-(2a_2h+2a_1\tau+b_1+b_2)\pm\sqrt{4a_1a_2(h+\tau)^2+4(a_2b_1+a_1b_2)(h+\tau)+(b_1+b_2)^2}}{2(a_1-a_2)}. \tag{4.27}$$

From equations (4.24) and (4.25) presented earlier we then have the following:

$$a_1=\frac{20\alpha_1(h+\tau)^{\beta_1-2}(\beta_1-1)}{(\beta_1+3)(\beta_1+2)}, \tag{4.28}$$

$$b_1=\frac{12\alpha_1(h+\tau)^{\beta_1-1}(2-\beta_1)}{(\beta_1+3)(\beta_1+2)}, \tag{4.29}$$

$$a_2=\frac{20\alpha_2(h+\tau)^{\beta_2-2}(\beta_2-1)}{(\beta_2+3)(\beta_2+2)}, \tag{4.30}$$

and

$$b_2=\frac{12\alpha_2(h+\tau)^{\beta_2-1}(2-\beta_2)}{(\beta_2+3)(\beta_2+2)}. \tag{4.31}$$

It is straightforward in principle to find the range of values of h for which the global minimum of $C(x)$ is at $x = x'$ given by equation (4.27), though closed form expressions are not obtainable.

4.6. Comparison between the first order and second order approximations

To compare the first order and the second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ we apply the fixed planning horizon model with at most two replacements. For this comparison we apply the model with some arbitrary values for α, β and the planning horizon h. The comparison is based on the minimum cost obtained from each approximation and the minimum cost obtained by applying the maintenance cost per unit time in the form αt^{β}. The results obtained are shown in Tables 4.2-4.7. From the results in these tables one can observe that the minimum cost obtained in the case of the second order approximation is close to the minimum cost obtained by using the maintenance cost per unit time αt^{β}. Also the value of x^{*} (time for replacement at which the cost is minimum) is reasonably close to the value obtained using the second order approximation. Therefore we can conclude that the second order approximation is a suitable approximation to the maintenance cost per unit time αt^{β}, and that the first order approximation could be used if desired.

Table 4.2. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=10; α_1=20, α_2=40, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	7.0	1262	935	10	7.7	1344	966	10	8.1	1360	967	10
4	6.6	1325	1027	10	7.4	1411	1050	10	7.8	1424	1042	10
6	6.2	1378	1110	10	7.1	1465	1120	10	7.6	1477	1107	10
8	5.8	1422	1185	10	6.8	1508	1180	10	7.4	1524	1167	10

Table 4.3. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=15; α_1=20, α_2=40, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	11	1532	1280	15	11.9	1675	1339	15	12.5	1710	1347	15
4	10.6	1621	1400	15	11.7	1772	1453	15	12.3	1798	1448	15
6	10.2	1698	1511	15	11.4	1853	1552	15	12.1	1875	1537	15
8	9.9	1767	1614	15	11.1	1922	1640	15	11.9	1943	1619	15

Table 4.4. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=10; α_1=20, α_2=30, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	6.3	1233	935	10	6.9	1319	966	10	7.3	1336	967	10
4	5.8	1290	1027	10	6.5	1381	1050	10	6.9	1394	1042	10
6	5.3	1335	1110	10	6.1	1428	1120	10	6.6	1442	1107	10
8	4.8	1372	1185	10	5.7	1465	1180	10	6.3	1483	1167	10

Table 4.5. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=15; α_1=20, α_2=30, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	10.1	1486	1280	15	10.9	1636	1339	15	11.4	1672	1347	15
4	9.6	1566	1400	15	10.5	1726	1453	15	11.1	1754	1448	15
6	9.1	1636	1511	15	10.2	1801	1552	15	10.8	1825	1537	15
8	8.6	1696	1614	15	9.8	1863	1640	15	10.5	1887	1619	15

Table 4.6. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=10; α_1=40, α_2=20, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	3.2	1303	1420	3.2	2.5	1413	1482	2.5	2.3	1441	1483	2.3
4	2.2	1350	1605	2.2	1.4	1458	1650	1.4	1.2	1477	1634	1.2
6	1.1	1370	1770	1.1	0.3	1470	1789	0.3	0.1	1490	1765	0.1
8	0.1	1367	1921	0.1	0.0	1457	1910	0.0	0.0	1484	1883	0.0

Table 4.7. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=15; α_1=40, α_2=20, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	5.6	1633	2109	5.6	4.8	1842	2228	4.8	4.6	1907	2244	4.6
4	4.6	1716	2350	4.6	3.7	1935	2456	3.7	3.6	1982	2445	3.6
6	3.7	1773	2571	3.7	2.6	1992	2654	2.6	2.5	2031	2624	2.5
8	2.7	1809	2777	2.7	1.6	2021	2829	1.6	1.5	2060	2788	1.5

Table 4.8. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=10; α_1=30, α_2=20, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	4.0	1256	1177	10	3.7	1360	1224	10	3.8	1384	1225	10
4	3.2	1305	1316	3.2	2.8	1413	1350	10	2.9	1432	1338	10
6	2.3	1335	1440	2.3	1.9	1442	1454	1.9	2.1	1462	1436	10
8	1.4	1349	1553	1.4	1.1	1454	1545	1.1	1.3	1480	1525	1.3

Table 4.9. Replacement results representing the total costs and time to first replacement (x^*) in the case of first order and second order approximations to the maintenance cost per unit time $M(t) = \alpha t^{\beta}$ along with the replacement results using the maintenance cost per unit time as αt^{β}. Fixed planning horizon h=15; α_1=30, α_2=20, β_1=0.5 and β_2=0.7.

Current age	1st. order approx. to αt^{β}				2nd. order approx. to αt^{β}				$M(t) = \alpha t^{\beta}$			
τ	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*	x	$C(x)$	$C(h)$	x^*
2	6.9	1540	1694	6.9	6.5	1732	1783	6.5	6.7	1786	1795	6.7
4	6.0	1619	1875	6.0	5.6	1824	1955	5.6	5.9	1865	1946	5.9
6	5.2	1679	2041	5.2	4.8	1890	2103	4.8	5.1	1925	2081	5.1
8	4.3	1724	2195	4.3	4.0	1936	2234	4.0	4.3	1972	2204	4.3

4.7. Numerical study of the behaviour of optimal policy with non-like-with-like replacement

In this part we study the behaviour of optimal policy of non-like-with-like numerically. The cost function is as given by the function (4.14) from which we have that

$$C(x)=\begin{cases}\dfrac{\alpha_1}{\beta_1+1}\left\{(x+\tau)^{\beta_1+1}-\tau^{\beta_1+1}\right\}+\dfrac{\alpha_2}{\beta_2+1}(h-x)^{\beta_2+1}+2R, & 0\le x<h,\\ \dfrac{\alpha_1}{\beta_1+1}\left\{(h+\tau)^{\beta_1+1}-\tau^{\beta_1+1}\right\}+R, & x=h.\end{cases} \tag{4.32}$$

4.7.1. Some numerical results

We illustrate the behaviour of optimal policy by example. We present optimum policies for a range of values of maintenance cost per unit time parameters (α and β) and current age (τ). We also vary the control parameter, h, the length of the planning horizon. Thus, with a fixed planning horizon of length 10 years and 15 years and $R=450$ and different values of $\alpha_1,\alpha_2,\beta_1,\beta_2$ and τ (the age of the current equipment), Tables 4.10-4.12 show the minimum total cost at $x=x'$ and the cost at $x=h$ and the value of x^* (the optimal replacement time according to the global minimum of $x=x'$ and $x=h$).

Table 4.10. The results of optimal policy behaviour with fixed planning horizons of length 10 years and 15 years respectively; α_1=20, α_2=20, β_1=0.5 and β_2=0.7.

	Planning Horizon $h=10$				Planning Horizon $h=15$			
τ	x'	Min. cost at x'	Min. cost at h	x^*	x'	Min. cost at x'	Min. cost at h	x^*
0	6	1223.1	831.6	10	10	1503.1	1224.6	15
2	6	1288.2	966.5	10	9	1605.5	1346.9	15
4	5	1334.8	1041.8	10	9	1665.7	1417.6	15
6	5	1372.0	1107.2	10	8	1724.0	1537.2	15
8	4	1400.0	1166.5	10	8	1773.2	1619.0	15

Table 4.11. The results of optimal policy behaviour with fixed planning horizons of length 10 years and 15 years respectively; α_1=20, α_2=20, β_1=0.7 and β_2=0.5.

	Planning Horizon $h=10$				Planning Horizon $h=15$			
τ	x'	Min. cost at x'	Min. cost at h	x^*	x'	Min. cost at x'	Min. cost at h	x^*
0	4	1220.2	1039.6	10	5	1503.1	1624.7	5
2	2	1287.7	1215.7	10	4	1595.6	1865.0	4
4	1	1317.3	1370.5	1	2	1648.2	2081.6	2
6	1	1334.1	1513.5	1	1	1672.6	2284.0	1
8	1	1349.5	1648.1	1	1	1687.9	2476.0	1

Table 4.12. The results of optimal policy behaviour with fixed planning horizons of length 10 years and 15 years respectively; α_1=20, α_2=30, β_1=0.7 and β_2=0.65.

	Planning Horizon $h=10$				Planning Horizon $h=15$			
τ	x'	Min. cost at x'	Min. cost at h	x^*	x'	Min. cost at x'	Min. cost at h	x^*
0	6	1326.5	1039.6	10	9	1742.	1624.7	15
2	6	1442.1	1215.7	10	8	1902.3	1865.0	15
4	5	1527.5	1370.5	10	8	2030.6	2081.6	8
6	4	1591.8	1513.5	10	7	2135.6	2284.0	7
8	3	1640.7	1648.1	3	6	2223.8	2476.0	6

From Table 4.10 it is noted that the optimal policy is always to replace at the end of the horizon in the case of $h=10$ and $h=15$. Therefore, we find that $x^*=10$ in the case of $h=10$ and $x^*=15$ in the case of $h=15$. This decision is affected, of course, by the fact that the maintenance cost per unit time of the current equipment is less expensive than that of the new equipment and the physical meaning of this behaviour seems to be to keep the current equipment as long as possible.

Table 4.11 shows that the optimal policy varies when the maintenance cost per unit time of the new equipment is cheaper than that of the current equipment. Thus, the optimum policy at the lower current ages ($\tau=0, \tau=2$) is always replace at the end of the horizon. For the older current ages, the maintenance cost per unit time of the current equipment becomes very expensive so that the policy tends to replace soon. In the case of $h = 15$ the optimal policy is always to replace as soon as possible.

Table 4.12 shows that the optimal policy in the case of $h = 10$ is replace at the end of the horizon except when $\tau = 8$ in which case the maintenance cost per unit time of the current equipment becomes expensive enough to replace within the horizon. In the case of $h = 15$ the optimal policy is to replace at the end of the horizon at the early current ages but this policy changes to replace within the horizon when the current equipment gets older showing the effect of the horizon length on the behaviour of optimal policy.

The above results show that the optimal policy is affected by the model parameters. It is noted that the most influential parameters are the current ages and the length of the planning horizon. Therefore, the optimum policy is sensitive to h so that h should be chosen with care. For brevity, we consider further examples graphically in the next section.

4.7.2. Graphical results

We now investigate the behaviour of optimal policy for non-like-with-like replacement. Rather than using approximations to αt^{β} to obtain explicit values for x^*, we explore the behaviour numerically through the relation between the two ratios x^*/h and α_1/α_2. We present the results graphically. This investigation is done by varying the values of α_1 and α_2 and the values of β_1 and β_2, determining x^* and then plotting x^*/h against α_1/α_2. For a fixed planning horizon h, the optimum value of x (the time for replacement) is determined. If the cost is a minimum at h, then $x^{*} = h$ leading to replacement once at the end of the horizon. Otherwise the cost is a minimum at $x' < h$ and $x^{*} = x'$, leading to replacement twice at $x = x'$ and $x = h$. Moreover, $x^{*}/h = 1$ indicates that there is only one replacement at the end of the horizon and $x^{*}/h < 1$ indicates that there are two replacements one at x' and the other at h.

In Figures 4.5 to 4.16 we show the behaviour of optimal policy according to the maintenance costs per unit time parameters and the length of the planning horizon.

Figure 4.5 shows that the optimal policy changes when α_1 and τ (the age of the current equipment) increase. Thus, for $\alpha_1 = 20$ the optimal decision is always to replace at the end of the horizon when $\tau = 0$. Also when $\tau = 2$ the optimal decision changes from replace at the end of the horizon to replace within the horizon as α_1 increases (the maintenance cost per unit time of the current equipment becomes more expensive than that of the purchased equipment). For $\tau = 4$ and $\tau = 8$ it is obvious that the decision is to replace within the horizon most of the time except when $\alpha_1 = 20$ but for some values of α_2 ($\alpha_1 / \alpha_2 > 1.05$) the decision is to replace at the end of the horizon.

Figure 4.6 shows that the condition $\beta_1 < \beta_2$ affects the optimal policy since the maintenance cost per unit time of the current equipment is increasing more slowly. Thus, for $\tau = 0$ the decision is always to replace at the end of the horizon. When $\tau = 2$ the decision is to replace at the end of the horizon when $\alpha_1 = 20$ or 30 but the policy changes when $\alpha_1 = 40$ because of the expensive maintenance cost per unit time for current equipment. For $\tau = 4$ the optimal policy is always to replace at the end of the horizon when $\alpha_1 = 20$ and replace within the horizon when $\alpha_1 = 30$ or 40 with the same values for α_2. For $\tau = 8$ the maintenance cost per unit time of the current equipment is increasing gradually for $\alpha_1 = 20$, $\alpha_1 = 30$ and $\alpha_1 = 40$ respectively and that affects the optimal policy decision to replace within the horizon at certain values for α_2.

Figure.4.7 shows that the optimal policy is affected by the age of the current equipment. Thus, for $\tau = 0$ the decision is always to replace at the end of the horizon except for $\alpha_1 = 40$ when $\alpha_1 / \alpha_2 > 1.8$. For $\tau = 2$ the decision is always to replace at the end of the horizon when $\alpha_1 = 20$ but it is noticed that the optimal policy changes to replace within the horizon when $\alpha_1 = 30$ for $\alpha_1 / \alpha_2 > 1.88$ and when $\alpha_1 = 40$ for $\alpha_1 / \alpha_2 > 1.33$. For $\tau = 4$ the optimal policy is nearly the same as for $\tau = 2$. For $\tau = 8$ the optimal policy starts with replace at the end of the horizon and turns to replace within the horizon.

Figure 4.8 shows that the optimal policy rapidly changes being affected by all the parameters of the maintenance cost per unit time. Thus the behaviour of optimal policy illustrated in Figure 4.8 is nearly the same as the behaviour of optimal policy illustrated in Figure 4.5. The difference between Figure 4.8 and Figure 4.5 is that the decision replace within the horizon is taken at earlier stages because $\beta_2 = 0.5$ in Figure 4.8 and $\beta_2 = 0.65$ in Figure 4.5. The case $\beta_1 = 0.7$ and $\beta_2 = 0.5$ in Figure 4.8 makes the maintenance cost per unit time of the current equipment much more expensive than that of the purchased equipment compared with the case $\beta_1 = 0.7$ and $\beta_2 = 0.65$ in Figure 4.5.

Figure 4.9 shows that the optimal policy is affected by the length of the planning horizon compared with the behaviour of optimal policy in Figure 4.5 when $h=10$. Therefore, the optimal policy is to replace within the horizon most of the time for $\tau=0$, $\tau=2$ and $\tau=4$ respectively and all the time for $\tau=8$.

Figure 4.10 shows that the behaviour of optimal policy is affected by the length of the planning horizon $h=15$ compared with $h=10$ in Figure 4.6. It is noticed that the optimal policy is nearly the same in both Figures. The difference is that the decision replace within the horizon takes place earlier in the case $h=15$ (more expensive maintenance cost per unit time for the current equipment).

Figure 4.11 shows the effect of $h=15$ on the optimal policy compared with the optimal policy shown in Figure 4.7 with $h=10$. Notice that there is a slight difference between the effect of $\beta_2=0.55$ in Figure 4.10 and $\beta_2=0.5$ in Figure 4.11.

Figure 4.12 shows that in addition to the effect of $h=15$ the optimal policy is affected by the condition $\beta_1>\beta_2$. Therefore, the optimal policy behaviour is almost to replace within the horizon except at the beginning only for $\tau=0$ when $0.5\le\alpha_1/\alpha_2<0.8$ and for $\tau=2$ when $0.5\le\alpha_1/\alpha_2<0.6$. Compared with Figure 4.8 the difference is obvious.

Figure 4.13 shows that the behaviour of optimal policy is greatly affected by the length of the horizon $h=20$ compared with the case $h=10$ in Figure 4.5 and the case $h=15$ in Figure 4.9. This effect of the longer planning horizon leads to the decision replace within the horizon except for a very short period when $\tau=0$ and $0.5\le\alpha_1/\alpha_2<0.6$.

Figure 4.14 shows that $h = 20$ has an influence on the optimal decision compared with that illustrated in Figure 4.6 for $h = 10$ and Figure 4.10 for $h = 15$. Figure 4.14 also shows that although the optimal policy is affected by the longer horizon $h = 20$ the condition $\beta_1 < \beta_2$ (cheaper maintenance cost per unit time for current equipment) allows replace at the end of the horizon for some time. It is noticed that the decision replace at the end of the horizon is ignored (replace within the horizon) as τ and α_1 increase.

Figure 4.15 shows that although $\beta_1 = \beta_2$ makes the maintenance cost per unit time of the current equipment cheaper the length of the horizon has an influence on the behaviour of optimal policy compared with that shown in Figure 4.7 and Figure 4.11. The figure also shows that the decision replace at the end of the horizon changes to replace within the horizon as τ and α_1 increase. There is also a slight difference between Figure 4.15 ($\beta_2 = 0.5$) and Figure 4.14 ($\beta_2 = 0.55$).

Figure 4 .16 shows straightforward behaviour of the optimal policy that the decision is always to replace within the horizon; it is more economical not to keep the current equipment until the end of the horizon. Figure 4.16 also shows that how the optimal policy is affected by the length of the horizon compared with that illustrated in Figures 4.8 and 4.12.

On the whole, we can see that as h increases it is more likely that it is optimal to replace within the horizon. This is likewise for $\beta_2 > \beta_1$ and $\alpha_2 > \alpha_1$. Thus the behaviour of optimal policy is as expected. The figures also illustrate the non-smooth behaviour of optimal policies for these replacement models. This is their major drawback.

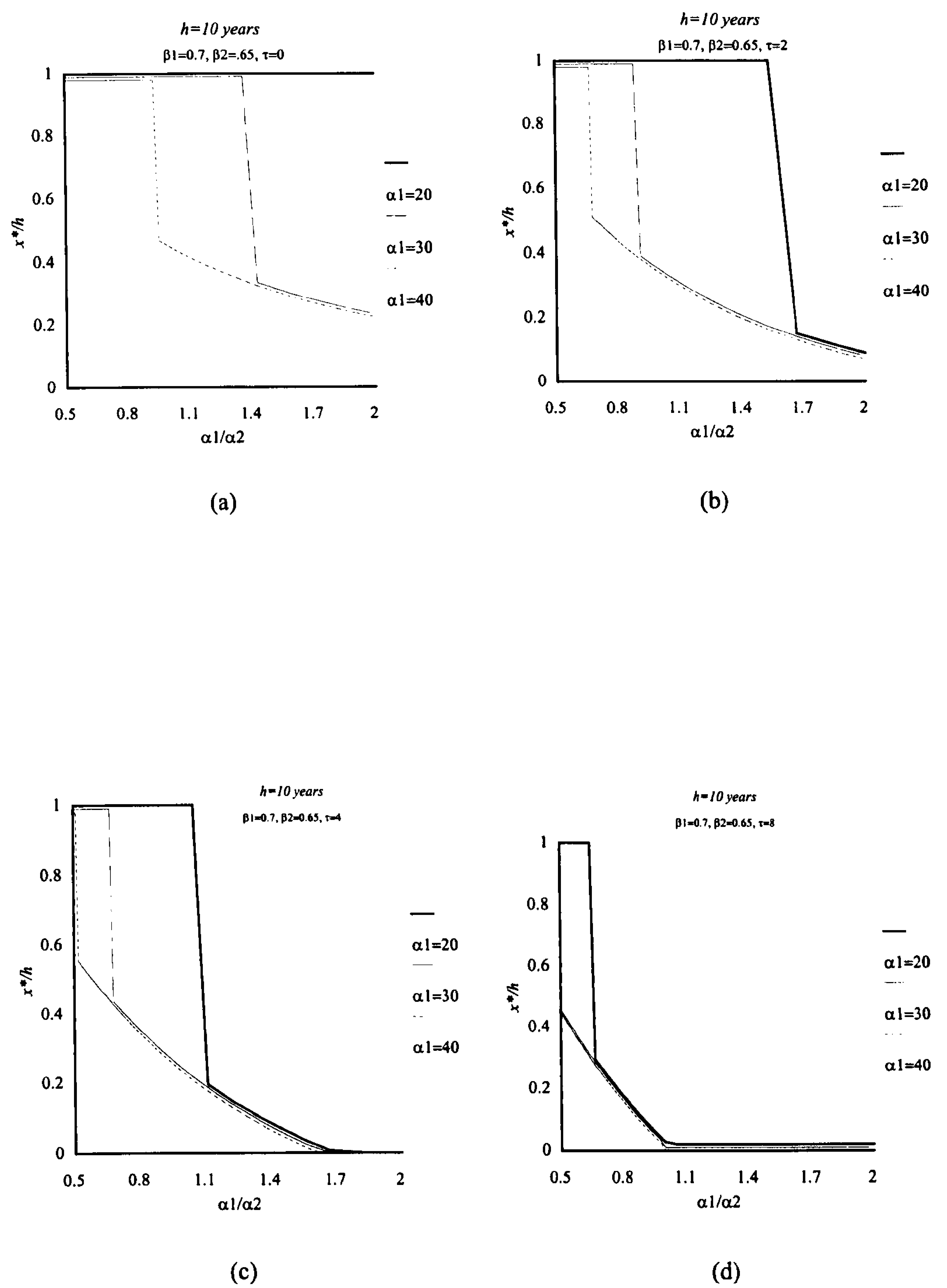


Figure 4.5. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 10$ years.

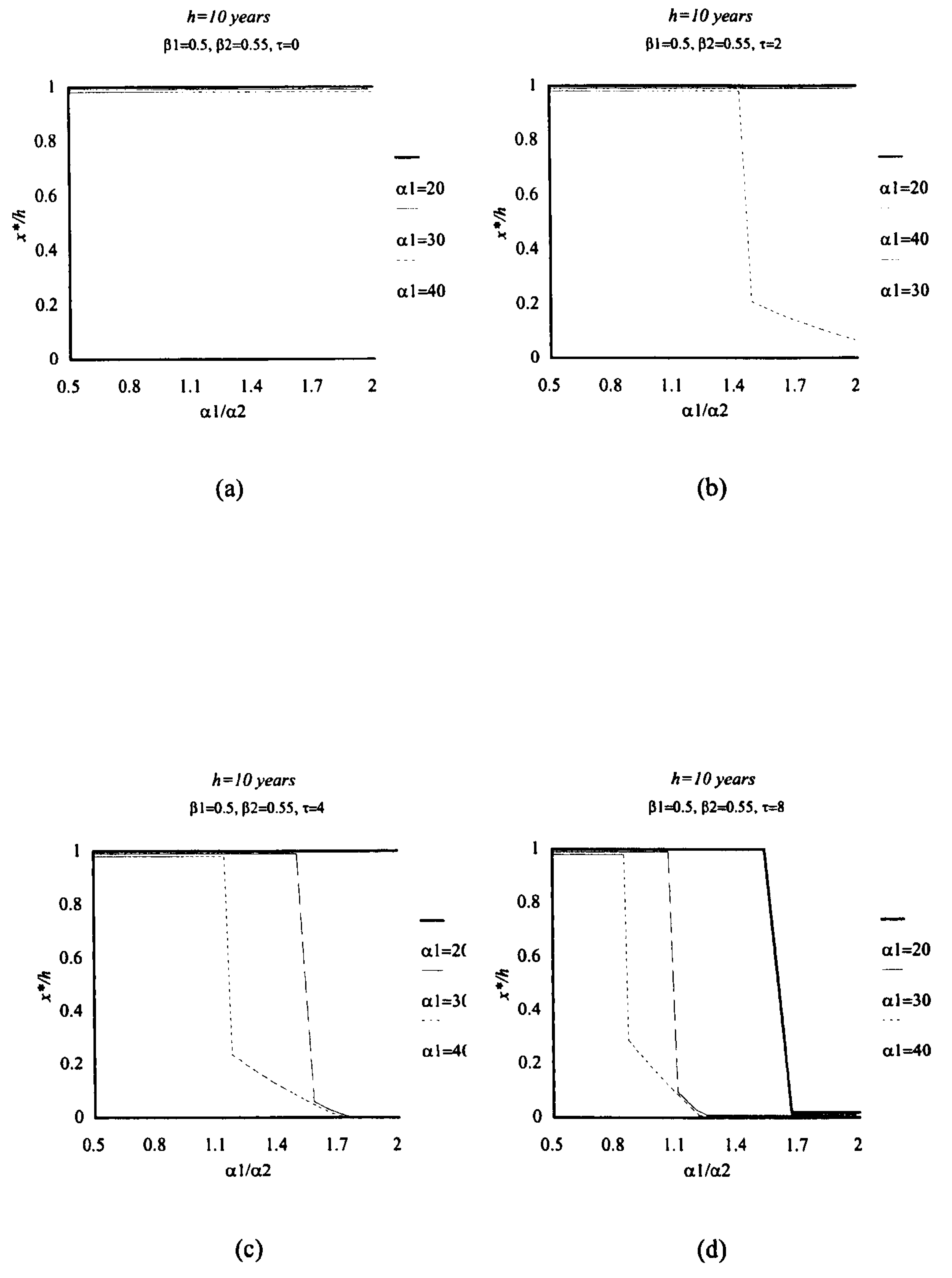


Figure 4.6. The behaviour of optimal policy for $\beta_1 / \beta_2 < 1$ and $h = 10$ years.

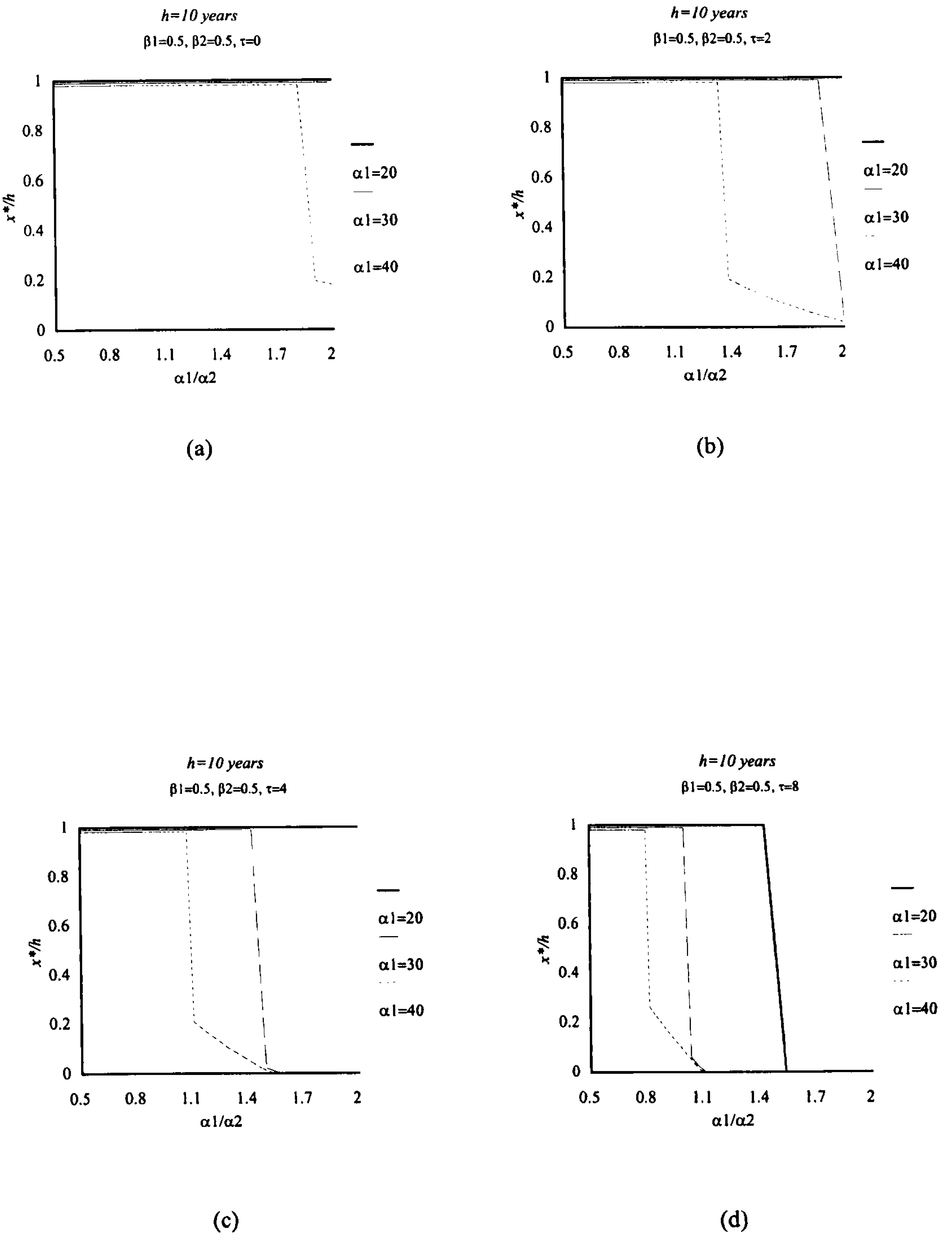


Figure 4.7. The behaviour of optimal policy for $\beta_1 / \beta_2 = 1$ and $h = 10$ years.

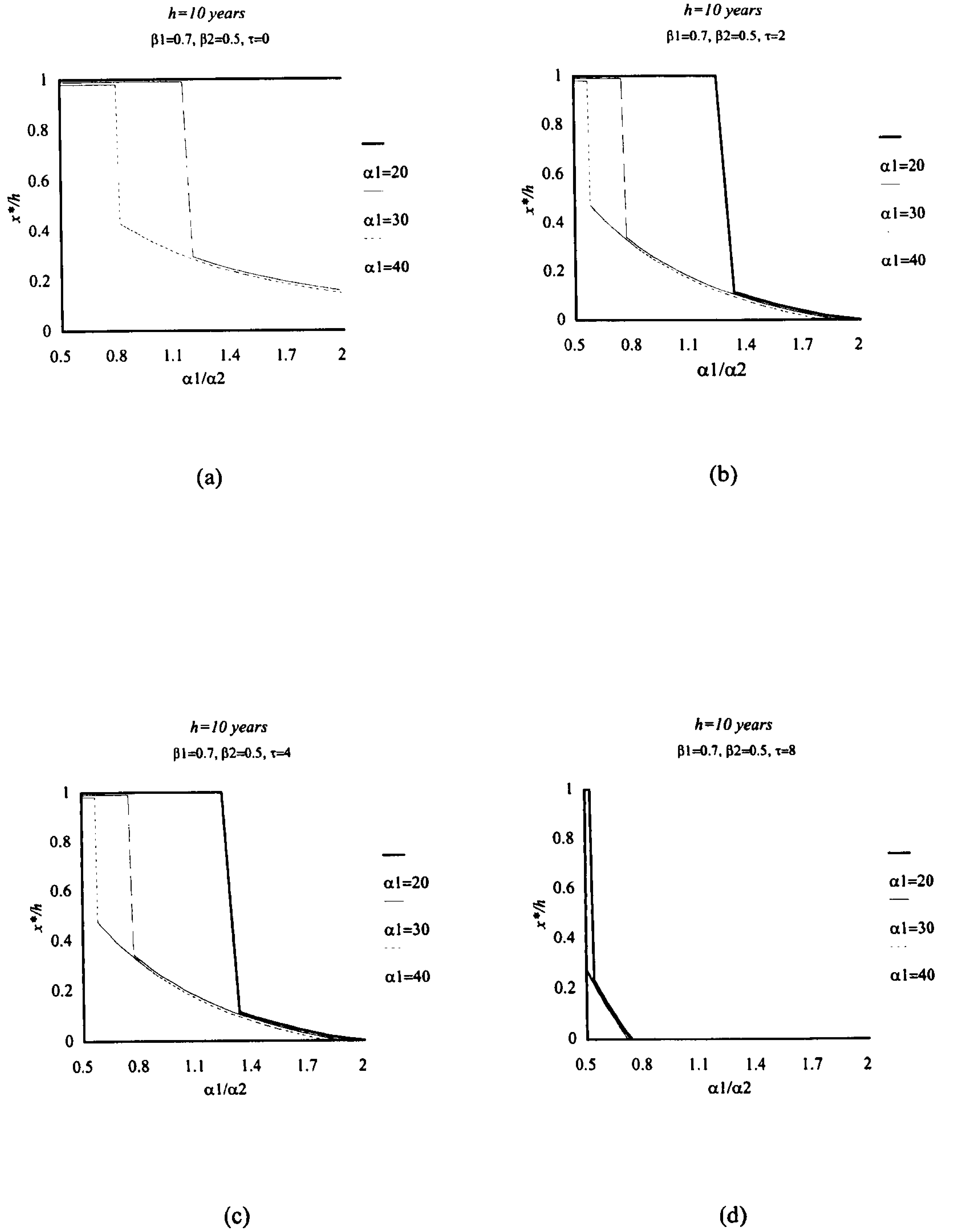


Figure 4.8. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 10$ years.

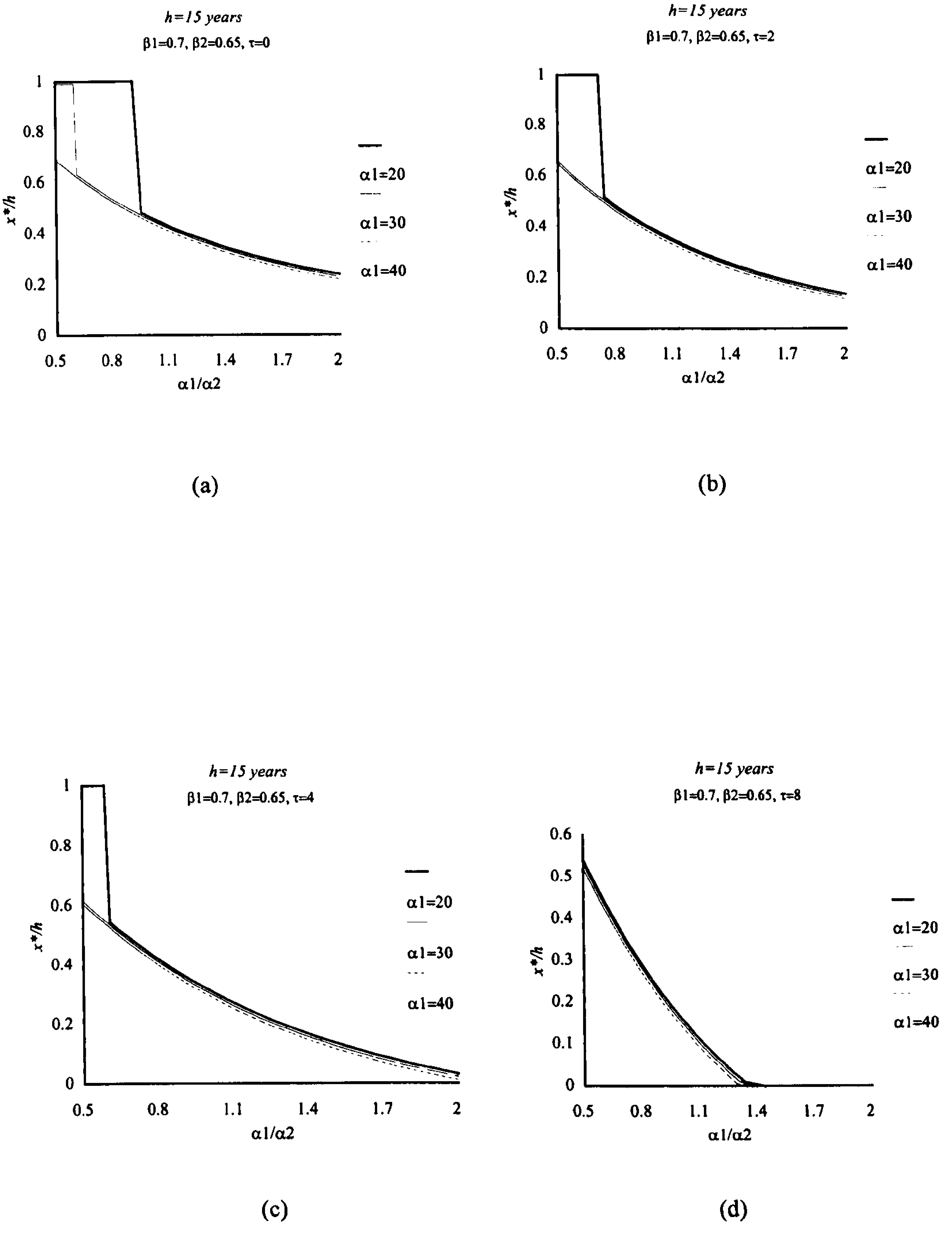


Figure 4.9. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 15$ years.

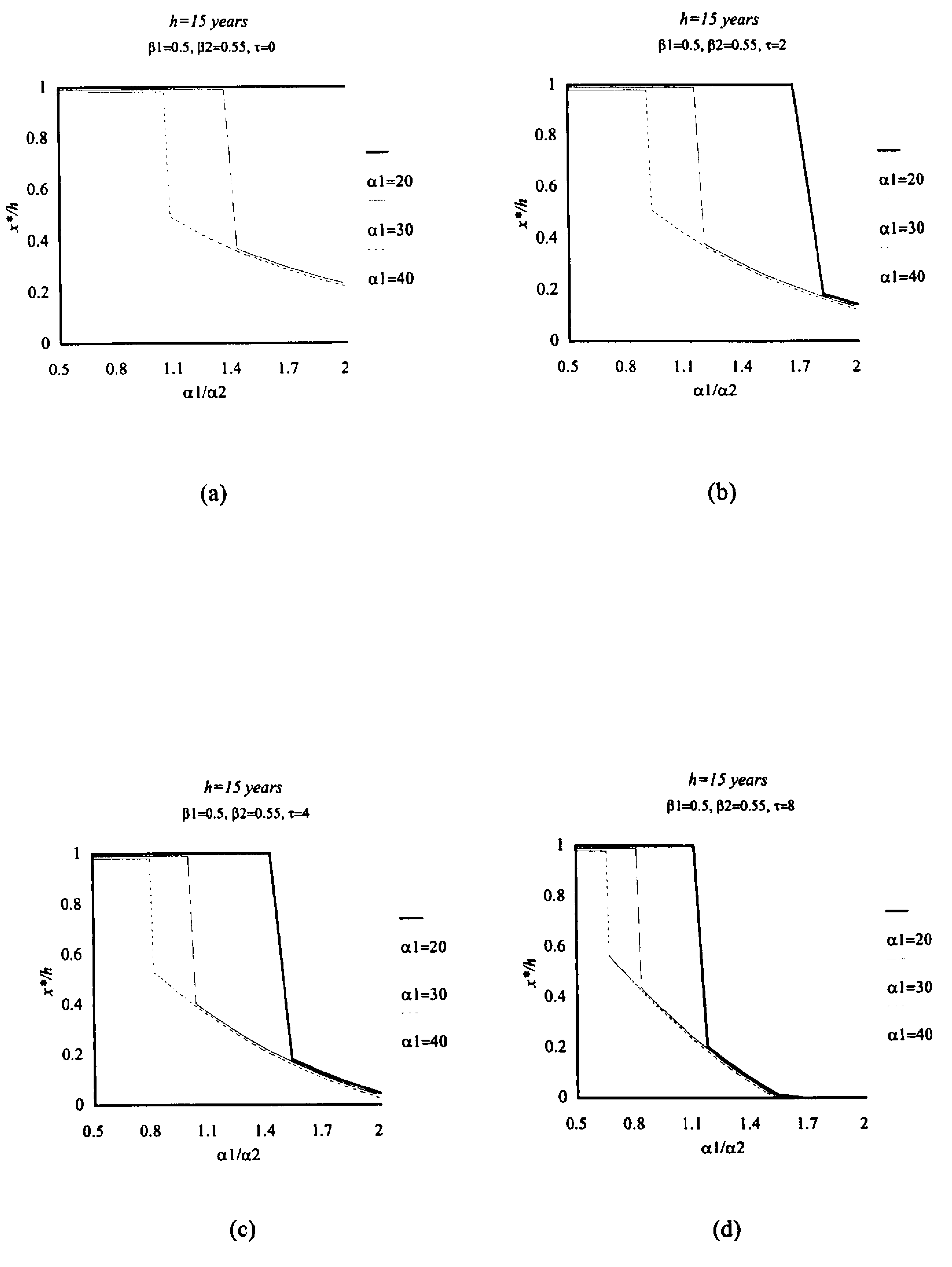


Figure 4.10. The behaviour of optimal policy for $\beta_1 / \beta_2 < 1$ and $h = 15$ years .

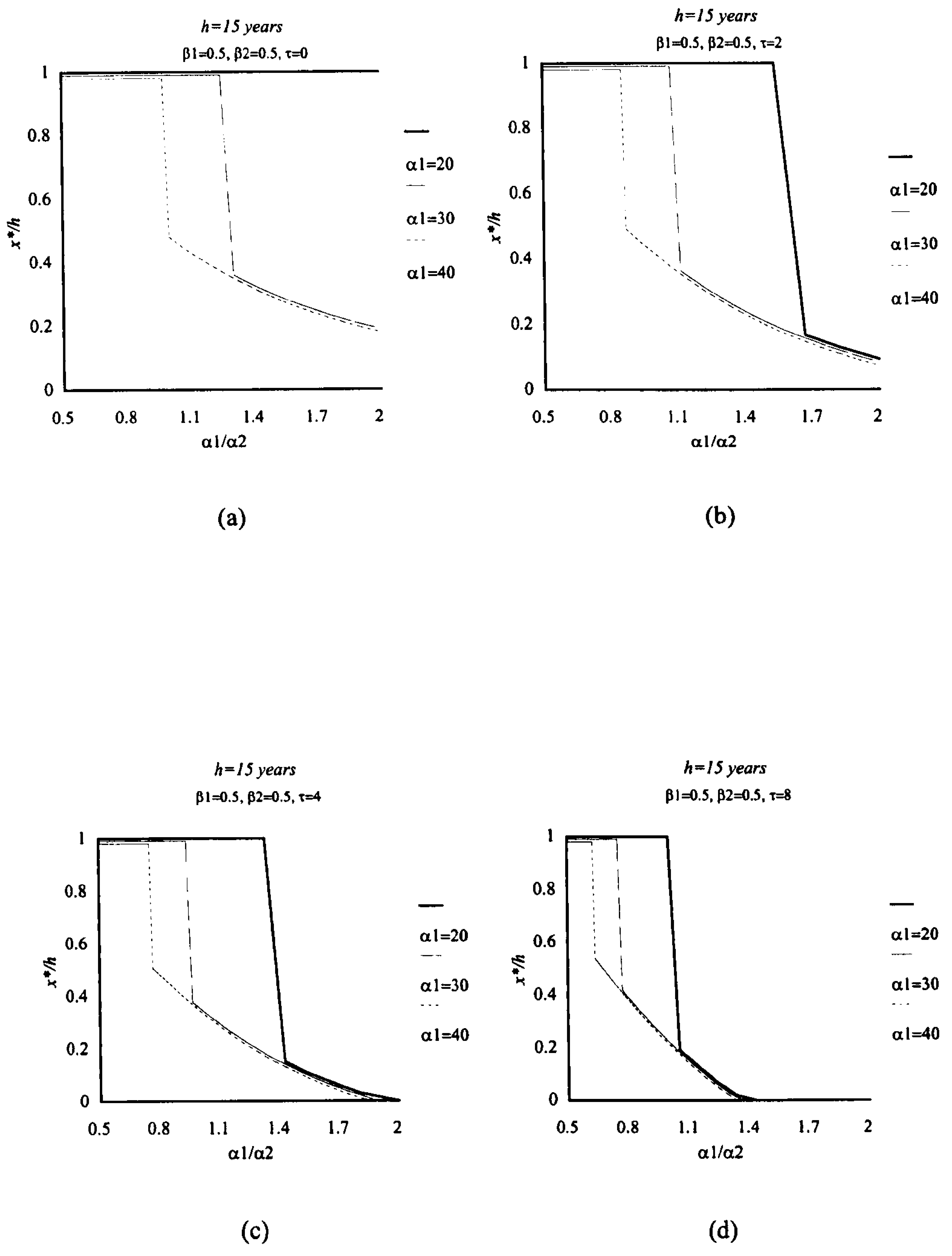


Figure 4.11. The behaviour of optimal policy for $\beta_1 / \beta_2 = 1$ and $h = 15$ years.

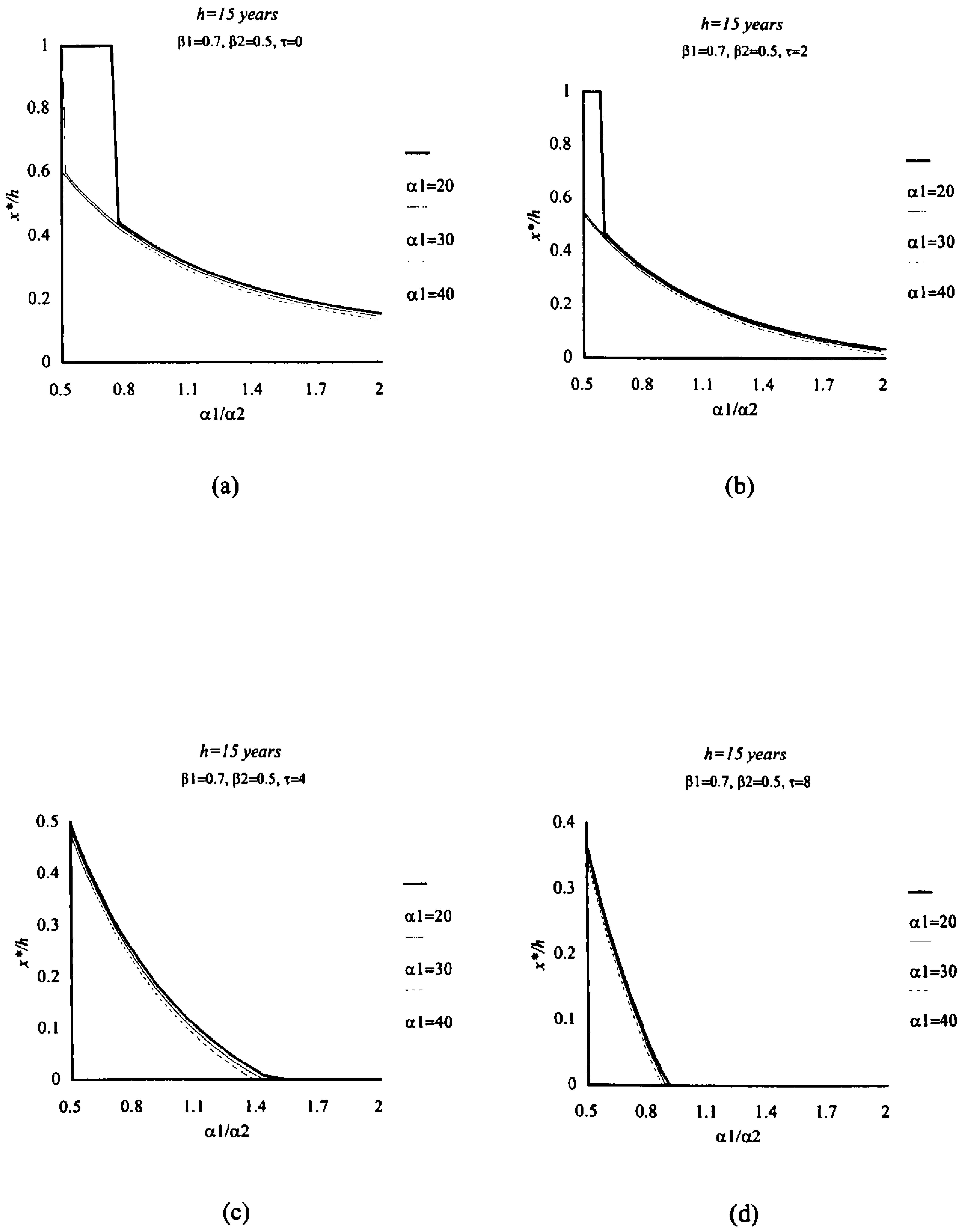


Figure 4.12. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 15$ years.

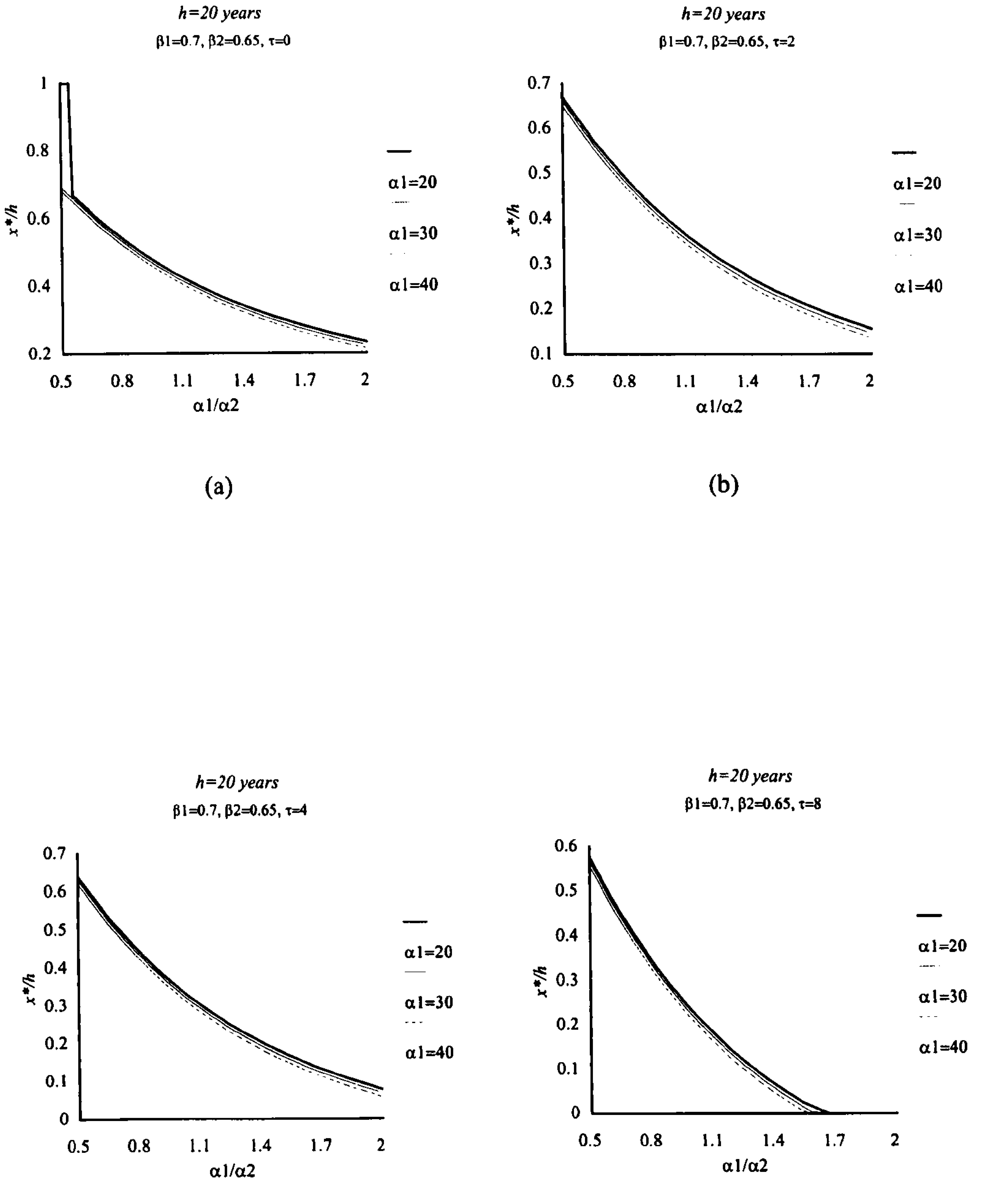


Figure 4.13. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 20$ years.

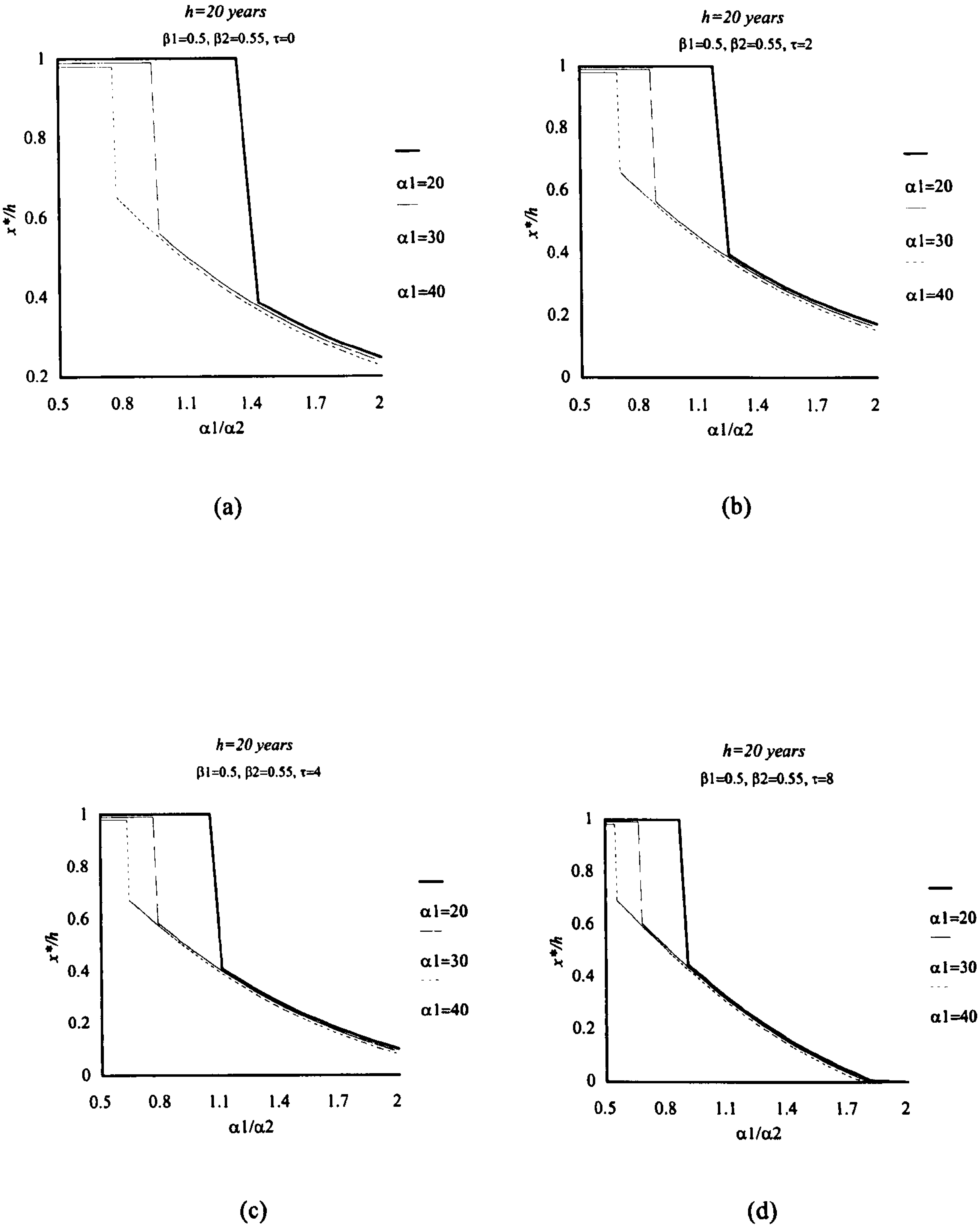


Figure 4.14. The behaviour of optimal policy for $\beta_1 / \beta_2 < 1$ and $h = 20$ years.

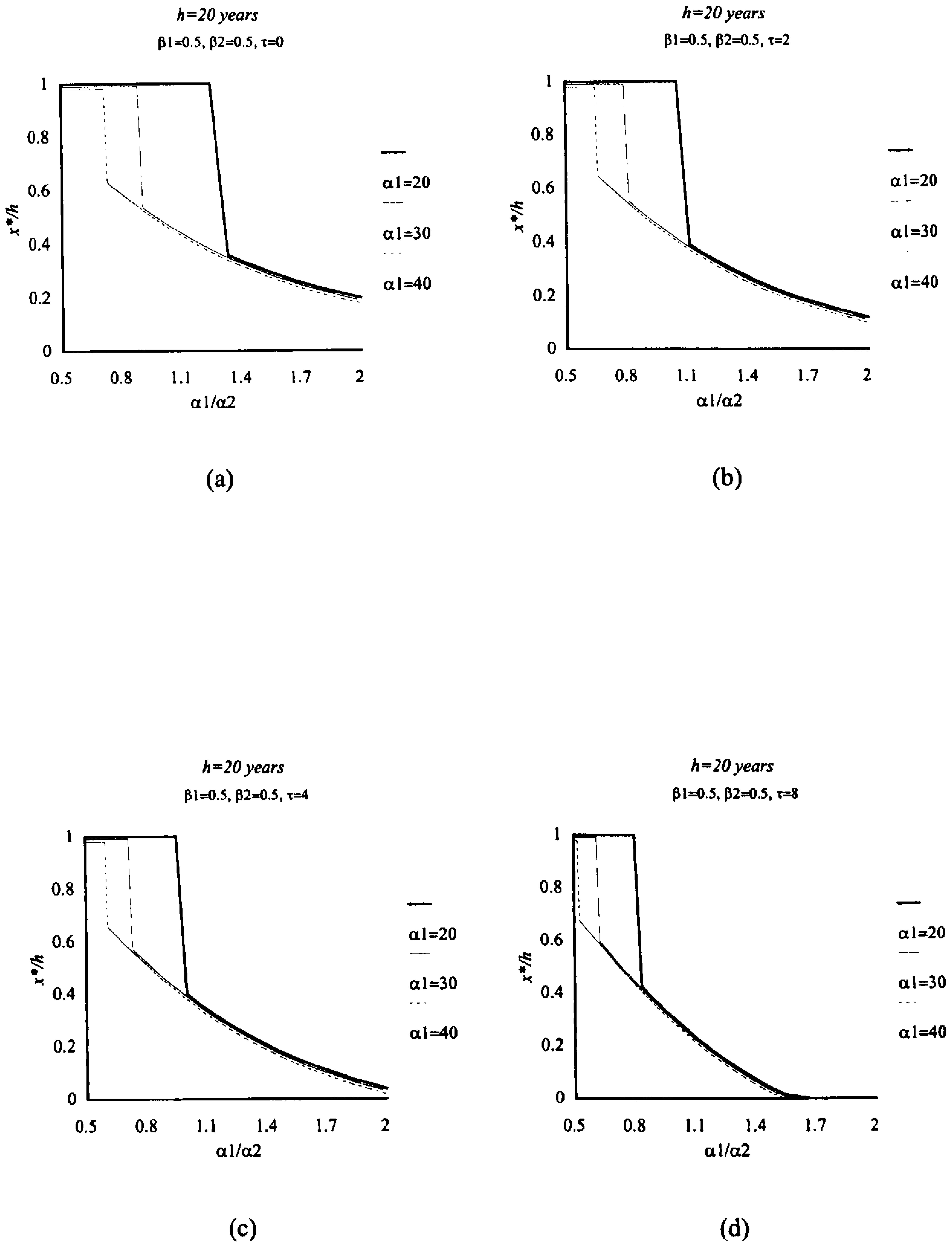


Figure 4.15. The behaviour of optimal policy for $\beta_1 / \beta_2 = 1$ and $h = 20$ years.

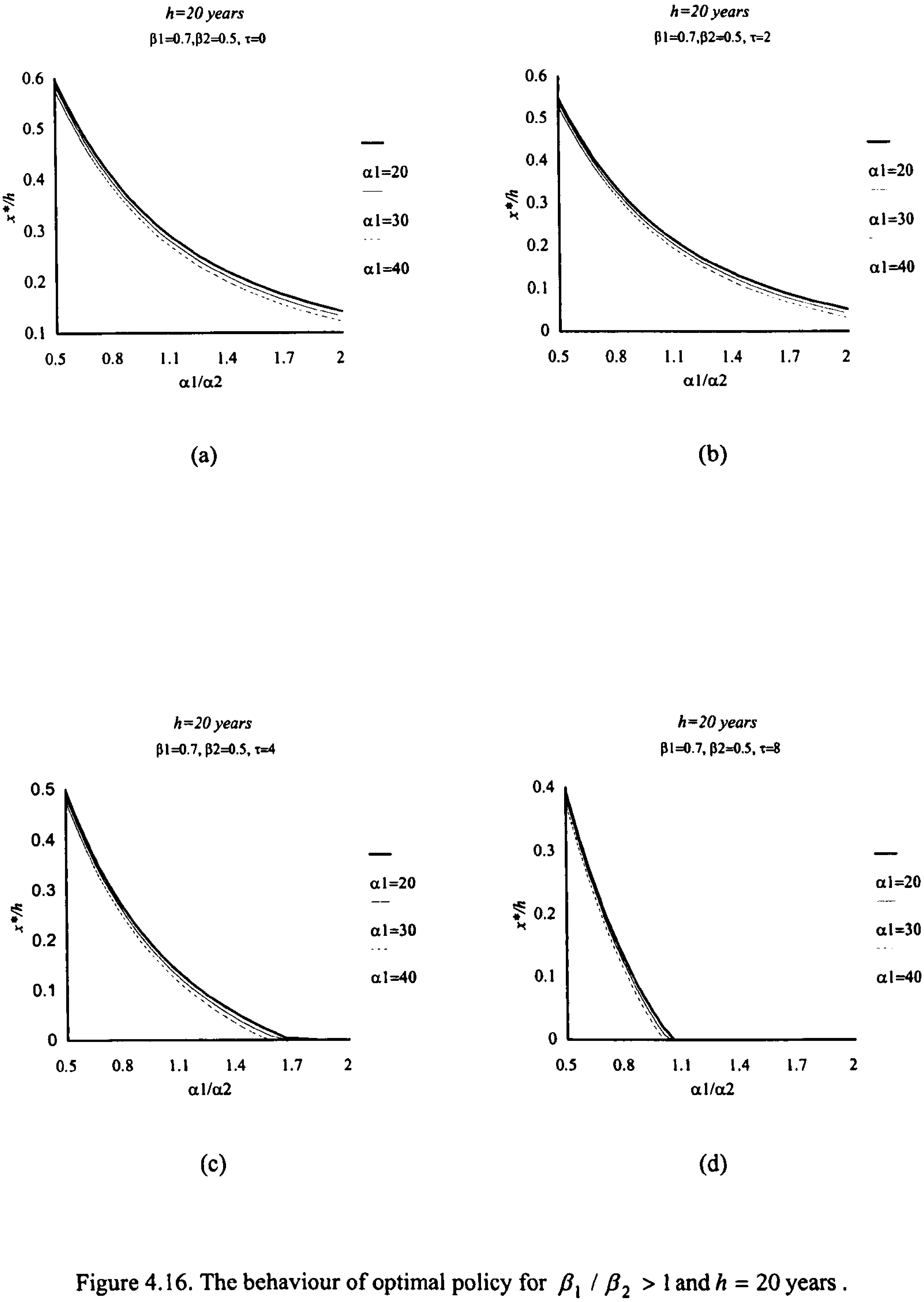


Figure 4.16. The behaviour of optimal policy for $\beta_1 / \beta_2 > 1$ and $h = 20$ years .

4.8. Finite planning horizon models of variable length

In addition to the fixed planning horizon modelling, we can consider the behaviour of variable length finite horizon models. This is done by considering models with a fixed number of cycles, and focusing on the form of maintenance cost per unit time for the current and future plant, and the replacement costs. For variable planning horizon models with a fixed number of cycles the situation considered is, given a plant currently aged τ years, when should it be replaced? That is how much longer the plant should be operated until replacement? This decision is the most important task for decision-makers. The on-going requirement for a plant is represented by subsequent replacement cycles (Christer & Goodbody, 1980). We start with a simple one cycle model before focusing on the two-cycle model of principal interest.

4.8.1. Notation and assumptions

Consider a planning horizon for which the length is a function of decision variables (lengths of replacement cycles). Also a resale and replacement are imposed at the end of each cycle. For simplicity we assume that:

(i) an equipment of age τ is currently being used and a typical new equipment is of age zero on replacement;

(ii) the cost of the new equipment is R;

(iii) the maintenance cost per unit time of the equipment is αt^{β}, where $\alpha > 0$, $0 < \beta \leq 1$; (see Figure 4.4).

(iv) the discount rate is unity.

4.8.2. One cycle replacement model

Here, we assume a plant of age τ and it is required to operate it until a certain time n before replacement, so we seek the optimum value of n, which minimises the cost. Figure 4.17 illustrates the model outline.

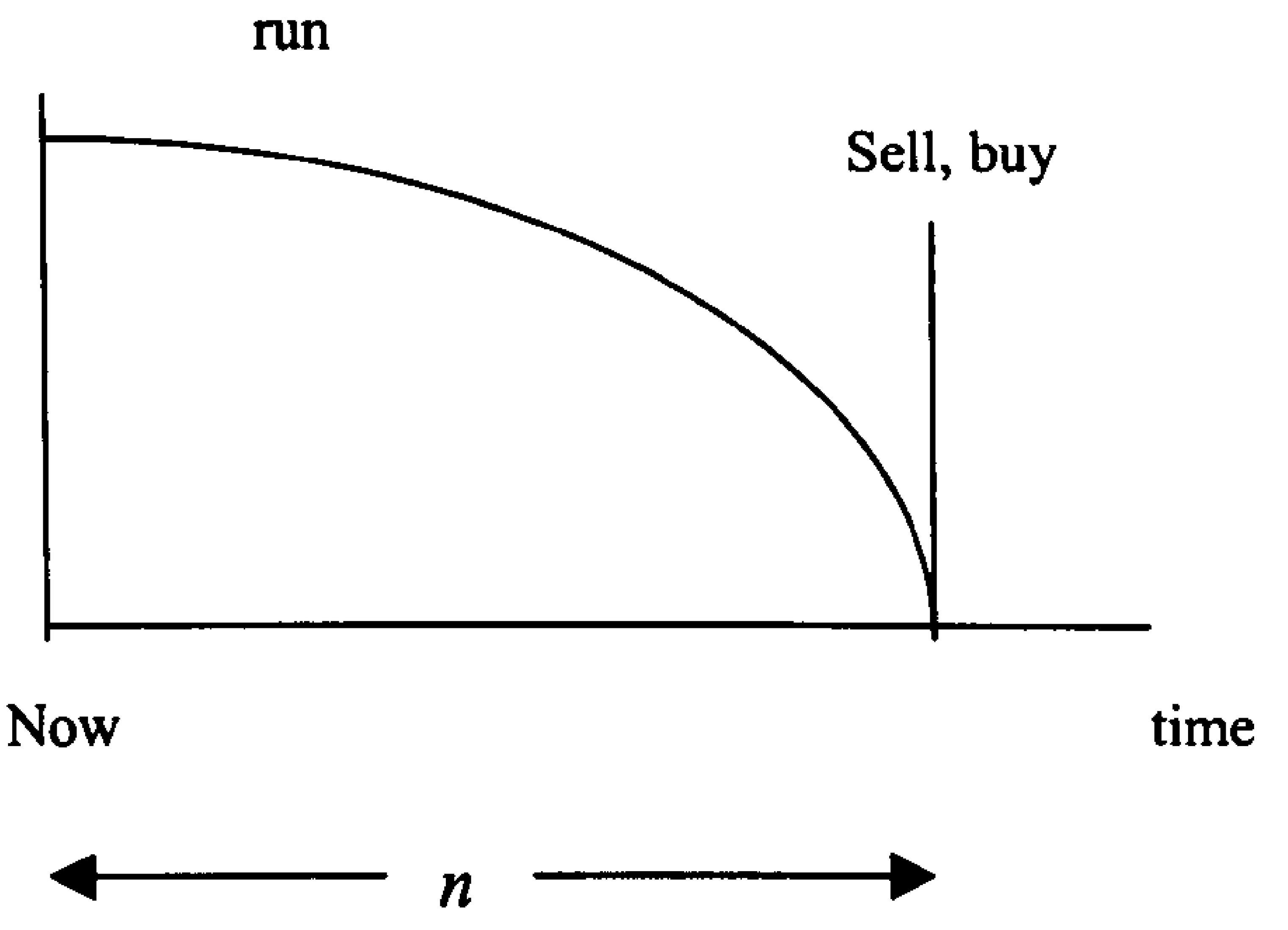


Figure 4.17. One cycle replacement behaviour.

The average cost per unit time is then

$$C(n) = \frac{1}{n}\left[\int_0^n f(t+\tau)dt + R\right], \tag{4.33}$$

where $f(\tau+t)$ is the maintenance cost per unit time of the existing equipment of current age τ; R is the purchase cost new. For the maintenance cost per unit time in the form $\alpha(\tau+t)^\beta$, the cost function is as follows:

$$C(n) = \frac{1}{n}\left[\frac{\alpha(n+\tau)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{\beta+1} + R\right]. \tag{4.34}$$

When $\tau \neq 0$ we obtain

$$nC(n) = \left[\frac{\alpha(n+\tau)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{\beta+1} + R\right].$$

Setting $dC/dn = 0$ we obtain

$$C(n^*) = \alpha(n^* + \tau)^\beta,$$

then from equation (4.34) we have

$$\alpha(n^* + \tau)^\beta = \frac{1}{n^*}\left[\frac{\alpha(n^* + \tau)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{\beta+1} + R\right], \tag{4.35}$$

as a non linear equation of n^* so that we move to the case $\tau = 0$.

Putting $\tau = 0$ in equation (4.35) we obtain

$$\alpha(n^*)^\beta = \frac{1}{n^*}\left[\frac{\alpha(n^*)^{\beta+1}}{(\beta+1)} + R\right], \tag{4.36}$$

hence,

$$(n^*)^{\beta+1} = R/\alpha(1 - 1/(\beta+1)), \tag{4.37}$$

from which we get

$$n^* = [R(\beta+1)/\alpha\beta]^{1/(\beta+1)},$$

where n^* is the optimal time at which replacement should be made. This result is the same as the result obtained from infinite cycle model given by equation (4.2). Thus n^* is the economic life of the existing equipment.

Notice that equation (4.37) indicates that n^* exists when $(1 - 1/(\beta+1)) > 0$, so $\beta > 0$ is the condition for existence.

For $\beta = 0.7$, $\alpha = 25$, $R = 450$ and t measured in years, it is optimal to replace at n^*=9.2 which is the same as the infinite cycle model as expected.

4.8.3. Two-cycle replacement models

In this section we consider a model with two cycles. The aim is to determine the time at which the current equipment should be replaced, that is the length of the first replacement cycle. The on-going requirement for an equipment is represented by the second cycle (Christer & Goodbody, 1980). Figure 4.18 shows the model outline.

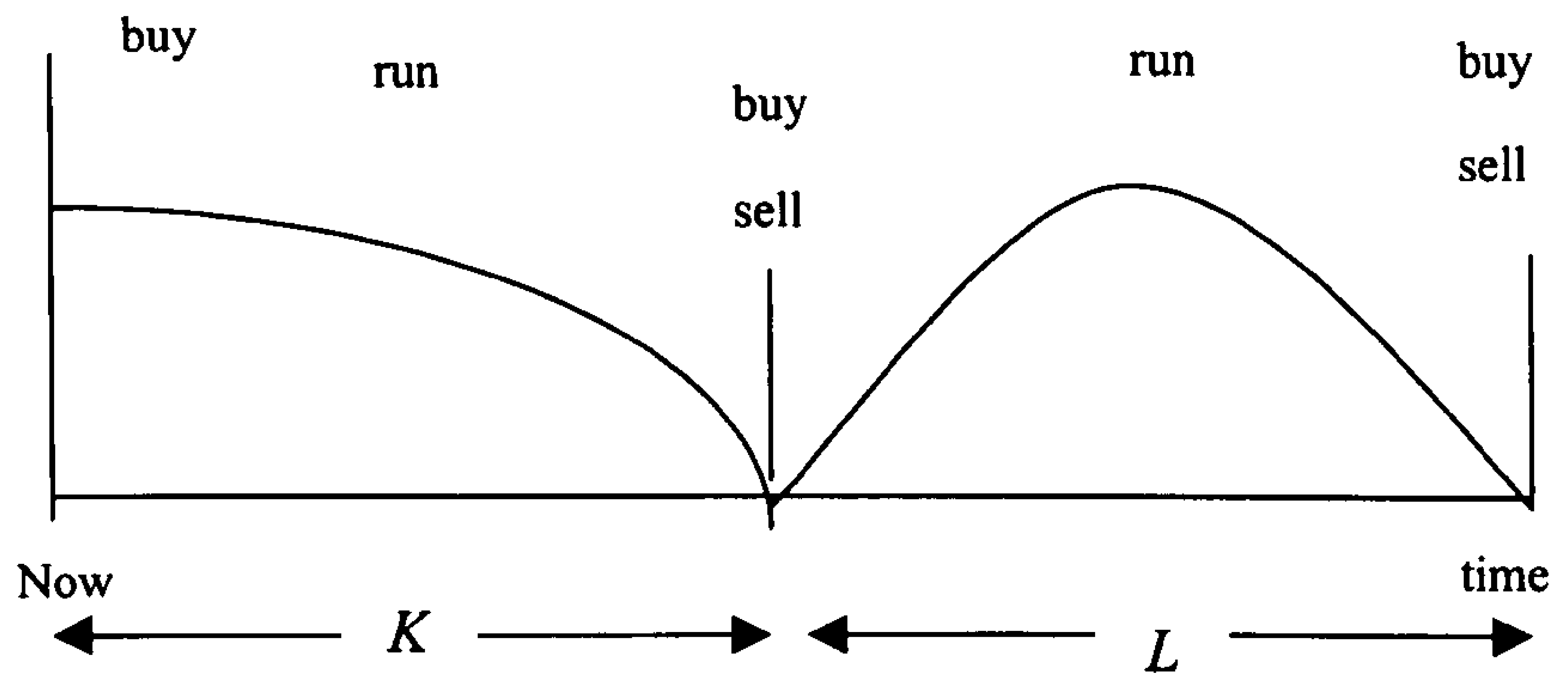


Figure 4.18. Two-cycle replacement procedure.

4.8.3.1. Like-with-like replacement

Here the new equipment is of the same type as the current equipment. The current equipment is operated for K years until resale (the first replacement cycle), the new equipment is purchased and run for L years (the second replacement cycle). Finally, a new equipment is purchased at the end of the second cycle. This model has a finite planning horizon of variable length $K+L$. We seek K^* and L^* which minimise the total cost per unit time (or alternatively the equivalent rent). The average cost per unit time over the two cycles is given by:

$$C(K,L)=\frac{1}{(K+L)}\left[\int_0^K f(t+\tau)dt+\int_0^L f(t)dt+2R\right], \qquad (4.38)$$

where $f(t)$ is the maintenance cost per unit time of the equipment of age t; and R is the purchasing cost new. Using the maintenance cost per unit time function of the form $f(t) = \alpha t^{\beta}$, we obtain

$$C(K,L) = \frac{1}{(K+L)}\left[\frac{\alpha(\tau+K)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{(\beta+1)} + \frac{\alpha L^{\beta+1}}{(\beta+1)} + 2R\right]. \tag{4.39}$$

Thus we obtain

$$(K+L)C(K,L) = \left[\frac{\alpha(\tau+K)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{(\beta+1)} + \frac{\alpha L^{\beta+1}}{(\beta+1)} + 2R\right]. \tag{4.40}$$

Differentiating both sides with respect to K and L we obtain

$$(K+L)\frac{\partial C}{\partial K} + C(K,L) = \alpha(K+\tau)^{\beta},$$

$$(K+L)\frac{\partial C}{\partial L} + C(K,L) = \alpha L^{\beta}.$$

(K^*, L^*) is the solution of $\partial C/\partial K = \partial C/\partial L = 0$. Hence

$$C(K^*, L^*) = \alpha(K^* + \tau)^{\beta}, \tag{4.41}$$

and

$$C(K^*, L^*) = \alpha(L^*)^{\beta}. \tag{4.42}$$

Solving equation (4.41) and equation (4.42) simultaneously we obtain

$$K^* + \tau = L^*.$$

Note that this implies that the ages at replacement for each cycle are the same and is a consequence of equal maintenance costs per unit time.

Substituting in equation (4.39) for $C(K^*, L^*)$ we have

$$\alpha(\tau + K^*)^{\beta} = \frac{1}{(2K^* + \tau)}\left[\frac{2\alpha(\tau+K^*)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{(\beta+1)} + 2R\right],$$

which is a nonlinear equation in K^* and may be solved mumerically to determine K^*.

When $\tau = 0$ we seek the decisions variables K^* and L^* by following the typical steps for minimizing $C(K,L)$. Whence $\alpha(K^*)^\beta = \alpha(L^*)^\beta$ from which $K^* = L^*$. Hence we obtain

$$\alpha(K^*)^\beta = [\alpha(K^*)^{\beta+1} + R(\beta+1)]/K^*(\beta+1),$$

hence

$$K^* = [R(\beta+1)/\alpha\beta]^{1/(\beta+1)} = L^*, \qquad \beta > 0. \quad (4.43)$$

Thus K^* and L^* are just the economic life of the equipment. This is the same result obtained from the infinite cycle replacement model in section 4.2 as expected.

Notice that the value of K^* and L^* in equation (4.43) is obtained when replacement is imposed at the end of the second cycle. Notice that if replacement is not imposed at the end of the second cycle then the difference is purchasing only once at the end of the first cycle. This results in

$$K^* = \{R(\beta+1)/2\alpha\beta\}^{1/\beta+1} = L^*, \qquad \beta > 0. \quad (4.44)$$

Therefore, not including the replacement cost at the end of the second cycle affects the decision. This is not the case in the fixed planning horizon models.

4.8.3.2. Non-like-with-like replacement

Here, we follow the same assumptions as mentioned earlier in section 4.3.1, thus the average total cost per unit time over the two cycles is given by:

$$C(K,L) = \frac{1}{(K+L)}\left[\int_0^K \alpha(t+\tau)^\beta dt + \int_0^L \rho\alpha t^\beta dt + 2R\right],$$

Here ρ is a factor representing the ratio of the maintenance cost per unit time for the current equipment to that for the new equipment, so $\alpha_2 = \rho\alpha$ and $\alpha_1 = \alpha$.

Proceeding as in section 4.8.3.1, we have that:

$$C(K,L) = \frac{1}{(K+L)}\left[\frac{\alpha(\tau+K)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{(\beta+1)} + \frac{\rho\alpha L^{\beta+1}}{(\beta+1)} + 2R\right]. \quad (4.45)$$

Thus we obtain

$$(K+L)C(K,L) = \left[\frac{\alpha(\tau+K)^{\beta+1}}{(\beta+1)} - \frac{\alpha\tau^{\beta+1}}{(\beta+1)} + \frac{\rho\alpha L^{\beta+1}}{(\beta+1)} + 2R\right]. \quad (4.46)$$

Differentiating both sides with respect to K and L we obtain

$$(K+L)\frac{\partial C}{\partial K} + C(K,L) = \alpha(K+\tau)^\beta,$$

$$(K+L)\frac{\partial C}{\partial L} + C(K,L) = \alpha\rho L^\beta.$$

(K^*, L^*) is the solution of $\partial C/\partial K = \partial C/\partial L = 0$. Hence

$$C(K^*, L^*) = \alpha(K^* + \tau)^\beta, \quad (4.47)$$

and

$$C(K^*, L^*) = \alpha\rho(L^*)^\beta. \quad (4.48)$$

Solving equation (4.47) and equation (4.48) simultaneously we obtain

$$K^* + \tau = \rho^{1/\beta} L^*. \quad (4.49)$$

From equation (4.45) we have that

$$(\beta+1)\rho(L^*)^\beta = \frac{1}{(1+\rho^{1/\beta})L^* - \tau}\left[(L^*)^{\beta+1}\rho(1+\rho^{1/\beta}) - \tau^{\beta+1} + 2R\alpha^{-1}(\beta+1)\right].$$

$$(4.50)$$

Again this is a nonlinear equation of L^* so that it must be solved numerically to determine L^* and consequently K^*. Therefore, we move to the easily interpreted case when $\tau = 0$.

Putting $\tau = 0$ in equation (4.50) we obtain

$$\rho\alpha(L^*)^\beta = \left[\alpha(\rho^{\frac{1}{\beta}}L^*)^{\beta+1} + \rho\alpha(L^*)^{\beta+1} + 2R(\beta+1)\right]/(\beta+1),$$

from which we obtain

$$L^* = \left[2R(\beta+1)/\alpha\rho\beta(1+\rho^{\frac{1}{\beta}})\right]^{1/(\beta+1)}. \qquad (4.51)$$

Substituting from equation (4.51) and putting $\tau = 0$ in equation (4.49) we obtain

$$K^* = \rho^{\frac{1}{\beta}}\left[2R(\beta+1)/\alpha\rho\beta(1+\rho^{\frac{1}{\beta}})\right]^{1/(\beta+1)}. \qquad (4.52)$$

This shows that $K^* > L^*$ when $\rho > 1$ and vice versa. The existence of K^* and L^* is satisfied for $\beta > 0$. Also if $\rho > 1$ then for $\beta = 0.7$, $\alpha = 25$, $R = 450$, $\rho = 2$ and t measured in years, we have that the length of the planning horizon is 15.8 with $K^* = 11.5$ and $L^* = 4.3$. Also for the same values used above, but $\rho = 1/2(2/3)$, we find that $K^* = 6.4$ and $L^* = 17.3$ for $\rho = 1/2$, but $K^* = 7.6$ and $L^* = 13.6$ for $\rho = 2/3$.

Note that there is a condition on τ for the existence of L^* in equation (4.50). This condition may be given explicitly when $\beta = 1$, when equation (4.50) simplifies to

$$(L^*)^2 - \frac{2}{1+\rho}L^*\tau + \frac{\tau^2}{\rho(1+\rho)} - \frac{4R}{\alpha\rho(1+\rho)} = 0. \qquad (4.53)$$

A solution to equation (4.53) exists provided $\tau^2 < 4(1+\rho)R\alpha^{-1}$. The condition for K^*, L^*>0 is stronger: namely $\tau^2 < 4\rho R\alpha^{-1}$. For

$$4\rho R\alpha^{-1} \le \tau^2 < 4(1+\rho)R\alpha^{-1},$$

optimum policy would be replace immediately. A similar condition for K^*, L^*>0 holds in the case of like-with-like replacement $(\rho = 1)$ in which case

$$\tau < \sqrt{2} \times (\text{economic life}).$$

For the case $\beta_1 = \beta_2 = 0$ the average total cost per unit time is given by

$$C(K,L) = \frac{1}{(K+L)}\left[\int_0^K \alpha dt + \int_0^L \rho\alpha dt + 2R\right],$$

from which

$$C(K,L) = \frac{1}{(K+L)}\left[\alpha K + \rho\alpha L + 2R\right].$$

Differentiating with respect to K and L we obtain that

$$\partial C / \partial K = \{\alpha(1-\rho)L - 2R\}/(K+L)^2,$$

and

$$\partial C / \partial L = \{\alpha(\rho-1)K - 2R\}/(K+L)^2.$$

Therefore for K, L>0 there is no solution for $\partial C/\partial K = \partial C/\partial L = 0$ and the variable planning horizon replacement decision model is degenerate.

In fact for $\rho > 1$

$$C(K,L) = \frac{1}{(K+L)}\left[\alpha K + \alpha L - (1-\rho)\alpha L + 2R\right],$$

$$= \alpha - \frac{(1-\rho)\alpha L}{K+L} + \frac{2R}{K+L},$$

$$= \alpha + \frac{(\rho - 1)\alpha L}{K + L} + \frac{2R}{K + L},$$

which decreases to α as $K \to \infty$ for any L. So it is optimal to operate the current equipment as long as possible.

For $\rho < 1$

$$C(K, L) = \frac{1}{(K + L)}\left[\rho\alpha K + \rho\alpha L + \alpha K - \rho\alpha K + 2R\right],$$

$$= \frac{1}{(K + L)}\left[\rho\alpha(K + L) + \alpha K(1 - \rho) + 2R\right],$$

$$= \alpha\rho + \frac{\alpha K(1 - \rho)}{K + L} + \frac{2R}{K + L},$$

which decreases to $\alpha\rho$ as $L \to \infty$ for any K. In a practical context this implies we should take $K^* = 0$. However when searching for an optimal in the range $K \in [0, K_l]$, $L \in [0, L_l]$, the result $K^* = 0$ is not guaranteed.

For $\rho = 1$

$$C(K, L) = \alpha + \frac{2R}{K + L},$$

which decreases to α as $K \to \infty$ and/or $L \to \infty$.

4.9. An insight into fixed and variable horizon models

Now we look at optimum policies for fixed planning horizon model if h is treated as a decision variable, that is if the fixed planning horizon model is treated as a variable planning horizon model.

In the case of a simple fixed planning horizon model of like-with-like replacement, the cost function as given earlier is

$$C(x) = \begin{cases} \int_0^x \alpha(t+\tau)^\beta dt + \int_0^{h-x} \alpha t^\beta dt + 2R, & 0 \le x < h, \\ \int_0^h \alpha(t+\tau)^\beta dt + R, & x = h, \end{cases}$$

In this case h is fixed and x is variable. If we consider h as a decision variable and consider the cost per unit time, the cost function will be

$$C(x,h) = \frac{1}{h}\left[\int_0^x \alpha(t+\tau)^\beta dt + \int_0^{h-x} \alpha t^\beta dt + 2R\right], \qquad 0 \le x < h.$$

For simplicity we consider $\tau = 0$ and that gives the following form:

$$C(x,h) = \frac{1}{h}\left[\frac{\alpha x^{\beta+1}}{\beta+1} + \frac{\alpha(h-x)^{\beta+1}}{\beta+1} + 2R\right], \qquad 0 \le x < h. \quad (4.54)$$

In order to achieve the minimum cost we differentiate C with respect to x and h as follows

$$h\partial C / \partial x = \alpha x^\beta - \alpha(h-x)^\beta .$$

Setting $\partial C / \partial x = 0$ we get:

$$x = h - x. \qquad (4.55)$$

Also

$$h\partial C / \partial h + C = \alpha(h-x)^\beta .$$

Setting $\partial C / \partial h = 0$ we have that:

$$C = \alpha(h-x)^\beta .$$

Substituting in equation (4.54) we obtain

$$\alpha x^{\beta+1} = \frac{\alpha x^{\beta+1}}{\beta+1} + R .$$

from which we obtain:

$$x = \left[R / \alpha\left(1 - 1/\beta + 1\right)\right]^{\beta+1} = h - x . \qquad (4.56)$$

Compared with the variable planning horizon model of like-with-like replacement in which the cost function equation was as follows

$$C(K,L)=\frac{1}{(K+L)}\left[\frac{\alpha K^{\beta+1}}{(\beta+1)}+\frac{\alpha L^{\beta+1}}{(\beta+1)}+2R\right],$$

from which we obtain $K=L$ and

$$K=\{R/\alpha(1-1/(\beta+1))\}^{\frac{1}{\beta+1}}=L, \tag{4.57}$$

which is the same as in equation (4.56) but with $K=x$ and $L=h-x$.

Also in the simple case of non-like-with-like replacement in which $\alpha_1\neq\alpha_2$, the cost function was as follows

$$C(x)=\begin{cases}\int_0^x \alpha(t+\tau)^\beta\,dt+\int_0^{h-x}\rho\alpha t^\beta\,dt+2R, & 0\le x<h,\\ \int_0^h \alpha(t+\tau)^\beta\,dt+R, & x=h,\end{cases}$$

Considering x and h as variables the cost function takes the following form:

$$C(x,h)=\frac{1}{h}\left[\int_0^x \alpha(t+\tau)^\beta\,dt+\int_0^{h-x}\rho\alpha t^\beta\,dt+2R\right], \qquad 0\le x<h.$$

Again $\tau=0$ gives

$$C(x,h)=\frac{1}{h}\left[\frac{\alpha x^{\beta+1}}{(\beta+1)}+\frac{\rho\alpha(h-x)^{\beta+1}}{(\beta+1)}+2R\right], \tag{4.58}$$

from which we get:

$$h\partial C/\partial x=\alpha x^\beta-\rho\alpha(h-x)^\beta.$$

Setting $\partial C/\partial x=0$ yields:

$$x=\rho^{1/\beta}(h-x), \tag{4.59}$$

and also

$$h\partial C/\partial h+C=\rho\alpha(h-x)^\beta.$$

Setting $\partial C / \partial h = 0$ gives:

$$C = \rho\alpha(h-x)^{\beta}.$$

Substituting in equation (4.51) we obtain

$$\alpha x^{\beta} = \frac{1}{(1+1/\rho^{1/\beta})x}\left[\alpha\frac{x^{\beta+1}}{\beta+1} + \alpha\rho\frac{x^{\beta+1}}{(\beta+1)\rho^{1+1/\beta}} + 2R\right],$$

from which we obtain:

$$x = \left[2R/\alpha(1+1/\rho^{1/\beta})(1-1/(\beta+1))\right]^{1/\beta+1}. \qquad (4.60)$$

In the case of variable planning horizon model of non like-with-like replacement the cost function equation was:

$$C(K,L) = \left[\alpha(K^{\beta+1} + \rho L^{\beta+1}) + 2R(\beta+1)\right]/(K+L)(\beta+1). \qquad (4.61)$$

Also we obtained

$$K = \rho^{1/\beta}L, \qquad (4.62)$$

$$K = \left[2R/\alpha(1+1/\rho^{1/\beta})(1-1/(\beta+1))\right]^{1/\beta+1}. \qquad (4.63)$$

Thus the results from the variable planning horizon model are identical to the results presented in equation (4.59) and equation (4.60). So the two models are mathematically equivalent. The fixed horizon model has fixed h, variable x and variable number of cycles; the variable planning horizon model has variable $h = K + L$, variable K and fixed number of cycles.

One of the principal ideas of this thesis is that the at-most-two-replacements fixed planning horizon model is a natural alternative to the variable planning horizon two-cycle model of Christer and Goodbody (1980). The fixed horizon model behaves sensibly even when $\beta = 0$, and also is a natural choice when management set the planning horizon in a strategic plan.

4.10. Fixed planning horizon models: a Dynamic programming approach

4.10.1. Introduction

Dynamic programming (DP) is a particular approach to optimisation. It is not a particular algorithm in the sense that the simplex method for linear programming is one. Rather, DP is a way of structuring certain problems so that a particular methodology can be used. The name dynamic programming evolved because of its use with applications involving decision making over time. However, other situations which are static can also be solved successfully by DP. Dynamic programming is a way of looking at a problem, that may contain a large number of interrelated decision variables, in which the problem is regarded as if it consisted of a sequence of problems, each of which required the determination of only one (or few) variables. Whenever it is possible to structure a problem in this way, it is usually the case that very much less computational effort is required. The computations at the different stages are linked through recursive computations in a manner that yields an optimal solution to the entire problem. Put simply, the DP approach transforms or considers an n-dimensional problem as n sequential one-dimensional problems. This transformation of the problem to one that requires much less computational effort is one of the main advantages of DP and will be discussed later in this section.

Problems to which DP has been applied are usually stated in the following terms. A physical, operational, or conceptual system is considered to progress through a series of consecutive stages. At each stage the system can be described or characterised by a relatively small set of parameters called the state-variables. At each stage, and no matter what state the system is in, one or more decisions must be made. The decisions may depend on either stage or state or both. It is usually assumed that the past history of the system, i.e., how it got to the current stage and state, are of no importance. In other words, the decisions are assumed to depend only upon the current stage and state. When a decision is made a return (value of the relevant part of the objective function) or reward is obtained and the system undergoes a transformation or transition to the next stage. The return is determined by a known single valued function of the input state. Similarly, the transformed state results from a known single-valued function of the decision acting upon the current state. The overall purpose of the staged process is to maximise or minimise some function of the state and decision variables. Therefore one can say that the key elements one associates with a DP problem are stages, states, decisions, transformations, and returns (Cooper. L & Cooper. M.W, 1981).

The principle or point of view that enables us to carry out the transformation we have just discussed is known as the principle of optimality. It was first enunciated by (Bellman, 1957). This "principle of optimality" is: An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with respect to the state which results from the initial decision.

One of the advantages of DP is that it determines absolute (global) maxima or minima rather than relative (local) optima. DP programming can be applied to problems whose initial mathematical representations are quite different.

The disadvantage of DP is that although a problem can be decomposed properly, a numerical answer still may not be attainable because of the complexity of the optimisation process at each stage. We must point out, however, that in spite of this disadvantage, the solution of many problems has been facilitated greatly through the use of DP.

To put in context what has been discussed above, a simple equipment replacement example is: Let us suppose that we have a system in operation which is one year old. If a decision is made annually to keep the current system or to replace it, what is the policy that minimises the total cost or maximises the profit over the next five years?. We wish to find a sequence of five annual decisions, i.e., to keep or to replace which minimise the total cost from five years of operations. This analysis can be facilitated by numbering the stages backward, i.e., stage 1 means we have one year left. Hence we start with stage 5. Figure (4.19) indicates the notation using (t_j) for system age and d_j for decisions which are keep (K) or replace (R).

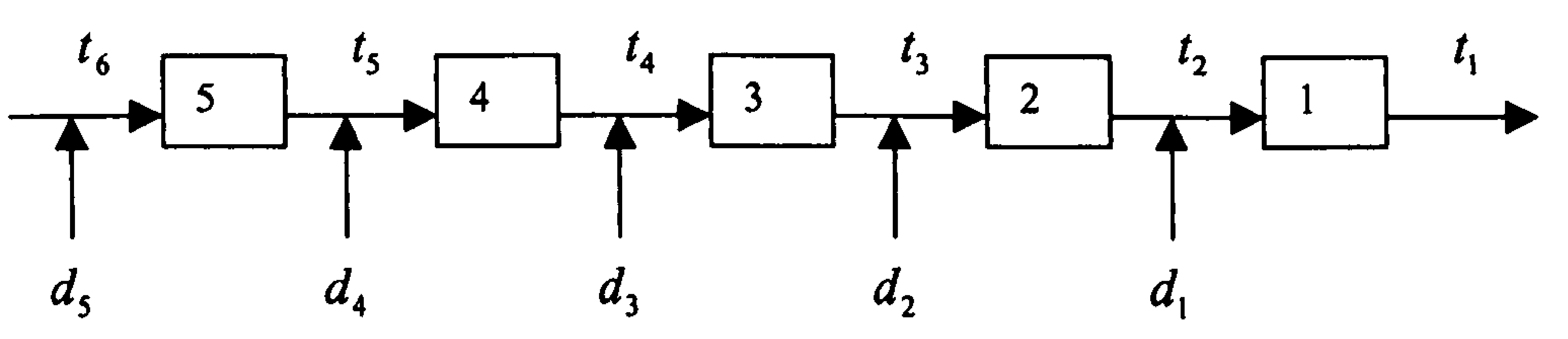


Figure 4.19. Stages of the decision problem.

It can be seen that t_j is given by

$$t_j = t_{j+1} + 1,$$

if the decision is to keep the system, and $t_j = 1$ if the system is replaced. Figure (4.20) explains the above notation and conventions in further detail.

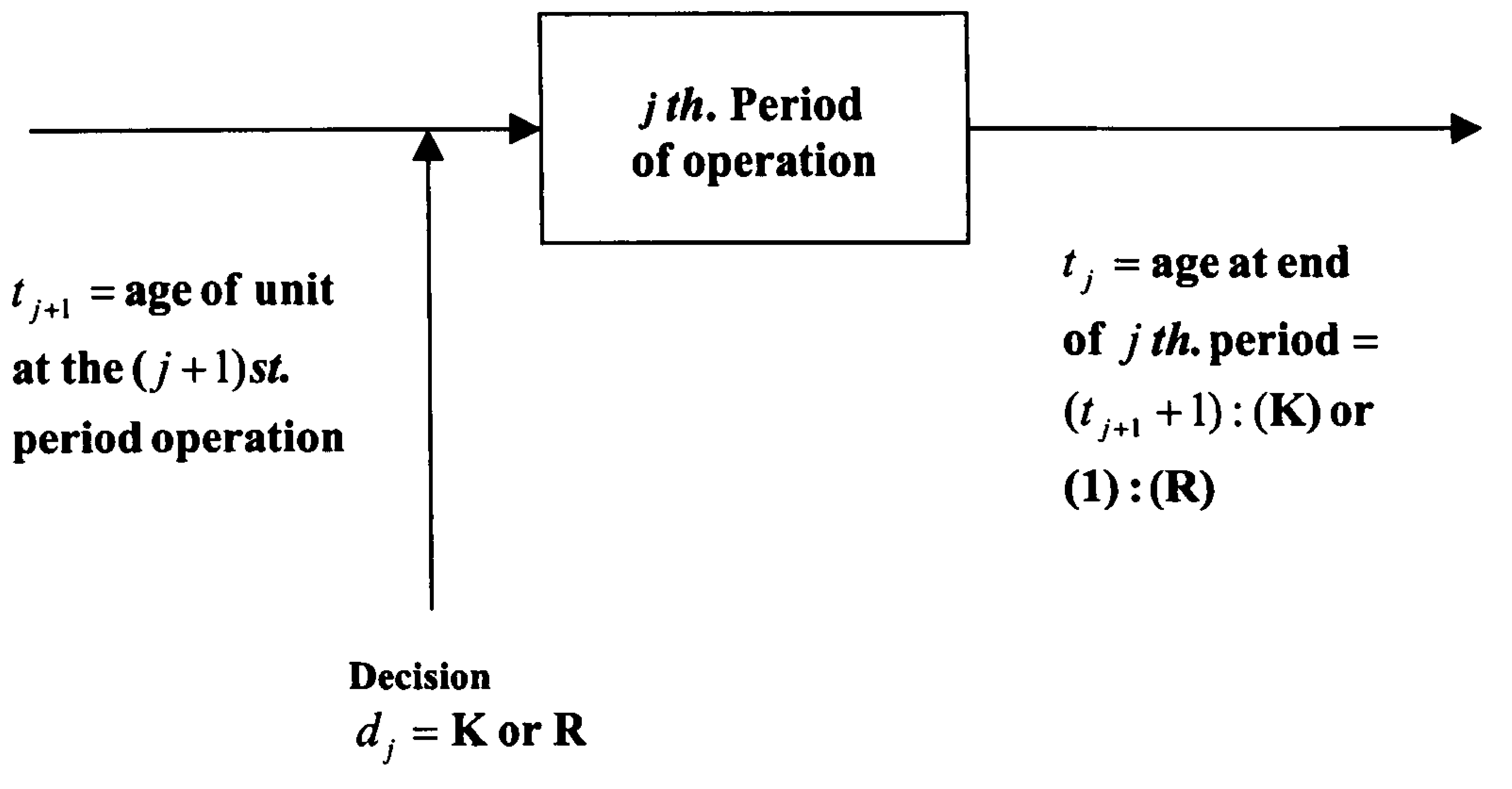


Figure 4.20. Decision problem notation.

For our study, we now describe a replacement model in which we are no longer restricted to the model of at most two replacements. This latter restriction has been used in the earlier sections. A drawback of the at most two replacements model is that we cannot presume that this will give us an "optimal" solution to the problem of interest. However for reasonably chosen h (the fixed planning horizon), we would not expect more than two replacements. However it is still important to attempt to determine a "globally" optimal policy, which may be done using a dynamic programming approach. The difficulty with this approach is that we can only investigate the behaviour of optimal policy numerically. Essentially, the solution obtained in the dynamic programming approach will not give insight into the problem in same way as the economic life models earlier in this chapter.

4.10.2. Like-with-like replacement

Here the planning horizon h is divided into N equally spaced time periods of length Δh. The current equipment is of age τ time periods. R is the purchase cost of new equipment and resale value is zero. Note that, because the DP procedure is computational we need not assume zero resale value.

The maintenance cost per period at age m time periods is

$$M(m) = \int_{(m-1)\Delta h}^{m\Delta h} \alpha t^{\beta} dt = \frac{\alpha}{\beta+1}\left[(m\Delta h)^{\beta+1} - ((m-1)\Delta h)^{\beta+1}\right]. \tag{4.64}$$

We have two decisions at the end of each period, that is, either keep (K) or replace (R) the current equipment. This decision minimises the cost incurred when there are n time periods remaining in the horizon and the current equipment is of age m time periods. A translation of the preceding paragraph, with the addition of choosing whichever total cost is smaller, give us a set of recurrence relations which solves the problem of minimising the total cost over a total time of N periods. These recurrence relations are

$$V(n,m) = \min\begin{bmatrix} (\mathrm{K}): M(m+1) + V(n-1, m+1) \\ (\mathrm{R}): M(1) + V(n-1,1) + R \end{bmatrix}, \tag{4.65}$$

$$(n = 1,2,\ldots, N;\ \ m = 1,2,\ldots, \tau + N - n).$$

$V(n,m)$ is the total cost over n remaining periods when the current age of the equipment is m.

We also require the relation

$$V(0,m) = R \ (\ m = 1,2,\ldots\ldots\ldots\ldots\ldots\ldots, \tau + N). \qquad (4.66)$$

Notice that replacement is imposed (by convention) at the end of the horizon h and equation (4.66) represents this fact.

At a particular epoch, the optimal policy is that decision (keep or replace) which minimises $V(n,m)$. Over the complete horizon, optimal policy will consist of a sequence of keep (K) or replace (R) decisions at each epoch. This will imply an optimal number of replacements N^* over the planning horizon and optimal times to replacement (for each replacement epoch), $x_1^*,\ldots\ldots,x_{N^*}^*$. This should be compared with earlier policies (at most two replacements) with optimal policy $(N^*,x^*)(N^* = 1,2; x^* = h, x^* < h)$.

4.10.2.1. Computational examples for like-with-like replacement

In this section we implement the like-with-like replacement using the dynamic programming approach. We determine the optimal policy over a fixed planning horizon h that is divided into equally spaced periods. To give an insight about the dynamic programming approach for solving the replacement problem we use different values for the fixed planning horizon h (say, $h = 10$, $h = 15$ and $h = 20$). Also we use different values for the maintenance cost per period parameters (αs and βs). Because the computations are large for $h = 15$ and $h = 20$ we present two computational examples for $h = 10$ with different αs and βs. The computations results are shown in Tables 4.13 and 4.14. The FORTRAN 77 program is given in appendix 1.

From Table 4.13 we can determine the optimal 10-year policy. If we currently have an equipment of age zero (new) at the beginning of the horizon we enter the table at $m = 0$ and $n = 10$, since we wish to calculate the optimal 10-year policy. We see that the minimum total cost is 871.6 and we keep the equipment one year more. We now move over to $n = 9$ and $m = 1$ since the equipment is now one year old. We see that the optimal policy is to keep the equipment one year more. Thus throughout the table one can follow the same way until reaching the last column ($n = 1$) to obtain the optimal policy actions and results over the 10 years. Therefore with the condition that replacement is compulsory at the end of the horizon one can observe that the optimal policy actions are (KKKKKKKKKK) with optimal policy results $N^* = 1$ (number of replacements over the horizon) and $x_1^* = h = 10$ (time to replacement) with minimum total cost $V(10,0) = 871.6$. Note that in this case there is only one replacement that is compulsory at the end of the horizon, so that $N^* = 1$.

Similarly if we start with equipment of age 2 years old from Table 4.13 we can find that the optimal policy actions are (KKKKKKKKKK) and also $N^* = 1$ and $x_1^* = h = 10$ with minimum total cost $V(10,2) = 966.5$.

If we start with an equipment of age 4 years old at the beginning of the horizon from the table we can see some change in the optimal policy obtained above. We enter the table at $m = 4$ and $n = 10$, since we wish to calculate the optimal 10-year policy. We see that the minimum total cost is 1287.2 and we keep the equipment one year more. Again by following the same way as in the cases $\tau = 0$ and $\tau = 2$ one can find that at $n = 7$ and $m = 7$ the equipment must be replaced and this will lead to year 6 ($n = 6$) with the equipment of age one year old ($m = 1$). Starting from $n = 6$ and following the same way as mentioned above we can obtain the optimal policy. Therefore the optimal policy actions are (KKKRKKKKKK) with optimal policy results $N^* = 2$, $x_1^* = 7$ and $x_2^* = h = 10$ with minimum total cost $V(10,4) = 1287.2$.

Similarly we can determine the optimal policy from Table 4.14. In the case of $\tau = 0$ we have the actions (KKKKKKKKKK) with optimal policy results $N^* = 1$ and $x_1^* = h = 10$ with minimum total cost $V(10,0) = 1335$. In the case of $\tau = 2$; (KKKKRKKKKK); $N^* = 2$, $x_1^* = 6$ and $x_2^* = h = 10$ with minimum total cost $V(10,2) = 1584.9$. Finally in the case of $\tau = 4$; (KKKRKKKKKK); $N^* = 2$, $x_1^* = 7$ and $x_2^* = h = 10$ with minimum total cost $V(10,4) = 1678.4$.

The computations results for $h = 10$, $h = 15$ and $h = 20$ with different maintenance cost per period parameters (αs and βs) and different current ages (τ) of the existing equipment are summarised in Tables 4.15, 4.16 and 4.17.

From the previous results we notice that for reasonable values of α and β there is at most two replacements over the planning horizon. Also we notice that the number of replacements $N^* = 3$ takes place only when h, α and β greatly increase.

Table 4.13. The results of optimal policy over the planning horizon $h = 10$ years divided into 10 periods each of one year length. The maintenance cost per period parameters are $\alpha = 20$ and $\beta = 0.5$; the replacement (purchase) cost $R =$ M$450K. The numbers represent the minimum total cost over remaining periods and K and R represent the action takes place at each period; that is K keep the equipment and R replace the equipment.

	Number of periods remaining, n.									
Age m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	463.3 (K)	487.7 (K)	519.3 (K)	556.7 (K)	599.1 (K)	646.0 (K)	696.9 (K)	751.7 (K)	810.0 (K)	871.6 (K)
1	474.4 (K)	505.9 (K)	543.3 (K)	585.8 (K)	632.6 (K)	683.6 (K)	738.4 (K)	796.7 (K)	858.3 (K)	923.1 (K)
2	481.6 (K)	519.0 (K)	561.4 (K)	608.3 (K)	659.2 (K)	714.0 (K)	772.3 (K)	833.9 (K)	898.7 (K)	966.5 (K)
3	487.4 (K)	529.8 (K)	576.7 (K)	627.7 (K)	682.4 (K)	740.7 (K)	802.4 (K)	867.2 (K)	935.0 (K)	1273.6 (K)
4	492.4 (K)	539.3 (K)	590.3 (K)	645.0 (K)	703.3 (K)	765.0 (K)	829.8 (K)	897.6 (K)	1236.2 (K)	1287.2 (K)
5	496.9 (K)	547.9 (K)	602.6 (K)	660.9 (K)	722.6 (K)	787.4 (K)	855.2 (K)	1193.8 (K)	1244.8 (K)	1299.6 (K)
6	501.0 (K)	555.7 (K)	614.0 (K)	675.7 (K)	740.5 (K)	808.3 (K)	1146.9 (K)	1197.9 (K)	1252.7 (K)	1307.4 (K)
7	504.8 (K)	563.1 (K)	624.7 (K)	689.51 (K)	757.3 (K)	1096.0 (R)	1146.9 (R)	1201.7 (K)	1256.5 (K)	1314.8 (K)
8	508.3 (K)	569.9 (K)	634.7 (K)	702.6 (K)	1049.1 (R)	1096.0 (R)	1146.9 (R)	1201.7 (R)	1260.0 (K)	1318.3 (K)
9	511.6 (K)	576.4 (K)	644.3 (K)	1006.7 (R)	1049.1 (R)	1096.0 (R)	1146.9 (R)	1201.7 (R)	1260.0 (R)	1321.6 (K)
10	514.8 (K)	582.6 (K)	969.3 (R)	1006.7 (R)	1049.1 (R)	1096.0 (R)	1146.9 (R)	1201.7 (R)	1260.0 (R)	1321.6 (R)
11	517.8 (K)	937.7 (R)	969.3 (R)	1006.7 (R)	1049.1 (R)	1096.0 (R)	1146.9 (R)	1201.7 (R)	1260.0 (R)	1321.6 (R)

Table 4.14. The results of optimal policy over the planning horizon $h = 10$ years divided into 10 periods each of one year length. The maintenance cost per period parameters are $\alpha = 30$ and $\beta = 0.7$; the replacement (purchase) cost $R =$ M\$450K. The numbers represent the minimum total cost over remaining periods and K and R represent the action takes place at each period; that is K keep the equipment and R replace the equipment.

	Number of periods remaining, n.									
Age m	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
0	467.6 (K)	507.3 (K)	564.2 (K)	636.3 (K)	722.2 (K)	821.1 (K)	932.3 (K)	1055 (K)	1189 (K)	1335 (K)
1	489.7 (K)	546.6 (K)	618.6 (K)	704.6 (K)	803.5 (K)	914.7 (K)	1037.6 (K)	1171.8 (K)	1316.8 (K)	1472.4 (K)
2	506.9 (K)	578.9 (K)	664.9 (K)	763.8 (K)	875.0 (K)	997.9 (K)	1132.1 (K)	1277.1 (K)	1432.7 (K)	1584.9 (K)
3	522.1 (K)	608.0 (K)	706.9 (K)	818.1 (K)	941.0 (K)	1075.2 (K)	1220.2 (K)	1375.8 (K)	1528.0 (K)	1639.2 (K)
4	535.9 (K)	634.8 (K)	746.0 (K)	869.0 (K)	1003.1 (K)	1148.2 (K)	1303.7 (K)	1456.0 (K)	1567.2 (K)	1678.4 (K)
5	548.9 (K)	660.1 (K)	783.0 (K)	917.2 (K)	1062.2 (K)	1217.8 (K)	1370.1 (K)	1481.2 (K)	1592.4 (K)	1715.3 (K)
6	561.2 (K)	684.1 (K)	818.3 (K)	963.3 (K)	1118.9 (K)	1271.1 (R)	1382.3 (R)	1493.5 (K)	1616.4 (K)	1739.3 (K)
7	572.9 (K)	707.1 (K)	852.1 (K)	1007.7 (K)	1172.2 (R)	1271.1 (R)	1382.3 (R)	1505.2 (K)	1628.2 (K)	1762.3 (K)
8	584.1 (K)	729.2 (K)	884.8 (K)	1050.6 (K)	1172.2 (R)	1271.1 (R)	1382.3 (R)	1505.2 (R)	1639.4 (R)	1773.6 (K)
9	595.0 (K)	750.6 (K)	916.4 (K)	1086.3 (R)	1172.2 (R)	1271.1 (R)	1382.3 (R)	1505.2 (R)	1639.4 (R)	1784.5 (K)
10	605.6 (K)	771.4 (K)	1014.2 (R)	1086.3 (R)	1172.2 (R)	1271.1 (R)	1382.3 (R)	1505.2 (R)	1639.4 (R)	1784.5 (R)
11	615.8 (K)	957.3 (R)	1014.2 (R)	1086.3 (R)	1172.2 (R)	1271.1 (R)	1382.3 (R)	1505.2 (R)	1639.4 (R)	1784.5 (R)

Table 4.15. The results of optimal 10-year policy for different maintenance cost per period parameters (αs and βs) and different current ages of the existing equipment. N^* is the number of replacements over the horizon of 10 years length divided into 10 periods. x_i^* ($i = 1,\ldots\ldots, N^*$) are optimal times to replacements take place over the horizon.

		$h = 10$		
		$\tau = 0$	$\tau = 2$	$\tau = 4$
$\alpha = 20$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 10$; Min. cost=871.6	$N^* = 1$; $x_1^* = h = 10$; Min. cost=966.5	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1287.2
	$\beta = 0.7$	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1039.6	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1215.7	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1418.9
$\alpha = 30$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1082.5	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1224.8	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1480.8
	$\beta = 0.7$	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1334.5	$N^* = 2$; $x_1^* = 6, x_2^* = 10$; Min. cost=1584.9	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1678.4
$\alpha = 40$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1293.3	$N^* = 1$; $x_1^* = h = 10$; Min. cost=1483.1	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1674.4
	$\beta = 0.7$	$N^* = 2$; $x_1^* = 5, x_2^* = 10$; Min. cost=1625.9	$N^* = 2$; $x_1^* = 6, x_2^* = 10$; Min. cost=1813.2	$N^* = 2$; $x_1^* = 7, x_2^* = 10$; Min. cost=1937.8

Table 4.16. The results of optimal 15-year policy for different maintenance cost per period parameters (αs and βs) and different current ages of the existing equipment. N^* is the number of replacements over the horizon of 15 years length divided into 15 periods. x_i^* ($i = 1,\ldots\ldots, N^*$) are optimal times to replacements take place over the horizon.

		$h = 15$		
		$\tau = 0$	$\tau = 2$	$\tau = 4$
$\alpha = 20$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 15$; Min. cost=1224.6	$N^* = 1$; $x_1^* = h = 15$; Min. cost=1346.9	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=1575.0
	$\beta = 0.7$	$N^* = 1$; $x_1^* = h = 15$; Min. cost=1624.7	$N^* = 2$; $x_1^* = 8, x_2^* = 15$; Min. cost=1758.2	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=1858.4
$\alpha = 30$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 15$; Min. cost=1611.9	$N^* = 1$; $x_1^* = h = 15$; Min. cost=1795.3	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=1912.5
	$\beta = 0.7$	$N^* = 2$; $x_1^* = 7, x_2^* = 15$; Min. cost=1987.6	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=2187.3	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=2337.6
$\alpha = 40$	$\beta = 0.5$	$N^* = 2$; $x_1^* = 7, x_2^* = 15$; Min. cost=1997.3	$N^* = 2$; $x_1^* = 8, x_2^* = 15$; Min. cost=2148.0	$N^* = 2$; $x_1^* = 9, x_2^* = 15$; Min. cost=2249.9
	$\beta = 0.7$	$N^* = 2$; $x_1^* = 7, x_2^* = 15$; Min. cost=2350.1	$N^* = 2$; $x_1^* = 8, x_2^* = 15$; Min. cost=2616.4	$N^* = 3$; $x_1^* = 6, x_2^* = 12$, $x_3^* = 15$; Min. cost=2734.4

Table 4.17. The results of optimal 20-year policy for different maintenance cost per period parameters (αs and βs) and different current ages of the existing equipment. N^* is the number of replacements over the horizon of 20 years length divided into 20 periods. x_i^* ($i = 1,\ldots\ldots, N^*$) are optimal times to replacements the times for replacements take place over the horizon.

		$h = 20$		
		$\tau = 0$	$\tau = 2$	$\tau = 4$
$\alpha = 20$	$\beta = 0.5$	$N^* = 1$; $x_1^* = h = 20$; Min. cost=1642.6	$N^* = 1$; $x_1^* = h = 20$; Min. cost=1788.1	$N^* = 2$; $x_1^* = 12, x_2^* = 20$; Min. cost=1901.9
	$\beta = 0.7$	$N^* = 2$; $x_1^* = 10, x_2^* = 20$; Min. cost=2079.3	$N^* = 2$; $x_1^* = 11, x_2^* = 20$; Min. cost=2248.5	$N^* = 2$; $x_1^* = 12, x_2^* = 20$; Min. cost=2383.6
$\alpha = 30$	$\beta = 0.5$	$N^* = 2$; $x_1^* = 10, x_2^* = 20$; Min. cost=2164.9	$N^* = 2$; $x_1^* = 11, x_2^* = 20$; Min. cost=2302.8	$N^* = 2$; $x_1^* = 12, x_2^* = 20$; Min. cost=2402.8
	$\beta = 0.7$	$N^* = 2$; $x_1^* = 10, x_2^* = 20$; Min.cost=2668.9	$N^* = 3$; $x_1^* = 7, x_2^* = 14$, $x_3^* = 20$; Min. cost=2862.6	$N^* = 3$; $x_1^* = 8, x_2^* = 16$, $x_3^* = 20$; Min. cost=2979.4
$\alpha = 40$	$\beta = 0.5$	$N^* = 2$; $x_1^* = 10, x_2^* = 20$; Min.cost=2586.6	$N^* = 2$; $x_1^* = 11, x_2^* = 20$; Min. cost=2770.3	$N^* = 2$; $x_1^* = 12, x_2^* = 20$; Min. cost=2903.7
	$\beta = 0.7$	$N^* = 3$; $x_1^* = 6, x_2^* = 13$, $x_3^* = 20$; Min. cost=3131.0	$N^* = 3$; $x_1^* = 7, x_2^* = 14$, $x_3^* = 20$; Min. cost=3366.7	$N^* = 3$; $x_1^* = 8, x_2^* = 16$, $x_3^* = 20$; Min. cost=3522.6

4.10.3. Non-like-with-like replacement

We now make the same assumptions as above in the like-with-like replacement but that there are j equipment types $(j = 1, \ldots, N)$. For the current equipment $j = 1$. The maintenance cost per period at age m time periods for the $jth.$ equipment type is

$$M_j(m) = \frac{\alpha_j}{\beta_j + 1}\left[(m\Delta h)^{\beta_j+1} - ((m-1)\Delta h)^{\beta_j+1}\right]. \tag{4.67}$$

We suppose that at $jth.$ replacement we replace with type $j+1$. Again we have the keep or replace decision for each period. Minimising the total cost when there are n time periods remaining in the horizon and the current equipment is of age m time periods leads to the recurrence relations

$$V_j(n,m) = \min\begin{bmatrix} (\mathrm{K}): M_j(m+1) + V_j(n-1, m+1) \\ (\mathrm{R}): M_{j+1}(1) + V_{j+1}(n-1, 1) + R_{j+1} \end{bmatrix}, \tag{4.68}$$

$$(n = 1,2,\ldots, N;\;\; m = 1,2,\ldots, \tau + N - n;\;\; j = 1,2,\ldots, N).$$

We also require the relation

$$V_j(0,m) = R_j \;\;(\; m = 1,2,\ldots, \tau + N;\;\; j = 1,2,\ldots, N). \tag{4.69}$$

Again replacement is compulsory at the end of the horizon h.

As we mentioned in the previous section the optimal policy will consist of two possible actions at each period (keep or replace with the different type j) which minimises $V_j(n,m)$. Therefore over the whole horizon, optimal policy will consist of a sequence of keep (K) or replace (R) decisions at each epoch. This will imply an optimal policy $(N^*, x_1^*, \ldots, x_{N^*}^*)$ with number of equipment types $N^* + 1$. To make this approach practical, we need to model the cost of as many equipment types as there are replacements. This is difficult unless we assume some functional form for technological improvements or cost reduction at each replacement as in Elton and Gruber (1976). Given such a model to describe technological change, a DP approach to this replacement problem can then be formulated and solved in a similar manner to the like-with-like case.

4.11. Discussion

This chapter has been concerned with studying the behaviour of optimal policy of simple capital replacement models. This behaviour was described by determining the value of the decision variable x^* (the time for first replacement) and the range of values of the control variable h (the length of the planning horizon) for which there would be two replacements over the planning horizon. Difficulties arise when studying the behaviour of optimal policy for non-like-with-like replacement. To overcome these difficulties some approximations for the maintenance cost per unit time function were used. A numerical study of the behaviour of optimal policy of non-like-with-like replacement was presented graphically for particular values of the cost parameters, and illustrates the alternative decisions: replace within the horizon; or replace at the end of the horizon. It should be noted that when we say replacement within the horizon we include the case $x = 0$ (immediate replacement); thus within the horizon means $x \in [0, h)$. The behaviour of optimal policy was also compared with that of the variable planning horizon model. Sensible results can be obtained for the fixed horizon model when $\beta = 0$ although this is not guaranteed for the variable planning horizon model. We describe also the relation between the fixed planning horizon model and variable planning horizon model. Finally, a dynamic programming approach is presented for like-with-like and non-like-with-like replacement with implementation for the like-with-like case.

Much of the work considered in this chapter is summarized in Scarf and Hashem (2000).

CHAPTER 5

CHAPTER 5

The Behaviour of Optimal Policy for a Mixed Fleet

5.1. Introduction

Here we are concerned with studying the behaviour of optimal replacement policy for a mixed fleet over a fixed planning horizon. The case we are concerned with is the mixed fleet with many subfleets, say n subfleets, and each subfleet consists of a single item/equipment. In the simplest case, the mixed fleet consists of two subfleets and each subfleet consists of a single item/equipment. In the more general problem the mixed fleet consists of many subfleets and each subfleet contains many items/equipment. We wish to consider costs over a certain fixed planning horizon. We assume that individual items/equipment fulfil the same function : for example, a computer laboratory contains computers of the similar capability; or a transport company runs a number of subfleets of buses (characterised by make and model say) and all buses are used for city transport without discrimination. In general it is required to determine which of the subfleets to replace first. In practice the alternative choices for the replacement policy over the planning horizon in the many subfleets single item case are: replace nothing, replace item i $(i = 1,\ldots\ldots,n)$ of the n items. The replacement decision is taken on the basis of which alternative gives the minimum cost over the same fixed planning horizon. Consequently, we can consider different replacement scenarios, each of them is related to one of the n items/equipment. The study can be generalised to the case of many subfleets each with many items.

Throughout we adopt the convention that a subfleet is replaced at the end of the planning horizon.

5.2. Many subfleets case

In this section we consider n equipment operating over a certain fixed planning horizon and it is required to study the behaviour of optimal policy for different replacement scenarios. For example we consider the scenario i ($i = 1,.....,n$) to replace subfleet/equipment i within the planning horizon. We number the new subfleets/equipment as $n+1$, $n+2$. We assume that if we replace a subfleet within the horizon, then we replace another subfleet at the end; that is, there are at most two replacements. This need not to be the case in general; however for the range of maintenance costs and replacement costs associated with typical equipment in practice this is likely to be the case. For clarity and brevity, we confine ourselves to likely scenarios.

For scenario i ($i = 1,.....,n$) the total cost over $[0,h]$ given replacement of subfleet i with subfleet $n+1$ at x_i, is

$$C_i(x_i) = \begin{cases} \sum_{j=1}^{i-1} \int_0^h M_j(t+\tau_j)dt + \int_0^{x_i} M_i(t+\tau_i)dt + \int_0^{h-x_i} M_{n+1}(t)dt + \\ \qquad \sum_{j=i+1}^{n} \int_0^h M_j(t+\tau_j)dt \;\; + R_{n+1} + R_{n+2}, & 0 \le x_i < h, \\ \sum_{j=1}^{n} \int_0^h M_j(t+\tau_j)dt + R_{n+1}, & x_i = h, \end{cases} \quad (5.1)$$

where M_j is the maintenance cost per unit time of equipment j $(j = 1,.....,n)$; M_{n+1} is the maintenance cost per unit time of the new equipment $n+1$; R_{n+1} is the purchase cost of the new equipment $n+1$; R_{n+2} is the purchase cost of the new equipment $n+2$; τ_j is the current age of equipment j; x_i is the time of first replacement for equipment i and h is the length of the planning horizon.

5.2.1. The behaviour of optimal policy

Using the maintenance cost per unit time form $M(t) = \alpha t^{\beta}$, the cost function of replacing equipment i takes the following form

$$C_i(x_i) = \begin{cases} \sum_{j=1}^{i-1} \int_0^h \alpha_j (t+\tau_j)^{\beta_j} dt + \int_0^{x_i} \alpha_i (t+\tau_i)^{\beta_i} dt + \int_0^{h-x_i} \alpha_{n+1} t^{\beta_{n+1}} dt + \\ \qquad \sum_{j=i+1}^{n} \int_0^h \alpha_j (t+\tau_j)^{\beta_j} dt + R_{n+1} + R_{n+2}, & 0 \le x_i < h, \\ \sum_{j=1}^{n} \int_0^h \alpha_j (t+\tau_j)^{\beta_j} dt + R_{n+1}, & x_i = h. \end{cases} \quad (5.2)$$

To study the behaviour of optimal policy in the case of mixed subfleet we assume that the maintenance cost per unit time forms are applied with different αs and equal βs to obtain theoretical results simply. Therefore, the total cost over planning horizon if we replace equipment i takes the following form

$$C_i(x_i) = \begin{cases} \sum_{j=1}^{i-1} \int_0^h \alpha_j (t+\tau_j)^\beta \, dt + \int_0^{x_i} \alpha_i (t+\tau_i)^\beta \, dt + \int_0^{h-x_i} \alpha_{n+1} t^\beta \, dt + \\ \qquad \sum_{j=i+1}^{n} \int_0^h \alpha_j (t+\tau_j)^\beta \, dt + R_{n+1} + R_{n+2}, & 0 \le x_i < h, \\ \sum_{j=1}^{n} \int_0^h \alpha_j (t+\tau_j)^\beta \, dt + R_{n+1}, & x_i = h. \end{cases} \quad (5.3)$$

Differentiating C_i with respect to x_i we obtain

$$\frac{\partial C_i}{\partial x_i} = \alpha_i (x_i + \tau_i)^\beta - \alpha_{n+1} (h - x_i)^\beta .$$

Setting $dC_i / dx_i = 0$ we obtain

$$x_i + \tau_i = (\alpha_{n+1} / \alpha_i)^{1/\beta} (h - x_i),$$

$$= \lambda_i (h - x_i) .$$

Hence

$$x_i' = \frac{\lambda_i h - \tau_i}{\lambda_i + 1} \quad (5.4)$$

is now the local minimum for C_i but not necessarily the global minimum. The condition for x_i' to be the global minimum is that

$$C_i(x_i') < C(h) . \quad (5.5)$$

If inequality (5.5) is true then the global minimum of C_i is $x_i^* = x_i'$ otherwise $x_i^* = h$. Inequality (5.5) can be written as

$$\int_0^{x_i} \alpha_i (t+\tau_i)^\beta \, dt + \int_0^{h-x_i} \alpha_{n+1} t^\beta \, dt < \int_0^h \alpha_i (t+\tau_i)^\beta \, dt + R_{n+2} .$$

This simplifies to

$$\left[\frac{\alpha_i \lambda_i^{\beta+1} + \alpha_{n+1}}{(\lambda_i + 1)^{\beta+1}} - \alpha_i \right] (\tau_i + h)^{\beta+1} + R_{n+2} (\beta + 1) < 0,$$

from which we obtain that

$$h > \left\{ R_{n+2}(\beta+1)/(\alpha_i - \frac{\alpha_i \lambda_i^{\beta+1} + \alpha_{n+1}}{(\lambda_i + 1)^{\beta+1}}) \right\}^{(1/\beta+1)} - \tau_i. \tag{5.6}$$

Inequality (5.6) represents the values of the fixed planning horizon as a control variable over which equipment i would be replaced within the planning horizon and another equipment at the end of the planning horizon. Therefore, the value of h given is the planning horizon over which there are two replacements, one within the horizon and the other at the end of the horizon. If

$$h \le \left\{ R_{n+2}(\beta+1)/(\alpha_i - \frac{\alpha_i \lambda_i^{\beta+1} + \alpha_{n+1}}{(\lambda_i + 1)^{\beta+1}}) \right\}^{(1/\beta+1)} - \tau_i, \tag{5.7}$$

then there is only one replacement at the end of the horizon.

Now we can compare $C_1^*(x_1^*)$, $C_2^*(x_2^*)$,..., $C_n^*(x_n^*)$ to determine which subfleet to replace first.

5.2.2. Comparison between the two replacements in the two subfleets case

In the two subfleets case we consider two equipment operating over a certain fixed planning horizon and it is required to study the behaviour of optimal policy for different replacement scenarios. For example the first scenario is to replace the first subfleet numbered by equipment 1 within the planning horizon and the second scenario is to replace the second subfleet numbered by equipment 2 within the planning horizon and these scenarios are illustrated in Figures 5.1 and 5.2. We number the new subfleet by equipment 3.

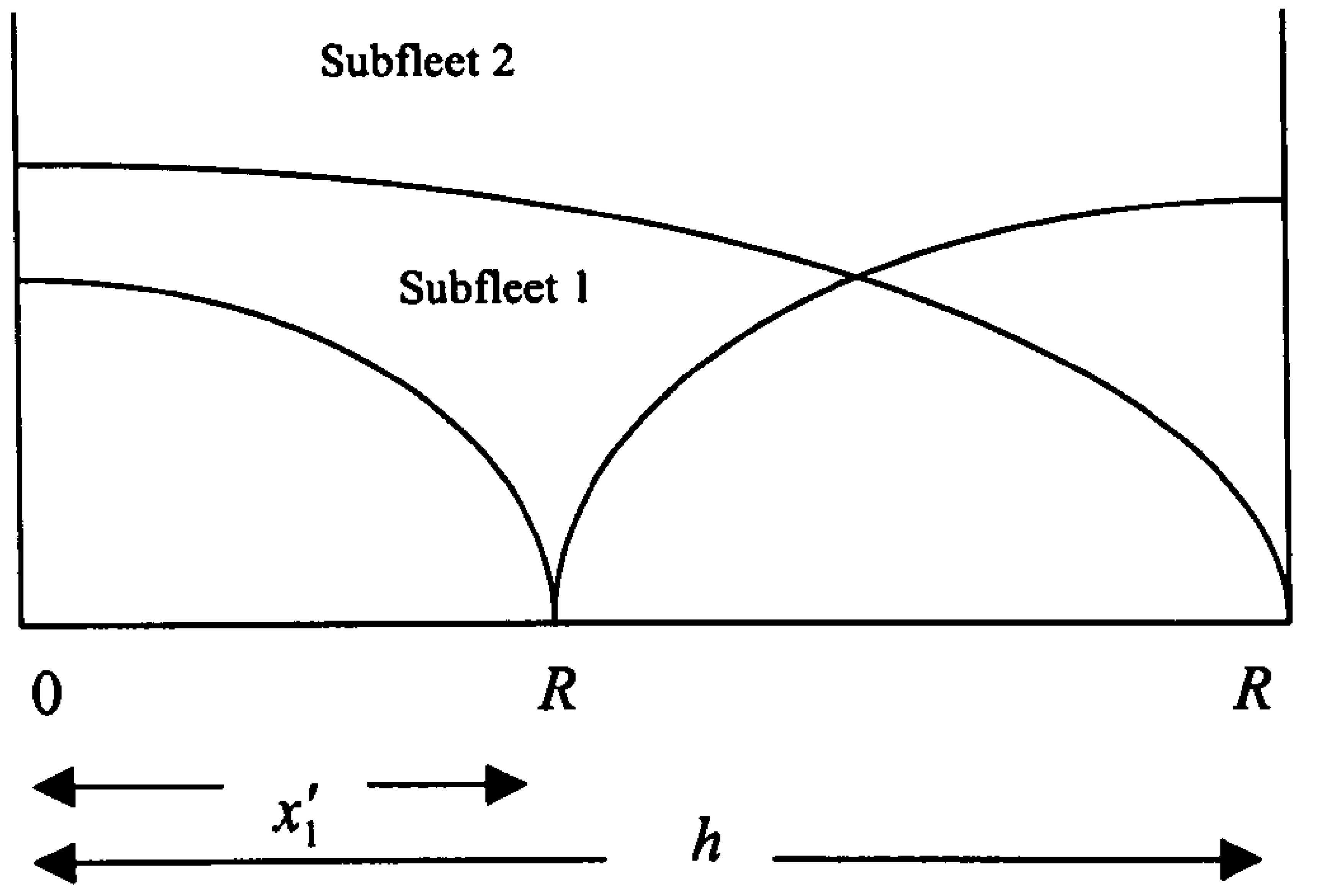


Figure 5.1. Replacement scenario for replacing subfleet 1 within the planning horizon.

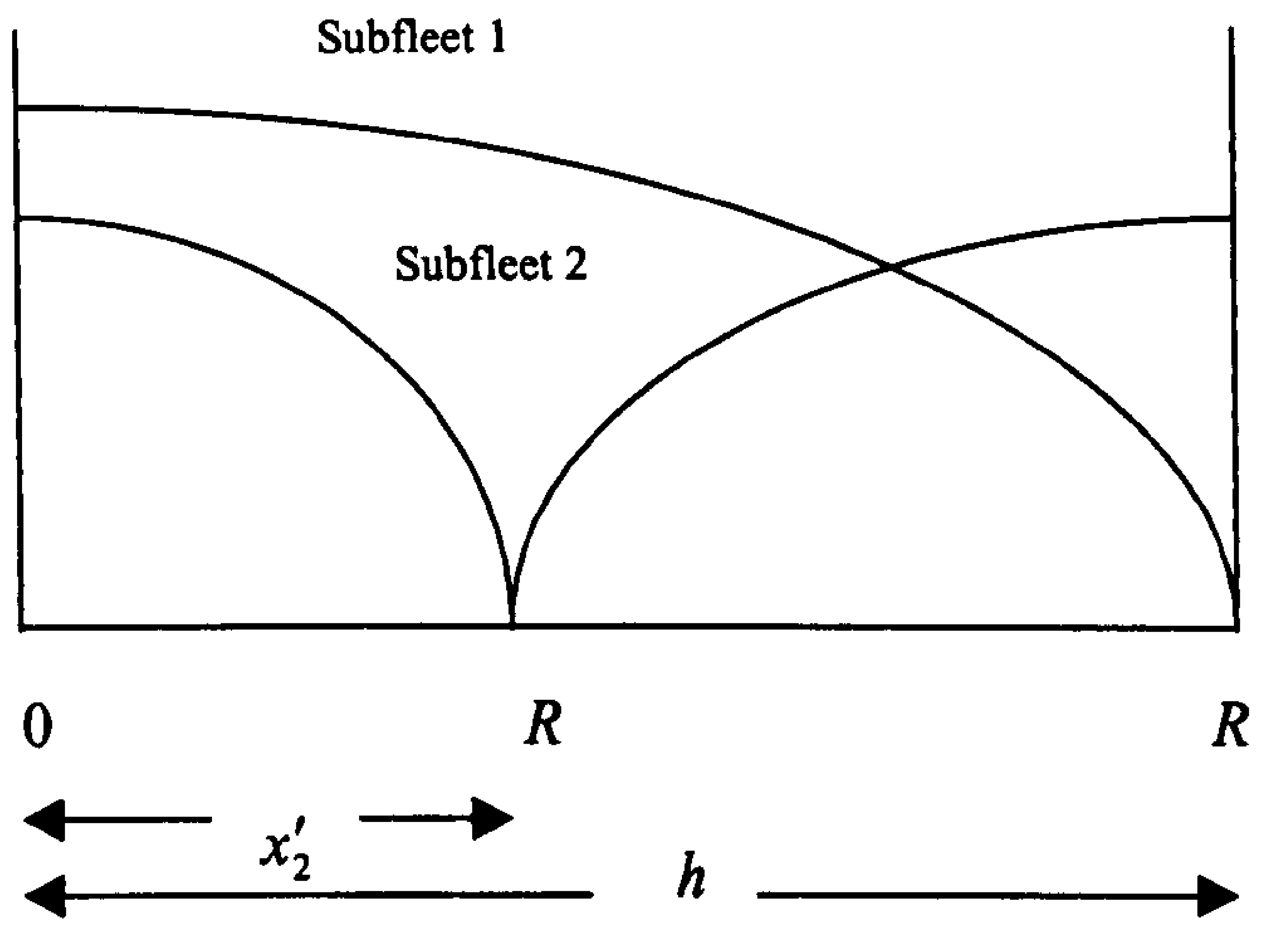


Figure 5.2. Replacement scenario for replacing subfleet 2 within the planning horizon.

We are thus concerned with comparing the cost functions $C_1(x)$ and $C_2(x)$. The aim is to determine which subfleet/equipment should be replaced first. We would choose the subfleet/equipment which gives the minimum cost according to the model parameters. The results are presented in Tables 5.1-5.12 for various values of the control variable h (h =10,15) and maintenance cost per unit time parameters α_1 (α_1=20,30,40), α_2(α_2=20,25,30), α_3(α_3=20,30,40) and current ages of the equipment τ_1(τ_1=0,2,4,6,8) and τ_2(τ_2=0,2,4,6,8) respectively.

Tables 5.1-5.4 show that for $\alpha_1 = 40$, $\alpha_2 = 30$ and $\alpha_3 = 20$ there is a difference between the results when $\tau_1 > \tau_2$ and the results when $\tau_1 < \tau_2$. This difference appears because of the great influence of the current ages of equipment 1 and equipment 2 on the behaviour of optimal policy. Thus, when $\tau_1 < \tau_2$ it is optimal to replace subfleet 2 first but it is optimal to replace subfleet 1 first when $\tau_1 > \tau_2$. The conclusion of these results is that when $\tau_1 < \tau_2$ the maintenance cost per unit time of equipment 2 becomes expensive gradually so that it is optimal to replace it first when the cost $C_2^*(x_2^*)$ is minimum compared with the cost $C_1^*(x_1^*)$ of replacing equipment 1. When $\tau_1 > \tau_2$, obviously the cost per unit time of equipment 1 becomes very expensive so that it is optimal to replace it first.

Tables 5.5-5.8 show that when $\alpha_1 = 20$, $\alpha_2 = 30$ and $\alpha_3 = 40$ it is optimal to replace equipment 1 at the end of the horizon when $\tau_1 = 0$, $\tau_2 = 2$ and $\tau_1 = 2$, $\tau_2 = 0$ because equipment 1 is still cheap so that it is optimal to keep (operate) it as long as possible. On the other hand, the policy changes for the other values of τ_1 and τ_2 leading to replace equipment 2 first when $\tau_1 < \tau_2$. (the maintenance cost per unit time of equipment 2 becomes expensive gradually) and replace equipment 1 first when $\tau_1 > \tau_2$ (the maintenance cost per unit time of equipment 1 becomes expensive gradually).

Tables 5.9-5.12 show that although $\alpha_1 (\alpha_1 = 20)$ is close to α_2 $(\alpha_2 = 25)$ the optimal policy is the same as the optimal policy obtained from Tables 5.1-5.4. These results show that the optimal policy is influenced by the conditions $\tau_1 < \tau_2$ and $\tau_1 > \tau_2$ leading to replace the more expensive equipment first.

Table 5.1. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 40, \alpha_2 = 30, \alpha_3 = 20, \beta = 0.5 \& \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	2133.0	5	1856.5	4	2019.9	4	Subfleet 2
2	4	2503.1	4	2221.3	3	3220.7	3	Subfleet 2
4	6	2841.3	3	2556.9	2	3771.1	2	Subfleet 2
6	8	3169.0	2	2876.2	2	3801.8	2	Subfleet 2

Table 5.2. The results of replacing the two subfleets over a planning horizon $h = 15, \alpha_1 = 40, \alpha_2 = 30, \alpha_3 = 20, \beta = 0.5 \& \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	4420.3	6	3730.4	4	5495.1	4	Subfleet 2
2	4	5212.3	4	4544.4	3	6925.0	3	Subfleet 2
4	6	5960.2	3	5287.1	3	8292.3	3	Subfleet 2
6	8	6680.0	3	6011.1	2	9637.8	2	Subfleet 2

Table 5.3. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 40, \alpha_2 = 30, \alpha_3 = 20, \beta = 0.5 \,\&\, \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	1856.5	4	2133.0	4	2019.8	4	Subfleet 1
4	2	2221.3	3	2503.1	3	2632.5	3	Subfleet 1
6	4	2556.9	2	2841.3	2	3220.7	2	Subfleet 1
8	6	2876.2	2	3169.0	2	3801.8	2	Subfleet 1

Table 5.4. The results of replacing the two subfleets over a planning horizon $h = 15, \alpha_1 = 40, \alpha_2 = 30, \alpha_3 = 20, \beta = 0.5 \,\&\, \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	3730.4	4	4420.3	6	5443.1	4	Subfleet 1
4	2	4544.4	3	5212.3	4	6876.5	3	Subfleet 1
6	4	5287.1	3	5960.2	3	8245.0	3	Subfleet 1
8	6	6011.1	2	6680.0	3	9591.1	2	Subfleet 1

Table 5.5. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 20, \alpha_2 = 30, \alpha_3 = 40, \beta = 0.5 \,\&\, \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	2257.9	6	2029.0	5	2019.8	10	Once at the end of h
2	4	2675.5	5	2435.1	4	2632.5	4	Subfleet 2
4	6	3055.1	4	2796.2	4	3220.7	4	Subfleet 2
6	8	3408.2	4	3142.4	3	3801.8	3	Subfleet 2

Table 5.6. The results of replacing the two subfleets over a planning horizon $h = 15, \alpha_1 = 20, \alpha_2 = 30, \alpha_3 = 40, \beta = 0.5 \& \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	4748.4	7	4131.3	6	5495.1	6	Subfleet 2
2	4	5613.3	6	5000.3	5	6925.0	5	Subfleet 2
4	6	6416.2	5	5791.6	5	8292.3	5	Subfleet 2
6	8	7184.5	5	6544.3	4	9637.8	4	Subfleet 2

Table 5.7. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 20, \alpha_2 = 30, \alpha_3 = 40, \beta = 0.5 \& \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	2029.0	5	2257.9	6	1985.9	10	Once at the end of h
4	2	2435.1	4	2675.5	5	2600.8	4	Subfleet 1
6	4	2796.2	3	3055.1	4	3189.7	4	Subfleet 1
8	6	3142.4	2	3408.2	4	3771.1	3	Subfleet 1

Table 5.8. The results of replacing the two subfleets over a planning horizon $h = 15, \alpha_1 = 20, \alpha_2 = 30, \alpha_3 = 40, \beta = 0.5 \& \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	4131.3	6	4748.4	7	5443.1	6	Subfleet 1
4	2	5000.3	5	5613.3	6	6876.5	5	Subfleet 1
6	4	5791.6	5	6416.2	5	8245.0	5	Subfleet 1
8	6	6544.3	4	7184.5	5	9591.1	4	Subfleet 1

Table 5.9. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 20, \alpha_2 = 25, \alpha_3 = 30, \beta = 0.5 \& \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	2204.6	6	1954.5	4	2019.9	4	Subfleet 2
2	4	2601.1	4	2337.2	4	2632.5	4	Subfleet 2
4	6	2957.1	4	2684.6	3	3220.7	3	Subfleet 2
6	8	3996.7	3	3018.9	3	3801.8	3	Subfleet 2

Table 5.10. The results of replacing the two subfleets over a planning horizon $h = 15, \alpha_1 = 20, \alpha_2 = 25, \alpha_3 = 30, \beta = 0.5 \& \tau_1 < \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
0	2	4597.5	7	3946.6	5	5495.1	5	Subfleet 2
2	4	5428.5	5	4788.6	4	6925.0	4	Subfleet 2
4	6	6204.5	4	5552.6	4	8292.3	4	Subfleet 2
6	8	6945.5	4	6291.6	3	9637.8	3	Subfleet 2

Table 5.11. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 20, \alpha_2 = 25, \alpha_3 = 30, \beta = 0.5 \& \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	1954.5	4	2204.6	6	1985.9	4	Subfleet 1
4	2	2337.2	4	2601.1	4	2600.8	4	Subfleet 1
6	4	2684.6	3	2957.1	3	3189.7	3	Subfleet 1
8	6	3018.9	3	3996.7	3	3771.1	3	Subfleet 1

Table 5.12. The results of replacing the two subfleets over a planning horizon $h = 10, \alpha_1 = 20, \alpha_2 = 25, \alpha_3 = 30, \beta = 0.5 \; \& \; \tau_1 > \tau_2$.

τ_1	τ_2	C_1	x_1^*	C_2	x_2^*	$C(h)$	x^*	**First replacement**
2	0	3946.6	5	4597.5	7	5443.1	5	Subfleet 1
4	2	4788.6	4	5428.5	5	6876.5	4	Subfleet 1
6	4	5552.6	4	6204.5	4	8245.0	4	Subfleet 1
8	6	6291.6	3	6945.5	4	9591.1	3	Subfleet 1

5.2.3. Influence of cost parameters on number of replacements

In this section we refer to inequality (5.6) which describes the number of replacements according to the cost parameters. Thus, inequality (5.6) represents the values of h (the length of the planning horizon) over which there are two replacements and there is only one replacement for the values of h satisfying inequality (5.7). Assuming given cost parameters and using the two inequalities mentioned above, the number of replacements can be determined.

For $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 20, \beta = 0.5, R = 450$, and t measured in years and equipment 1 would be replaced first, Table 5.13 shows the results of replacing equipment 1 for different values of τ_1 (the current age of equipment 1). For example Table 5.13 shows that at $\tau_1 = 0$ it is optimal to replace twice if $h > 13.7$ otherwise replace once only at the end of the horizon. Also the table shows that for $h > 13.7$, say $h = 14$, the decision will be replace within the horizon at $x = 4$ and replace once again after 10 years. For $\tau_1 = 2$ years it is optimal to replace twice if $h > 11.7$ otherwise replace once only at the end of the horizon. Therefore, for $h > 11.7$, say $h = 12$, the decision will be replace within the horizon at $x = 2$ and replace once again after 10 years.

Obviously, these results show the effect of τ_1 (the current age of equipment 1) on the value of h to be chosen and the value of x^*(the first replacement time).

For $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 30, \beta = 0.5, R = 450$, and t measured in years and equipment 1 would be replaced first, Table 5.14 shows the results of replacing equipment 1 for different values of τ_1 (the current age of equipment 1). For example Table 5.14 shows that at $\tau_1 = 0$ it is optimal to replace twice if $h > 18.1$ otherwise replace once only at the end of the horizon. Also the table shows that for $h > 18.1$, say $h = 19$, the decision will be replace within the horizon at $x = 9.5$ ($x = h/2$ as in the case of like with like replacement when $\tau = 0$; section 4.3.2) and replace once again after 9.5 years. For $\tau_1 = 2$ years it is optimal to replace twice if $h > 16.1$ otherwise replace once only at the end of the horizon. Therefore, for $h > 16.1$, say $h = 17$, the decision will be replace within the horizon at $x = 7.5$ ($x = (h - \tau)/2$ as in the case of like with like replacement when $\tau \neq 0$; section 4.3.2) and replace once again after 9.5 years.

Again, the results show how τ_1 (the current age of equipment 1) affects the value of h and the value of x^*.

For $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 40, \beta = 0.5, R = 450$, and t measured in years, Table 5.15 shows the results of replacing equipment 1 for different values of τ_1. For example Table 5.15 shows that at $\tau_1 = 0$ it is optimal to replace twice if $h > 23.3$ otherwise replace once only at the end of the horizon. Also the table shows that for $h > 23.3$, say $h = 24$, the decision will be replace within the horizon at $x = 15.4$ and replace once again after 8.6 years. For $\tau_1 = 2$ years it is optimal to replace twice if $h > 21.3$ otherwise replace once only at the end of the horizon. Therefore, for $h > 21.3$, say $h = 22$, the decision will be replace within the horizon at $x = 13.4$ and replace once again after 8.6 years.

The case of different βs is presented in chapter 4 for studying the behaviour of optimal policy for simple fleet. It is not straightforward to obtain theoretical results in two subfleets case for different βs.

Table 5.13. The number of replacements related to the length of the planning horizon affected by cost parameters for replacing equipment 1; $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 20, \beta = 0.5, R =$ M$450K.

τ_1	x^*	Two replacements	One replacement
0	4	$h > 13.7$	$h \le 13.7$
2	2	$h > 11.7$	$h \le 11.7$
4	0	$h > 9.7$	$h \le 9.7$
6	0	$h > 7.7$	$h \le 7.7$

Table 5.14. The number of replacements related to the length of the planning horizon affected by cost parameters for replacing equipment 1; $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 30, \beta = 0.5, R =$ M$450K.

τ_1	x^*	Two replacements	One replacement
0	9.5	$h > 18.1$	$h \le 18.1$
2	7.5	$h > 16.1$	$h \le 16.1$
4	5.5	$h > 14.1$	$h \le 14.1$
6	3.5	$h > 12.1$	$h \le 12.1$

Table 5.15. The number of replacements related to the length of the planning horizon affected by cost parameters for replacing equipment 1; $\alpha_1 = 30, \alpha_2 = 20, \alpha_3 = 40, \beta = 0.5, R =$ M\$450K .

τ_1	x^*	Two replacements	One replacement
0	15.4	h>23.3	$h \le 23.3$
2	13.4	h>21.3	$h \le 21.3$
4	11.4	h>19.3	$h \le 19.3$
6	9.4	h>17.3	$h \le 17.3$

5.3. Many subfleets problem with up to two replacements within the planning horizon

In the case of many subfleets with up to two replacements within the horizon the replacement scenario (see Figure 5.3) for replacing equipment i ($i = 1,.....,n$) and equipment j ($j = 1,.....,n$) with cost function $C(x_i, x_j)$ is as follows

$$C(x_i,x_j)=\begin{cases}\sum_{k=1}^{n}\int_0^h M_k(t+\tau_k)dt+R_{n+1}, & x_i=h,,x_j>h \text{ or } x_i>h,,x_j=h,\\ \sum_{k=1}^{i-1}\int_0^h M_k(t+\tau_k)dt+\int_0^{x_i}M_i(t+\tau_i)dt+\int_0^{h-x_i}M_{n+1}(t)dt+ & \\ \qquad \sum_{k=i+1}^{n}\int_0^h M_k(t+\tau_k)dt\ +R_{n+1}+R_{n+2}, & x_i<h,x_j=h,\\ \sum_{k=1}^{j-1}\int_0^h M_k(t+\tau_k)dt+\int_0^{x_j}M_j(t+\tau_j)dt+\int_0^{h-x_j}M_{n+1}(t)dt+ & \\ \qquad \sum_{k=j+1}^{n}\int_0^h M_k(t+\tau_k)dt\ +R_{n+1}+R_{n+2}, & x_i=h,x_j<h,\\ \sum_{k=1}^{i-1}\int_0^h M_k(t+\tau_k)dt+\int_0^{x_i}M_i(t+\tau_i)dt+\int_0^{h-x_i}M_{n+1}(t)dt+ & \\ \quad \sum_{k=i+1}^{j-1}M_k(t+\tau_k)dt+\int_0^{x_j}M_j(t+\tau_j)dt+\int_0^{h-x_j}M_{n+2}(t)dt+ & \\ \qquad \sum_{k=j+1}^{n}\int_0^h M_k(t+\tau_k)dt\ +R_{n+1}+R_{n+2}+R_{n+3}, & x_i<h,x_j<h,\end{cases} \tag{5.8}$$

where M_k is the maintenance cost per unit time of equipment k $(k=1,\ldots\ldots,n+2)$; R_{n+1} is the purchase cost of the new equipment $n+1$; R_{n+2} is the purchase cost of the new equipment $n+2$; R_{n+3} is the purchase cost of the new equipment $n+3$; τ_k is the current age of equipment k; x_i is the time of first replacement that is related equipment i; x_j is the time of second replacement that is related to equipment j and h is the length of the planning horizon.

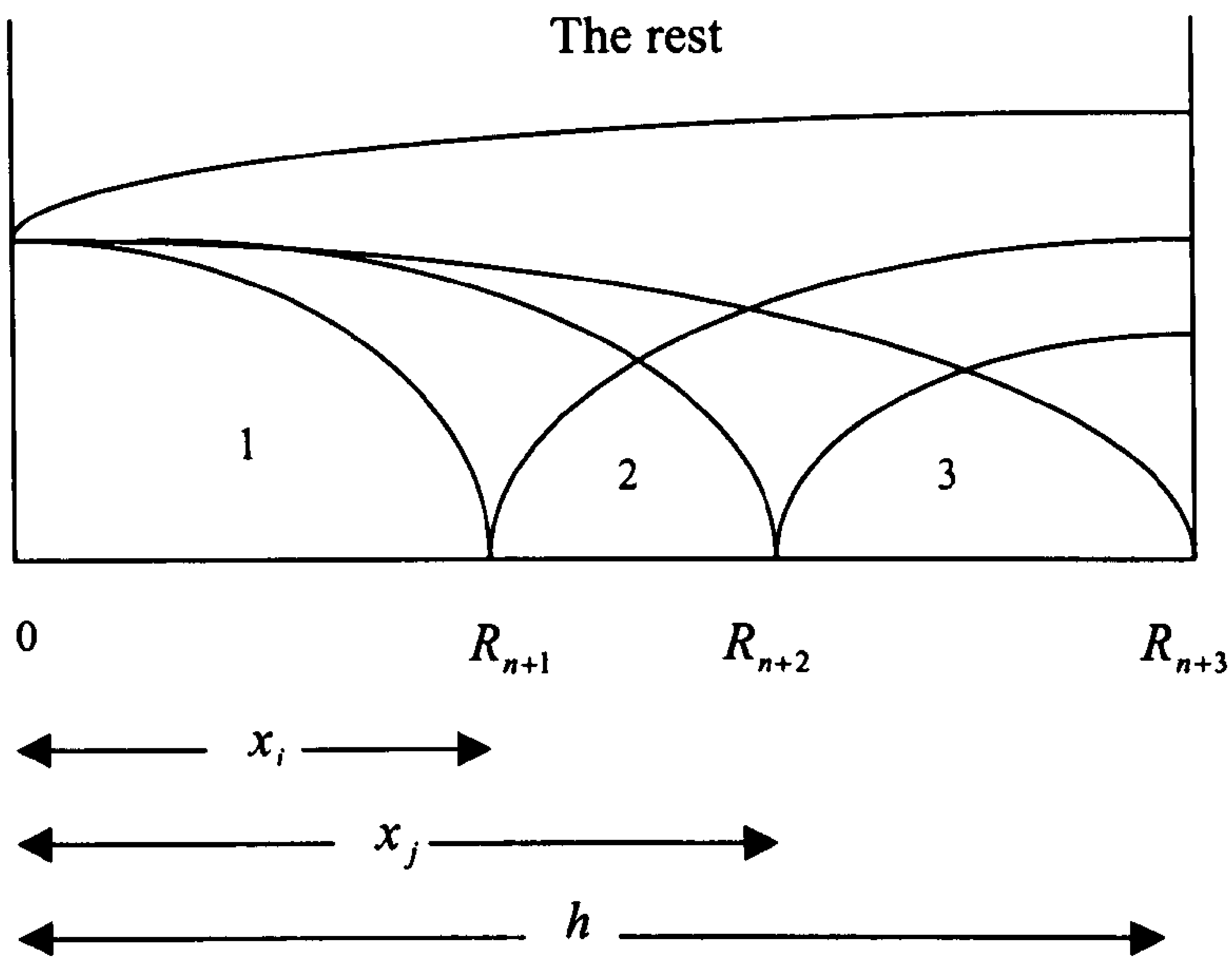


Figure 5.3. Replacement scenario for two replacements within the horizon.

Here x_i is time to first replacement, x_j is time for the second replacement and a replacement is made at the end of the planning horizon. We have that

$$C(x_i, x_j) = \begin{cases} C_0(x_i = h, x_j > h \text{ or } x_i > h, x_j = h) = C_0, \\ C_i(x_i; x_j = h) = C_i(x_i), \\ C_j(x_j;\ x_i = h) = C_j(x_j), \\ C_{ij}(x_i < h, x_j < h) = C_{ij}(x_i, x_j), \end{cases} \tag{5.9}$$

Now the cost function is a surface with discontinuities and it would be possible to find the local minima of each of the functions C_0, C_i, C_j and C_{ij}. The global minimum would then be found in a similar manner to that described in section 5.2.1.

The optimal policy has cost which is the minimum of (C_0, C_i, C_j, C_{ij}). If C_0 is the minimum of (C_0, C_i, C_j, C_{ij}) then $(x_i = h, x_j > h)$ or $(x_i > h, x_j = h)$ is the optimal policy. If C_i is the minimum of (C_0, C_i, C_j, C_{ij}) then (h, x_i^*) is the optimal policy. If C_j is the minimum of (C_0, C_i, C_j, C_{ij}) then (h, x_j^*) is the optimal policy. If C_{ij} is the minimum of (C_0, C_i, C_j, C_{ij}) then (x_i^*, x_j^*) is the optimal policy.

For these replacement scenarios, the behaviour of optimal policy becomes complex. This may be an interesting area for further study. Conditions on α_1, α_2, α_3 and h will be difficult to obtain.

5.4. Many subfleets problem with many items case

We now suppose that the mixed fleet consists of n subfleets and each subfleet contains many items. Also we assume that there are at most two replacements. If the n subfleets have sizes r_k $(k = 1,.....,n)$ then the cost of replacing subfleet i first is

$$C_i(x) = \begin{cases} \sum_{j=1}^{i-1} \int_0^h f_j(t)dt + \int_0^{x_i} f_i(t)dt + \int_0^{h-x_i} f_{n+1}(t)dt + \sum_{j=i+1}^{n} \int_0^h f_j(t)dt + \\ \qquad r_{n+1}R_{n+1} + r_{n+2}R_{n+2}, & 0 \le x_i < h, \\ \sum_{j=1}^{n} \int_0^h f_j(t)dt + r_{n+1}R_{n+1}, & x_i = h, \end{cases} \quad (5.10)$$

where

$$f_j(t) = \sum_{k=1}^{r_i} M_j(t + \tau_{jk}),$$

where f_j is the maintenance cost per unit time of subfleet j; M_j is the maintenance cost per unit time of each equipment in subfleet j; R_{n+1} is the purchase cost of the new items in subfleet $n+1$; R_{n+2} is the purchase cost of the new items in subfleet $n+2$, τ_{jk} is the current age of equipment k in subfleet

j; x_i is the time of the first replacement and h is the length of the planning horizon.

Essentially, the maintenance costs per unit time are summed over all items in the subfleet as discussed in Scarf & Bouamra (1994). We then proceed as in section 5.2.1.

Obviously, it is difficult to make a theoretical study on the behaviour of optimal policy for the case of many subfleets with many items because there are so many parameters relating to maintenance costs per unit time. A numerical study could be undertaken. A dynamic programming approach, in which the number of replacements is not restricted could be used. The three subfleets problem with many items in each subfleet was considered by Scarf & Hashem (1997) and applied to the Malaysian bus fleet (see chapter 6).

5.5. Discussion

Studying the behaviour of optimal policy here is an attempt to improve the current practice of modelling replacement and optimum number of replacements over the fixed planning horizon by considering different subfleet problems. The models considered can provide meaningful decision support for the operator for a number of decisions. For example, if the fleet consists of many subfleets then the optimal cost of replacing one of the subfleets within the horizon can be calculated indicating which subfleet should be replaced first.

We presented the many subfleets case with single items for theoretical study of the behaviour of optimal policy for different replacement scenarios. The results of theoretical study provided the value of the length of the planning horizon (control variable) over which we would expect to carry out a replacement within the planning horizon.

To choose the optimal policy among different scenarios of replacement, a numerical study has been done on the two subfleet case with two different replacement scenarios. This numerical study is concerned with comparing the different results related to different ages and maintenance cost per unit time parameters of the current equipment.

For more flexible view we described the many subfleets problem with the possibility of up to two replacements over the fixed planning horizon although here it is difficult to make progress with a description of the behaviour of optimal policy. Finally, we present the many subfleets problem with many items in each subfleet.

CHAPTER 6

CHAPTER 6

Application of a Fixed Planning Horizon Model

6.1. Background

Data are very important for work on replacement modelling to be carried out. It is assumed that data relating to maintenance cost are available and adequate for modelling purposes. Typical data are age related operating costs, such as fuel costs and failure costs. In the maintenance area it has been found that too little attention is paid to data collection and to consideration of the usefulness of models for solving real problems through model fitting and validation (Ascher and Feingold, 1984). Much attention is paid to the invention of new models rather than to the applicability of the models. Thus, the question is: "How is mathematical modelling in replacement to develop if it is to be justified by its success in tackling real problems when the information available to judge this success is sparse?" (Scarf, 1997).

In this chapter we describe a case study in which replacement modelling is carried out for a complex fleet of vehicles. Express National Berhad operates inter-city bus services in Malaysia. We have monthly data for a period of 4 years up to the time of study (early 1995) on 6 models (Mercedes, Isuzu CSA, Mitsubishi, Isuzu CJR, Cummins and MAN) of varying ages. The number of buses of each model is different with 36 Mercedes, 37 Isuzu CSA, 30 Mitsubishi, 33 Isuzu CJR, 16 Cummins and 44 MAN. The MAN is the most recently introduced model. The purpose of the study considered here was to determine which subfleet to replace first (described theoretically in chapter 5). We also illustrate the main economic role that the fixed planning horizon model plays in the decision problem. The work considered in this chapter was published in Scarf & Hashem (1997).

6.2. A fixed planning horizon model for a mixed fleet

We consider an extension of the model discussed in section 3.4 in which we have a variable number, N, of operate-sell-and-buy cycles. We also consider a discrete time model for computational simplicity.

Let the inhomogeneous fleet comprise of r sub-fleets, with the current sub-fleets indexed by $k = 1,..,r$. New replacement sub-fleets are indexed by $k = r+1,..,r+N$. A replacement schedule is a permutation of $1,\ldots,N$. For convenience we consider the schedule $1,\ldots,N$. That is replace subfleet 1 first then subfleet 2, …., then subfleet N. For a fixed planning horizon of length h, and given replacement schedule and choice of model for the replacement sub-fleets, the decision variables are then : number of cycles, $N(\geq 1)$; and time from beginning of *ith* cycle to the replacement of sub-fleet i, $L_i(i = 1,\ldots,N)$. Thus the whole fleet is operated over cycle i, which ends with the resale of sub-fleet i and purchase of sub-fleet $r+i$ $(i = 1,\ldots,N)$. Sub-fleets need not be homogeneous and the current ages of plant are denoted by τ_{ij} $(i = 1,..,r+N; j = 1,..,n_i)$, n_i is the number of buses in sub-fleet i. The fleet size may be constant $(n_i = n_{r+i} \forall\, i)$ or variable, with sub-fleet sizes n_{r+i} $(i = 1,..,N)$ given. A more complex model may even consider some subset of the sub-fleets sizes n_{r+i} $(i = 1,..,N)$ as decision variables. The model itself is a development of earlier models (J.F. De Sousa & R.C. Guimaraes, 1992; P.A. Scarf & O. Bouamra, 1995). The Scarf and Bouamra model was a variable planning horizon with two cycles; the length of each cycle is a decision variable. De Sousa & Guimaraes model was a fixed planning horizon model but they were not concerned with the replacement of a mixed fleet. The model outline is presented in Figure 6.1.

The model in chapter 5 also considers the mixed fleet problem but is somewhat simple because there is no consideration of resale values and it is not formulated with variable number of cycles, N although in number of cycles terms they are equivalent.

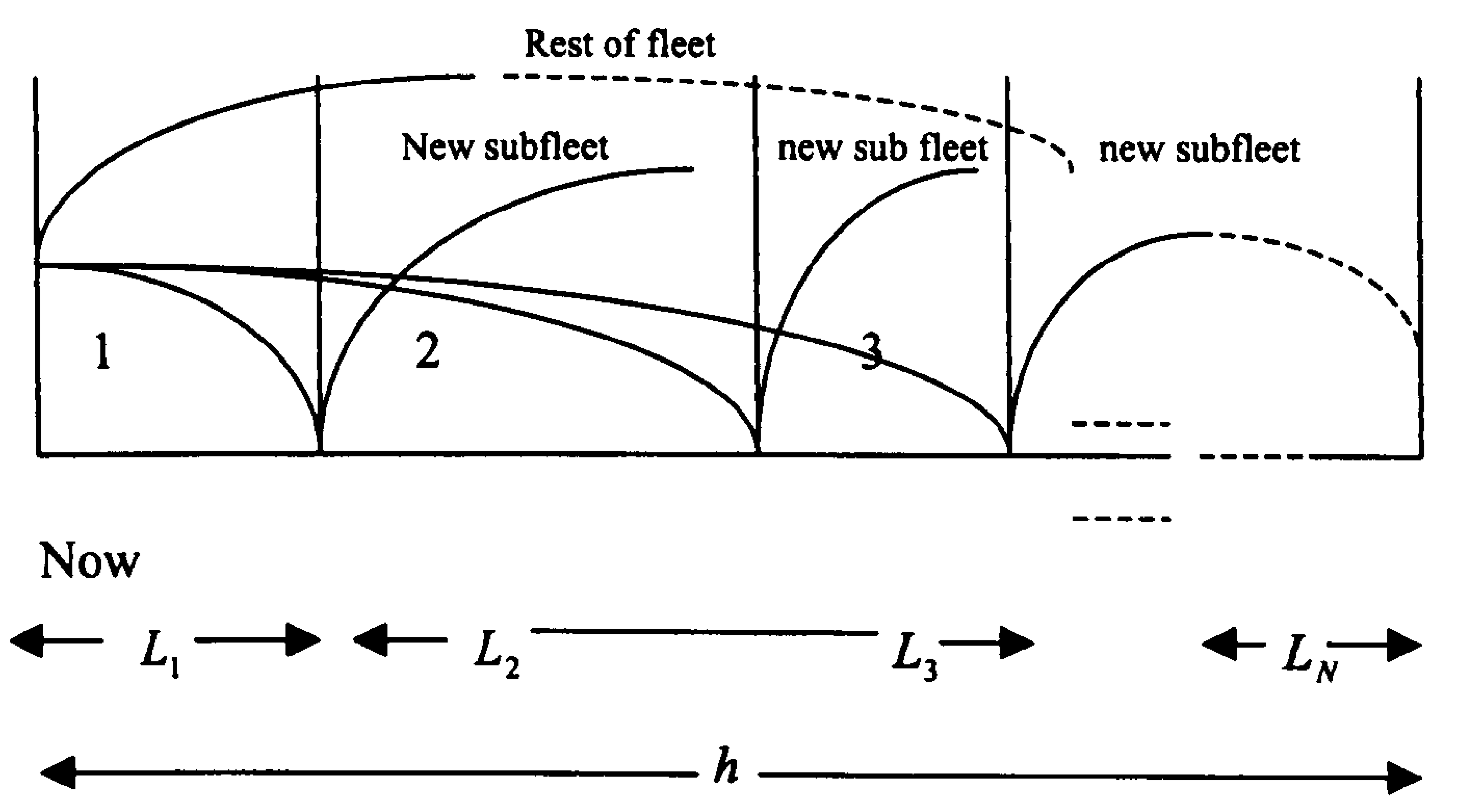


Figure 6.1. Fixed planning horizon for sub-fleet replacement.

For a given replacement schedule, the total discounted cost over the horizon h can be formulated as

$$C_{tdc}(N, L_1, .., L_N; h) = \sum_{i=1}^{N} \upsilon^{m_i} \left\{ \sum_{t=m_{i-1}+1}^{m_i} C_i(t) \upsilon^{t-m_i-1/2} + n_i R_{r+i} - S_i(m_i) \right\}, \qquad (6.1)$$

where

$$m_i = \sum_{j=0}^{i} L_j .$$

Here $C(.)$ is the age related maintenance cost per unit time of the whole fleet in cycle i; $S_i(.)$ is the age related resale of plant in sub-fleet i; and R_{r+i} is the cost of each of replacement plant in sub-fleet $r+i$ $(i = 1, .., N)$. Here υ is the discount rate. (We take $L_0 = 0$ for convenience). The costs $C_i(.)$ and $S_i(.)$ can be expressed as

$$C_i(t) = \sum_{k=i}^{r+i-1} \sum_{j=1}^{n_k} M_k(\tau_{kj} + t), \qquad (i = 1, .., N),$$

$$S_i(t) = \sum_{j=1}^{n_i} s_i(\tau_{ij} + t), \qquad (i = 1, .., N),$$

where $M_k(.)$ is the age related maintenance cost per unit time for an individual plant in sub-fleet k $(k = 1, .., r+N)$; and $s_i(.)$ is the age related resale value for individual plant in sub-fleet i. (Also, $\tau_{kj} = 0$ for $k > r$). Appropriate penalty costs (A.H. Christer & P.A. Scarf, 1994), associated with failures, may be incorporated into the operating costs which, as we mentioned earlier, included in the maintenance cost.

The objective function considered is the equivalent rent, namely

$$C_{tdc}(N, L_1, .., L_N; h) \Big/ \sum_{i=1}^{h} \upsilon^i,$$

which is minimised subject to the constraint

$$\sum_{i=1}^{N} L_i = h.$$

We denote the minimum by C^*_{rent}. Technological change is allowed for in that costs relating to proposed replacement plant for cycles $2,..,N$, may be assigned as appropriate.

The optimum replacement schedule may be obtained by minimising the objective function over all possible schedules. In practice the range of possibility for the choice of schedules would be narrowed greatly by the experience of the operator. Furthermore, as the decision maker will not have a firm value for the horizon length, the optimum policy must be "robust" to variation in h. Given that the fleet is mixed, both different replacement schedules and different planning horizon lengths will give rise to different age compositions of the fleet at the end of the horizon. Thus replacement policies need to be compared not just on the basis of cost but also on the basis of the age composition of the fleet at the end of the planning horizon. In fact this final age composition can be considered as quantifying the end-of-horizon effect. It then follows that the decision problem is strictly a multi-decision criteria one.

Non-uniform usage, particularly between sub-fleets, may be allowed for by varying the fleet size at replacements. For example, if older plant are under-utilised, a smaller number of new plant would be required to meet the demand currently placed on an older sub-fleet. This effectively reduces the replacement cost for that sub-fleet by factor which is the ratio of the utilisation of the old (current) to the new sub-fleet. Of course, other more complex methods of accounting for differing usage may be considered. Given sufficient data, maintenance costs could be quantified in terms of usage and optimum policy may be obtained given forecasts for usage of sub-fleets over the planning horizon.

The models may be extended to the case in which sub-fleets are retired as spares. The number of sub-fleets would simply increase by one at each replacement, with the costs associated with retired sub-fleet added. Predicting maintenance costs for a retired sub-fleet would be difficult however, as it is likely that no data would be available for this. Also it is assumed that equipment is bought new: in principle it is a simple matter to extend equation (6.1) to the case in which used equipment may be purchased (see Scarf, 1994).

Note that the formulation as presented allows for the possibility for a sub-fleet to be composed of a single unit of equipment. This may be appropriate if the fleet is small. The complexity of the computational problem increases rapidly as the number of sub-fleets increases.

6.3. Maintenance cost data

The maintenance cost data were available monthly for each bus. The behaviour of the data from 1990 until the end of 1994 is shown through a separate graph for each model. Also each graph contains two groups of data showing the data available until nearly 1992 (used in Scarf and Bouamra (1995)) and the data available after. Thus, Figure 6.2 illustrates the maintenance cost data for the Mercedes subfleet. It is observed that the maintenance cost of Mercedes buses was very low in 1993-1994; this is a consequence of the Mercedes sub-fleet having been retired as spares in 1995.

At the time of the study the Isuzu CSA was still in use but was in partial retirement and therefore a candidate for immediate replacement. Figure 6.3 illustrates the Isuzu CSA maintenance cost data.

Figure 6.4 illustrates the Mitsubishi maintenance costs. These indicate that Mitsubishi was very expensive and therefore a candidate for immediate replacement. The Isuzu CJR sub-fleet is relatively new and cost data are shown in Figure 6.5. Figure 6.6 illustrates that the maintenance costs for the Cummins sub-fleet increased rapidly over its early life. Figure 6.7 shows the maintenance cost data of Man. This model costing M$450K new, was introduced recently and data were limited. Figure 6.8 shows the cost data of all buses including Mercedes as a retired subfleet.

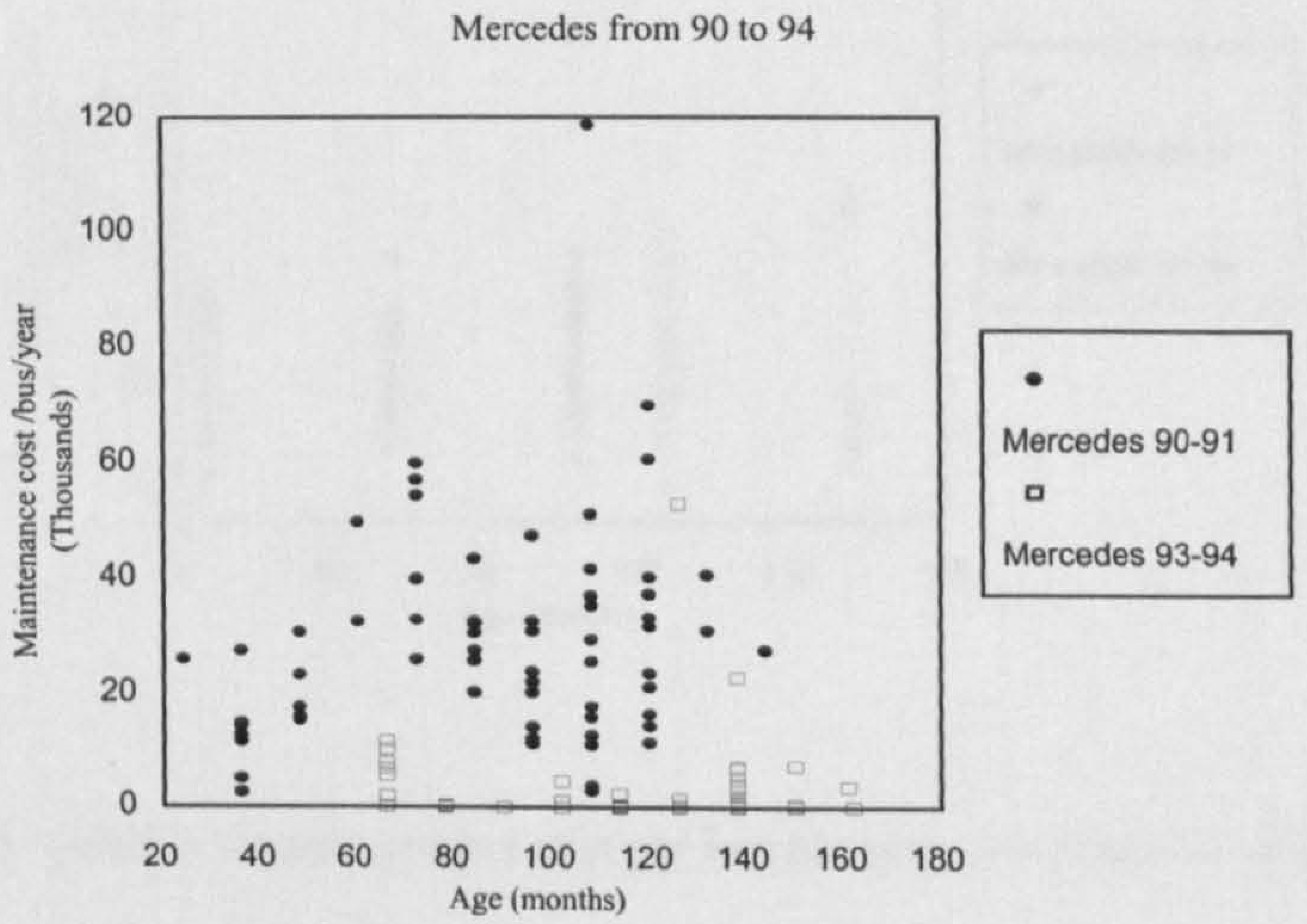


Figure 6.2. Mercedes maintenance cost per bus per year as a function of age in months.

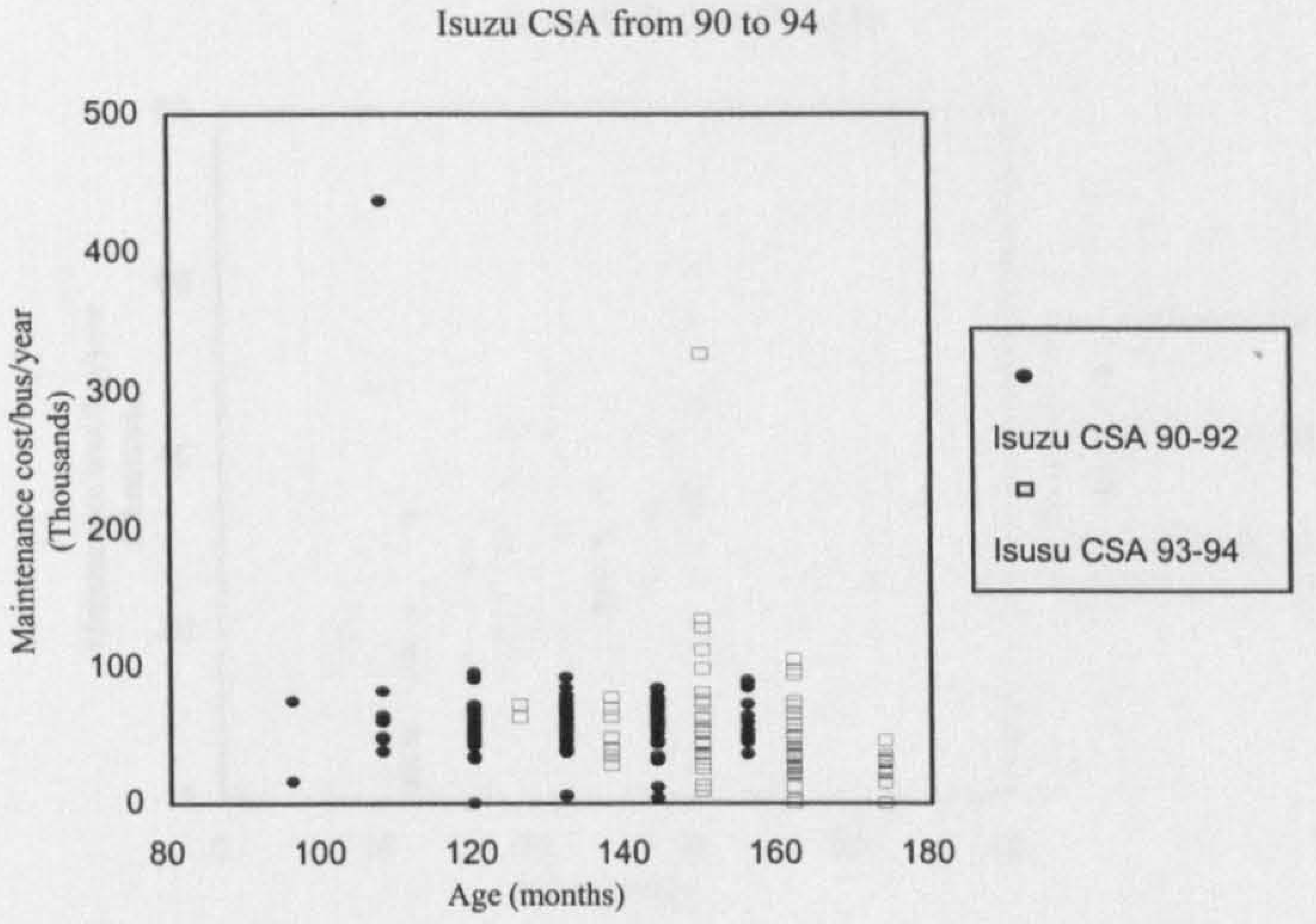


Figure 6.3. Isuzu CSA maintenance cost per bus per year as a function of age in months.

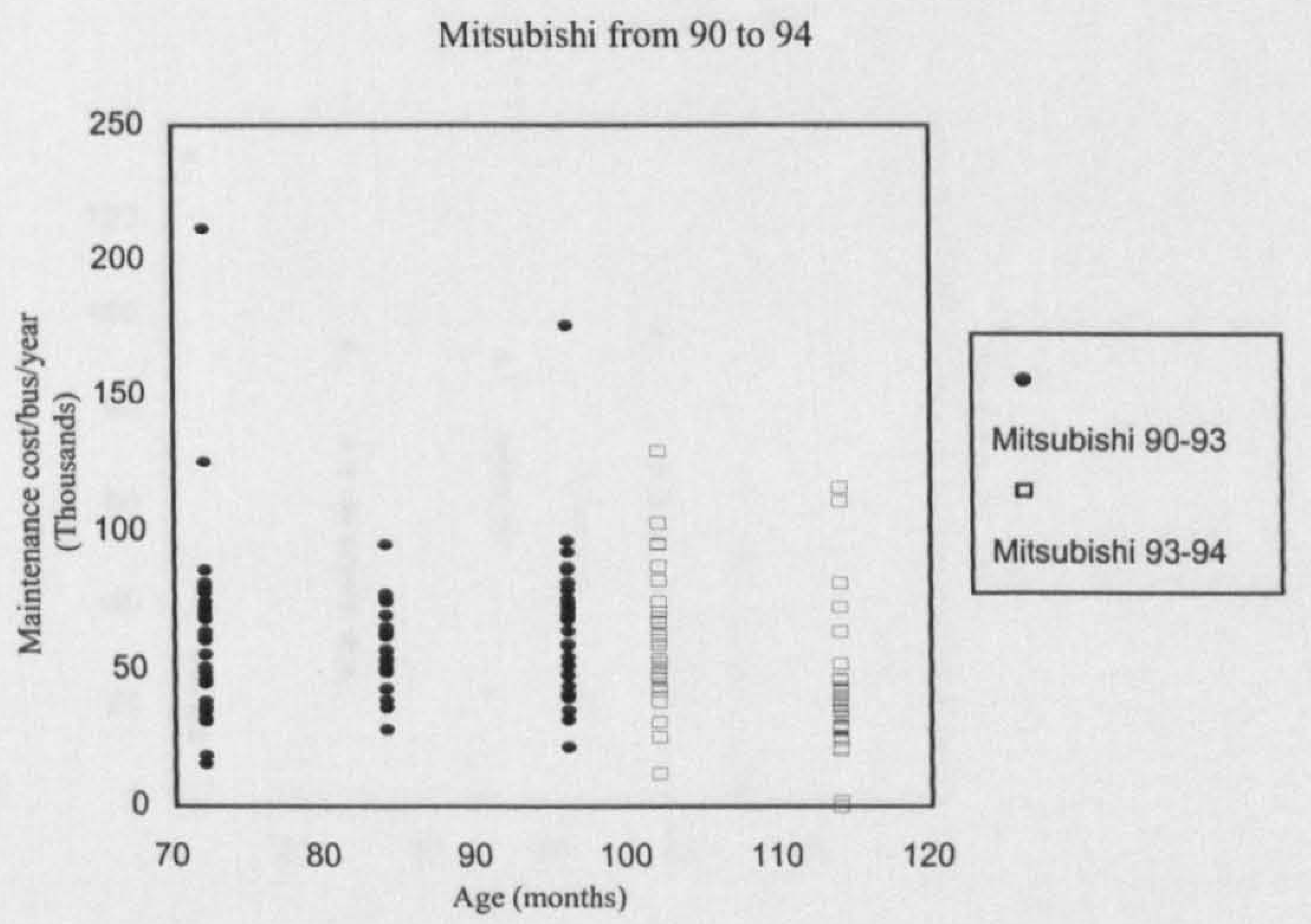


Figure 6.4. Mitsubishi maintenance cost per bus per year as a function of age in months.

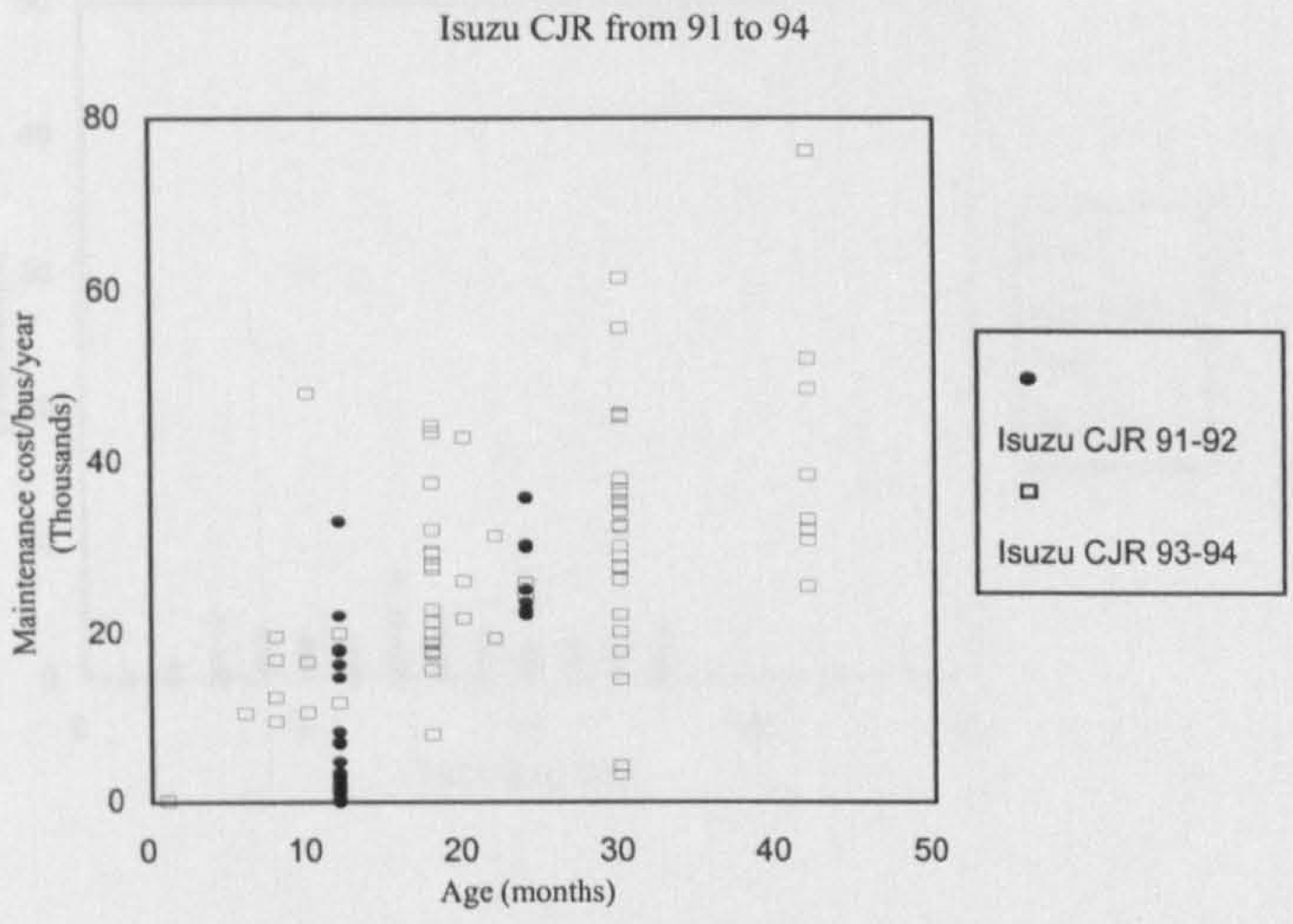


Figure 6.5. Isuzu CJR maintenance cost per bus per year as a function of age in months.

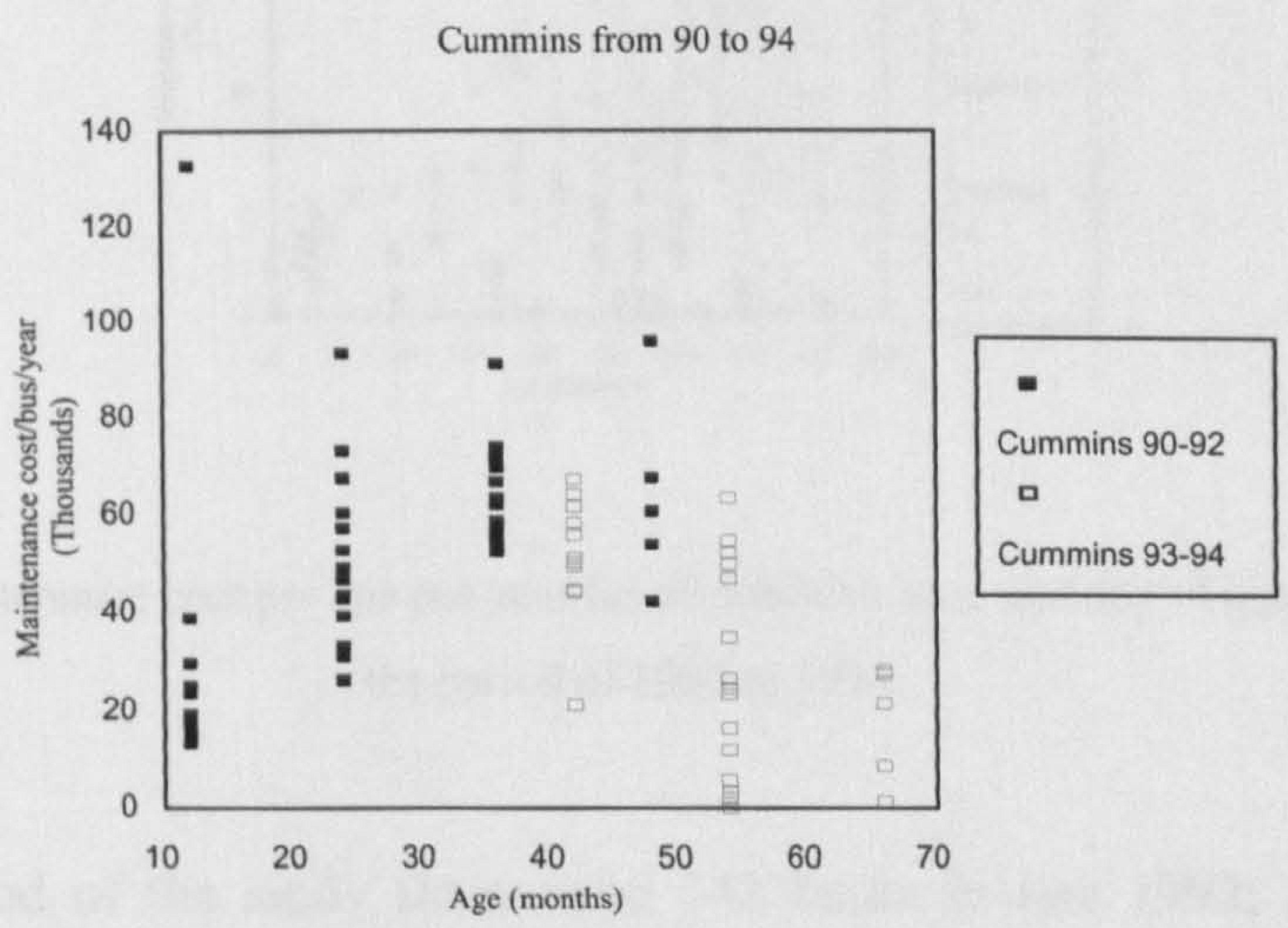


Figure 6.6. Cummins maintenance cost per bus per year as a function of age in months.

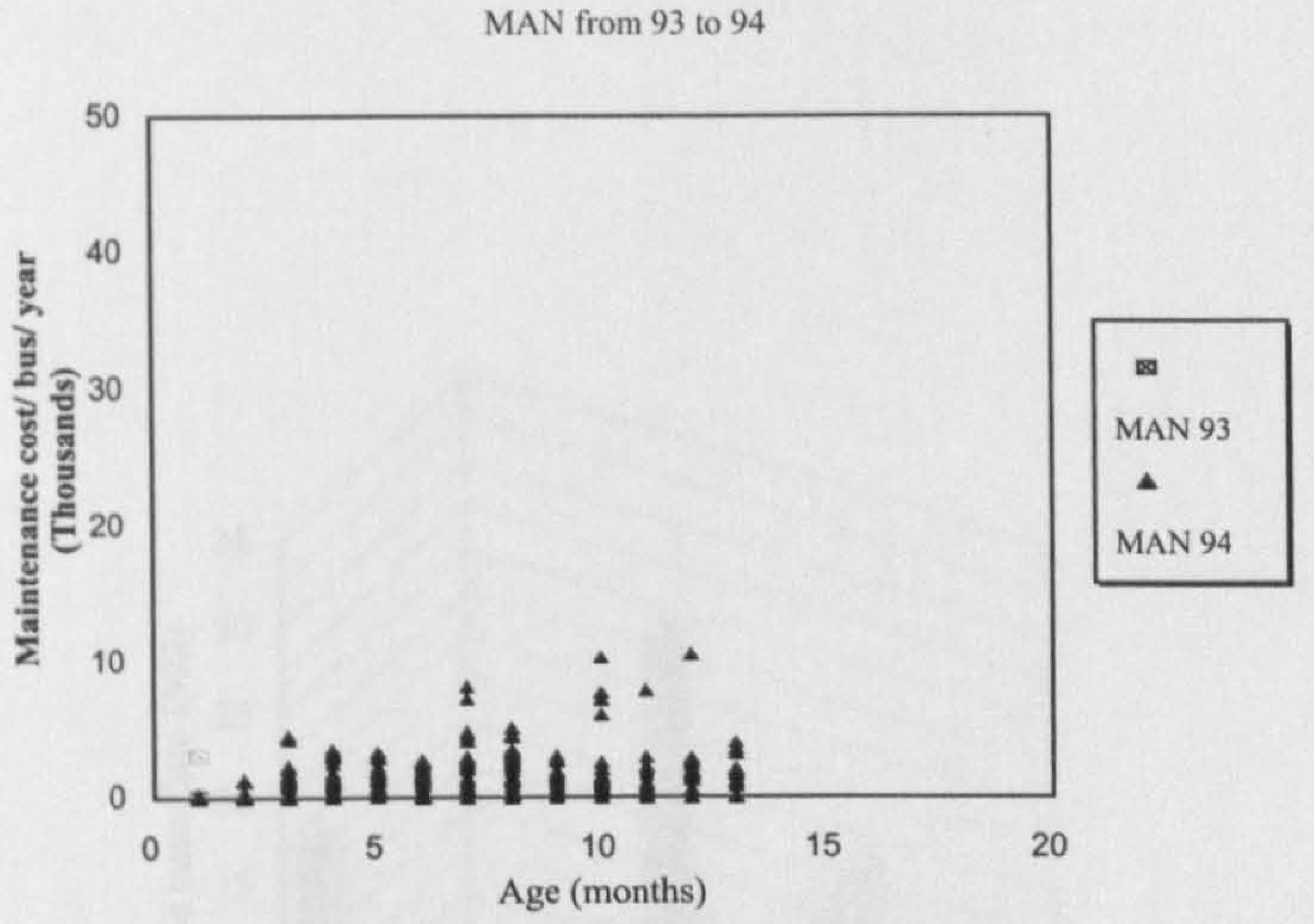


Figure 6.7. MAN maintenance cost per bus per year as a function of age in months.

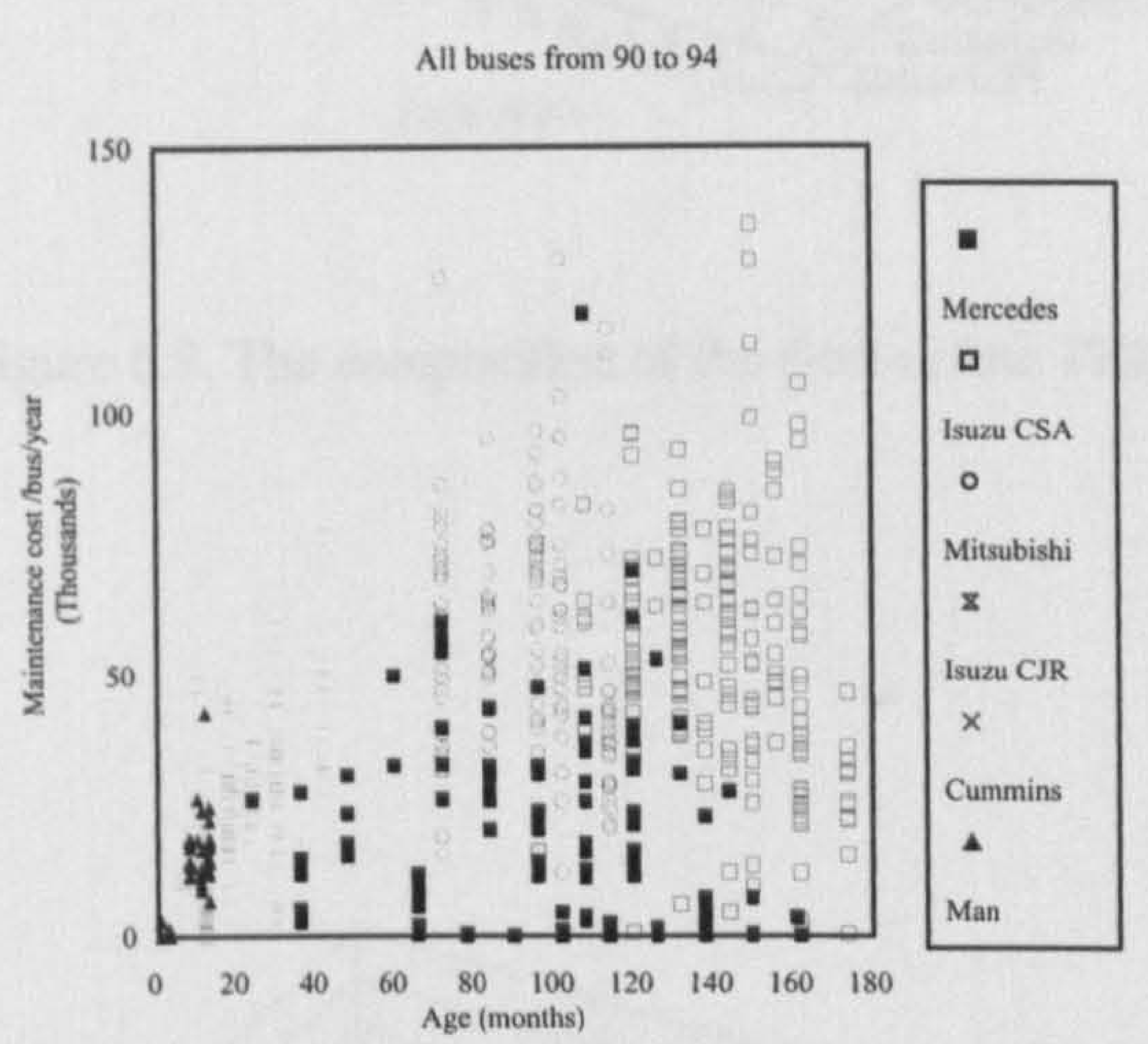


Figure 6.8. Maintenance cost per bus per year for all subfleets as a function of age in months over the period of 1990 to 1994.

Over the period of the study there were 142 buses in late 1992; as a result of replacing the Mercedes and buying MAN there were 160 buses in early 1995, and this is illustrated in Figures 6.9 and 6.10.

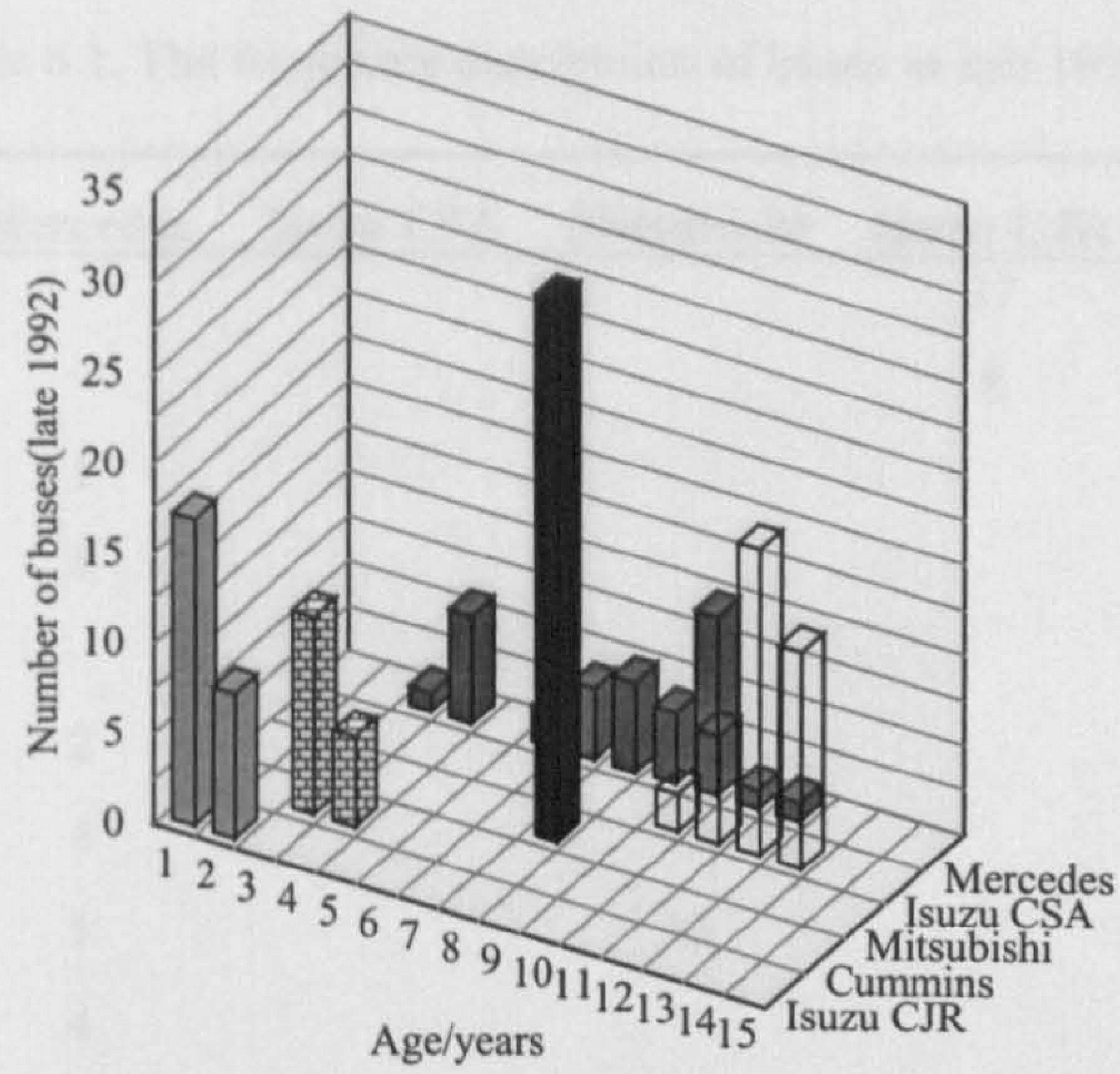


Figure 6.9. The composition of the fleet in late 1992.

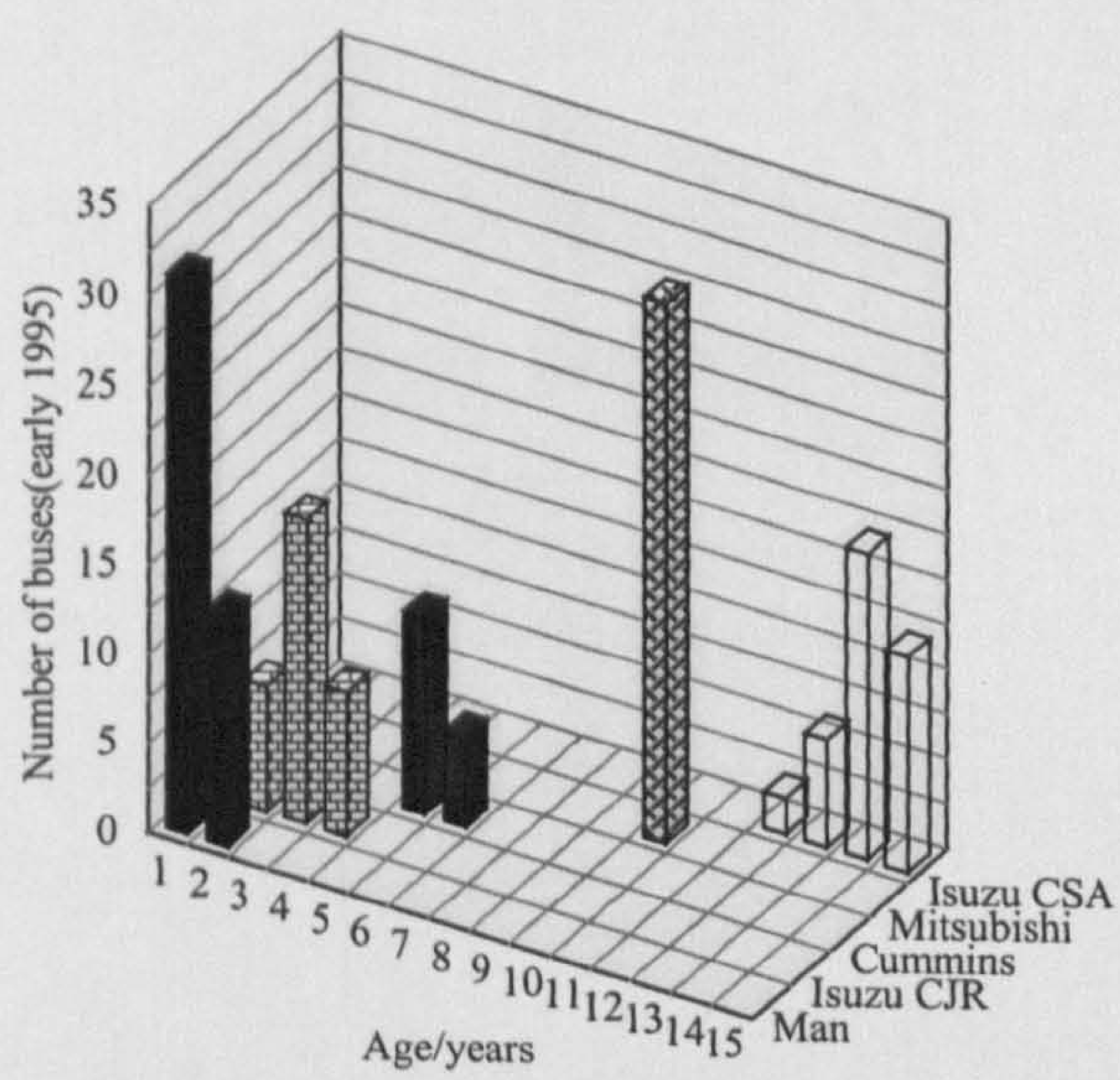


Figure 6.10. The composition of the fleet in early 1995.

The age distribution of the buses in late 1992 and early 1995 is presented in Tables 6.1and 6.2.

Table 6.1. The frequency distribution of buses in late 1992.

Age/year	Mercedes	Isuzu CSA	Mitsubishi	Isuzu CJR	Cummins
1				17	11
2				8	5
3	1				
4	6				
5					
6	2				
7	4				
8	5		30		
9	4				
10	10	2			
11	1	6			
12	1	17			
13		12			
14					
15					
Total	34	37	30	25	16

Table 6.2. The frequency distribution of buses in early 1995.

Age/year	Isuzu CSA	Mitsubishi	Isuzu CJR	Cummins	MAN
1			1		31
2			7		13
3			17		
4			8		
5				11	
6				5	
7					
8					
9					
10		30			
11					
12	2				
13	6				
14	17				
15	12				
Total	37	30	33	16	44

The purchase prices new (age zero) and resale values of the models were based on the model $S_i(\tau) = R_i \gamma \delta^{\tau}$ (Christer and Waller, 1987a). Discussion on this modelling with the company led to estimates of the parameters γ and δ of 0.613 and 0.811 respectively and these resale functions are illustrated in Figure 6.11.

6.4. Maintenance cost modelling

The model of bus earmarked to be introduced as a new sub-fleet was the MAN; made in Malaysia and costing M$450K new. Maintenance data records on a yearly basis over a five year period (Figure 6.12) were used to estimate maintenance costs for each vehicle-type. At first sight this dataset appears extensive. However, data on individual vehicle-types were not sufficient for obtaining the maintenance cost per unit time model for all vehicle-types. For example, for the MAN, only data relating to their first year of operation were available. Furthermore, for older vehicles the costs appeared to be decreasing. This could perhaps be put down to under-utilization (partial retirement) and also neglect of vehicles reaching the end of their useful life. It was therefore necessary to pool the data to obtain reasonable cost models. The fitted maintenance cost per unit time models for the Cummins, Isuzu CJR and MAN (Figure 6.12) were obtained by first fitting an overall cost model to data on vehicles up to 8 years-old, then scaling this model to the costs of the individual vehicle-types in the manner described in Christer (1988). The fitted cost models have the same β value 0.72; this simplification was introduced due to lack of objective data for individual subfleets. The maintenance costs for the older sub-fleets, the Mitsubishi and Isuzu CSA, were taken as constant.

The penalty cost for breakdown on the road was modelled as follows. Only limited breakdown data were available, and so the rate of occurrence of breakdowns was assumed to be proportional to the maintenance cost per unit time. The constant of proportionality was obtained using the known mean number of breakdowns on the road per month. Some refinement of this model is required, and we include it here purely to illustrate a potential method for quantifying penalty cost. Other age-related operating costs (e.g. fuel) were not quantified as no data were available.

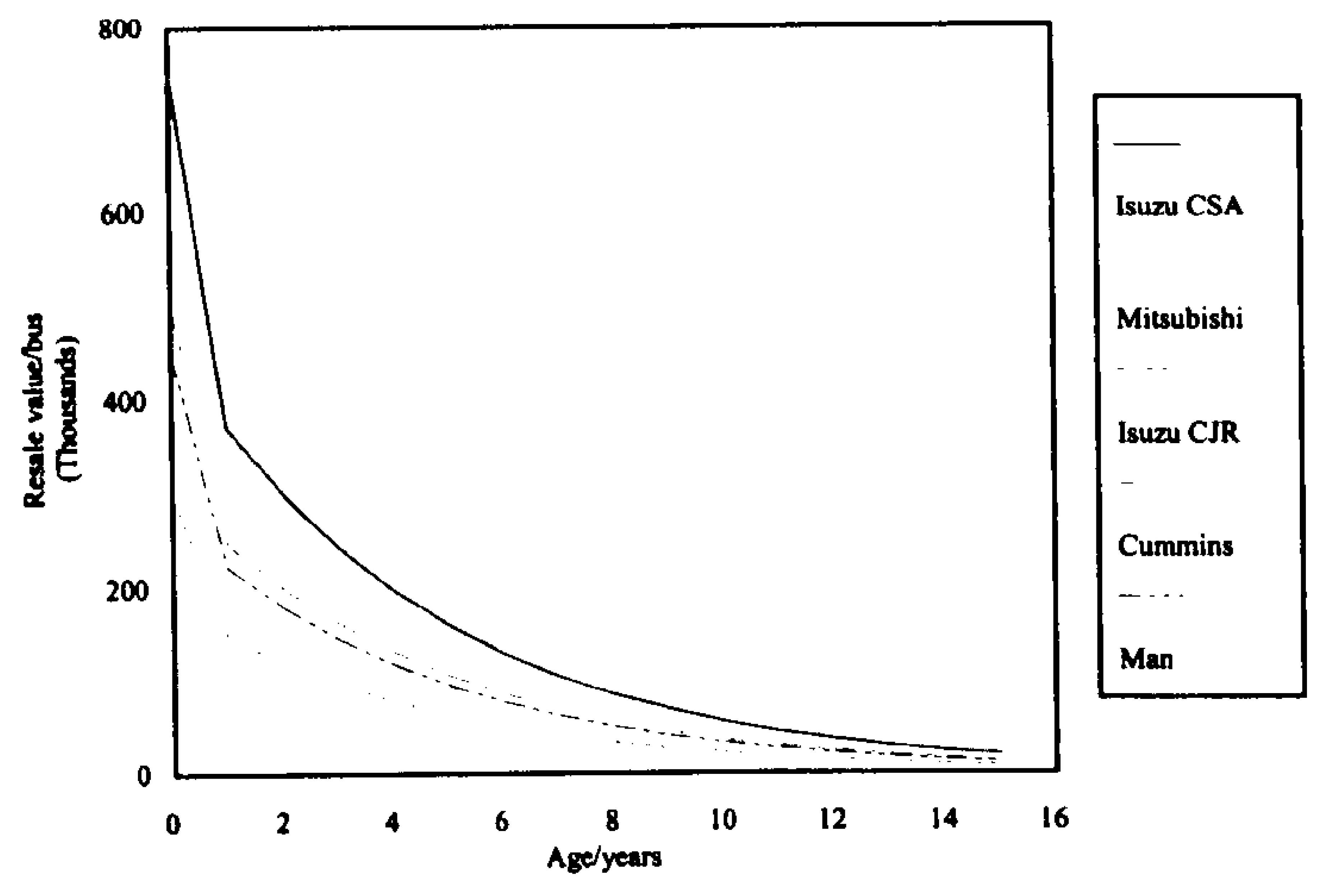


Figure 6.11. Resale values and prices new for each model of bus.

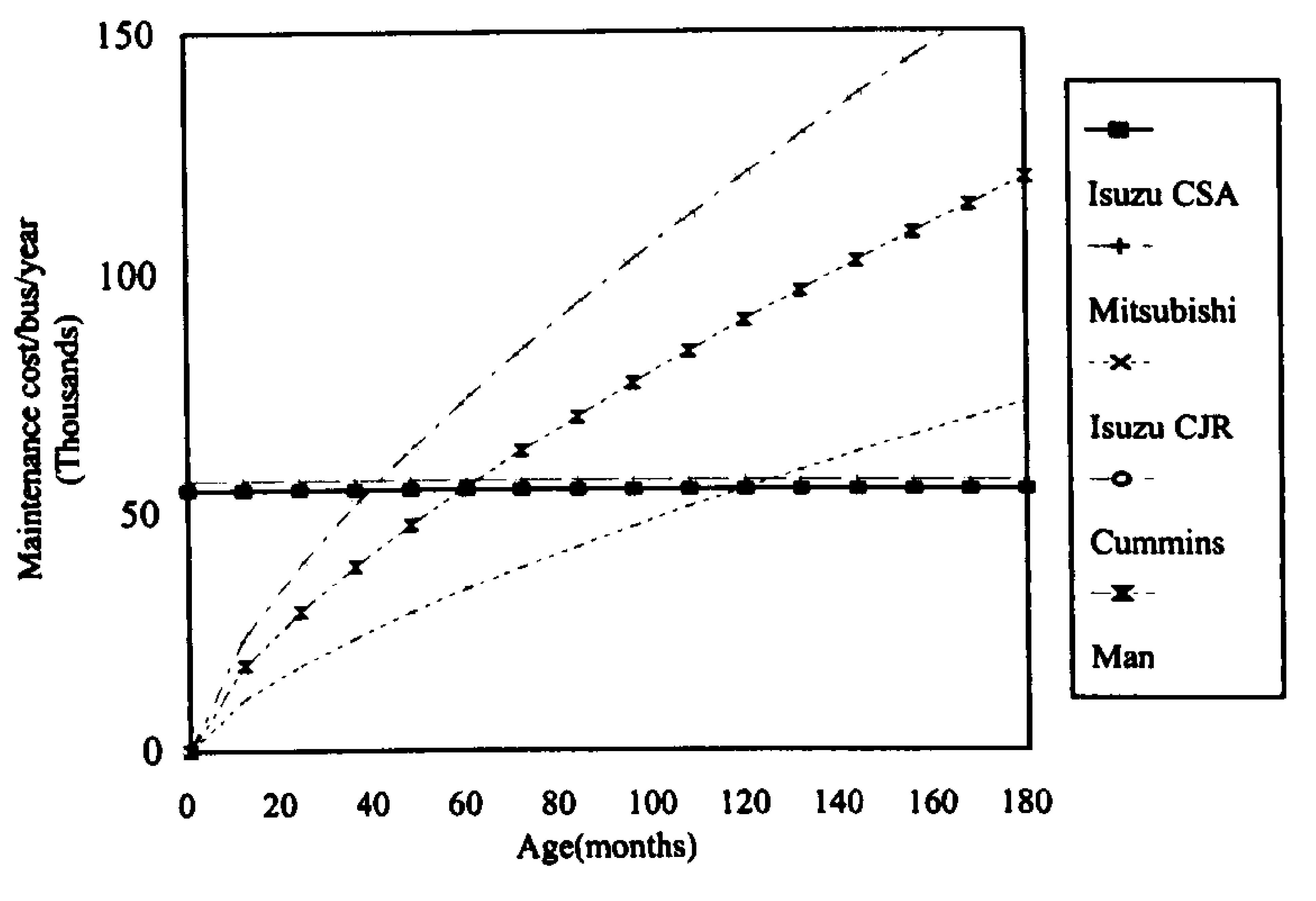


Figure 6.12. The fitted model for each sub-fleet:

Isuzu CSA, $M(t) = 55.6$; Mitsubishi, $M(t) = 57.8$; Cummins, $M(t) = 24.7t^{0.72}$;

Isuzu CJR, $M(t) = 11.1t^{0.72}$; Man, $M(t) = 18.4t^{0.72}$.

6.5. Application to a bus fleet

As mentioned above the data we have were obtained from Express National Berhad which operates inter-city bus services in Malaysia with a fleet which comprises of 160 vehicles of varying models (vehicle types). The composition of the fleet is mixed with, at the time of the study (early 1995), 5 vehicle-types of varying ages (see Figure 6.10 and Table 6.2).

It was known that the Mitsubishi and Isuzu CSA sub-fleets were in partial retirement and also candidates for immediate replacement, capital expenditure permitting. The usage of sub-fleets was unknown, but it was reasonable to suppose that the usage level for the Mitsubishi and Isuzu sub-fleet was about half that of the other newer sub-fleets. This assumption led to the optimal policy: replace the Mitsubishi and Isuzu CSA sub-fleets as soon as possible which proved uninteresting from a model validation point of view. Therefore in order to illustrate the replacement model, we consider the following sub-problem in detail: to investigate replacement policy for the fleet comprising of Cummins, Isuzu CJR and MAN, assuming a fixed fleet size (93 vehicles) and uniform usage.

6.5.1. Results for the sub-problem of interest

For the three sub-fleets problem, optimal policy is presented for each of the six replacement schedules (choice of order in which sub-fleets are replaced) in Table 6.3.

From Table 6.3 it can be seen that there is a significant variation in optimal cost and age with h, and no particular replacement schedule is everywhere optimal. Also, there is variation in cost and age with the length of the first cycle, the principal decision variable, for two particular replacement schedules. The table shows that for the optimum schedule over 15 years (180 months), Cummins-MAN-Isuzu CJR, a small increase in the time to first replacement does not markedly affect the cost or the mean age at the end of the horizon (see Figures 6.13 and 6.14). This suggests that this policy is a robust one. Table 6.4 shows the optimum policy of the schedule, Cummins-MAN-Isuzu CJR, as a function of the horizon h consisting of 4 cycles.

The effect of varying the penalty cost is presented in Table 6.5. The cost and the mean age at the end of the horizon h are affected by varying the penalty cost (see Figure 6.15). Table 6.6 considers optimal policy for a horizon model with two replacements exactly. For particular replacement schedule, that value of h which minimises the cost function represents the optimum horizon length ($L_1 + L_2$) for a two-cycle variable horizon model (see Figure 6.16).

Table 6.3. Optimum policy for each schedule for various horizon lengths, h=120, 150, 180 months; penalty cost, p = M\$2000; annual discount rate, υ=0.97. Cost of equivalent rent (M\$000s per month for whole fleet), average age of fleet at end of horizon, and optimum cycle lengths. Replacement schedules: CIM-Cummins-Isuzu CJR-MAN, etc.

Horizon (months)	Schedule	Cost/month (M\$000s)	Age (years)	L_1	L_2	L_3	L_4
120	CIM	745.77	7.0	6	114		
	CMI	763.54	9.9	120			
	ICM	816.66	8.4	120			
	IMC	816.66	8.4	120			
	MCI	782.10	5.1	24	6	90	
	MIC	838.05	6.5	42	78		
150	CIM	779.39	8.6	6	144		
	CMI	767.33	5.6	6	54	90	
	ICM	844.68	4.6	54	6	6	84
	IMC	848.07	4.4	60	6	6	78
	MCI	782.82	5.7	42	6	102	
	MIC	850.56	7.7	60	90		
180	CIM	787.88	6.0	18	72	6	84
	CMI	778.11	6.0	18	60	36	66
	ICM	840.97	5.2	72	6	6	96
	IMC	843.93	5.4	72	6	6	96
	MCI	794.58	6.7	54	6	120	
	MCI	859.93	4.4	72	6	6	96

Table 6.4. Optimum policy for schedule Cummins-MAN-Isuzu CJR as a function of horizon length, h; penalty cost, p = M\$2000; annual discount rate, υ=0.97. Cost of equivalent rent (M\$000s per month for whole fleet), average age of fleet at end of horizon, and optimum cycle lengths.

Horizon (months)	Cost/month (M\$000s)	Age (years)	L_1	L_2	L_3	L_4
60	661.19	5.8	60			
72	677.29	6.6	72			
84	696.83	7.5	84			
96	718.22	8.3	96			
108	740.62	9.1	108			
120	763.54	9.9	120			
132	765.09	5.1	6	42	84	
144	766.02	5.5	6	48	90	
156	769.21	5.9	6	54	96	
168	773.94	6.3	12	54	102	
180	778.11	6.0	18	60	36	66

Table 6.5. Optimum policy for schedule Cummins-MAN-Isuzu CJR for various penalty costs, p (M\$000s); horizon length h=120, 150, 180; annual discount rate, υ=0.97. Cost of equivalent rent (M\$000s per month for whole fleet), average age of fleet at end of horizon, and optimum cycle lengths.

Penalty cost	Horizon (months)	Cost/month (M\$000s)	Age (years)	L_1	L_2	L_3	L_4
0.0	120	625.43	9.9	120			
	150	665.17	5.6	6	54	90	
	180	668.27	6.3	18	60	102	
1.0	120	694.48	9.9	120			
	150	716.25	5.6	6	54	90	
	180	724.06	6.0	18	60	102	
2.0	120	763.54	9.9	120			
	150	767.33	5.6	6	54	90	
	180	778.11	6.0	18	60	36	66
5.0	120	905.31	4.7	6	36	78	
	150	916.68	5.1	6	54	36	54
	180	933.16	6.0	18	60	36	66

Table 6.6. Two-cycle model - optimum policy for three " best" schedules (Cummins-Isuzu CJR-MAN, Cummins-MAN-Isuzu CJR, MAN-Cummins-Isuzu CJR) for various horizon lengths, $h = 120, 150, 180$ months; penalty cost, p = M\$2000; annual discount rate, $\upsilon = 0.97$. Cost of equivalent rent (M\$000s per month for whole fleet), average age of fleet at end of horizon, and optimum cycle lengths.

Horizon (months)	Schedule	Cost/month (M\$000s)	Age (years)	L_1	L_2
120	CIM	745.77	7.0	6	114
	CMI	775.83	6.2	6	114
	MCI	784.92	7.5	42	78
150	CIM	779.39	8.6	6	144
	CMI	804.14	7.7	6	144
	MCI	809.27	9.0	60	90
180	CIM	818.73	10.0	18	162
	CMI	839.32	9.0	18	162
	MCI	839.32	10.5	78	102

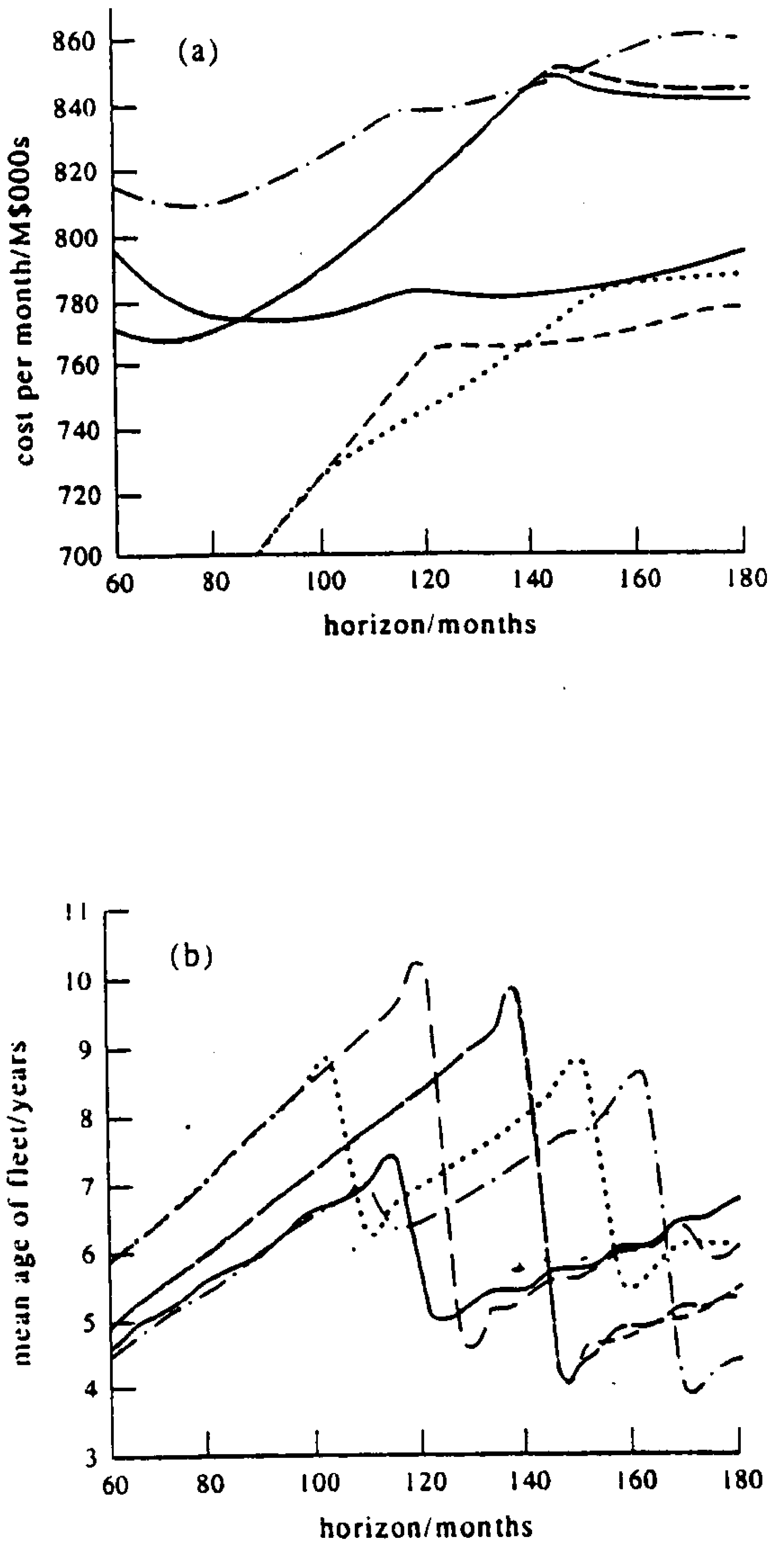


Figure 6.13. For all schedules: (a) cost of equivalent rent (M$000s per month for whole fleet), $C^*_{tdc}(h)$ vs h; (b) mean age of fleet at time h for optimal policy vs h. Penalty cost, p =M$000s; annual discount rate, υ =0.97. .... CIM; _ _ _ CMI; _____ ICM: _ _ _ IMC; _____ MCI: __ . __ MIC.

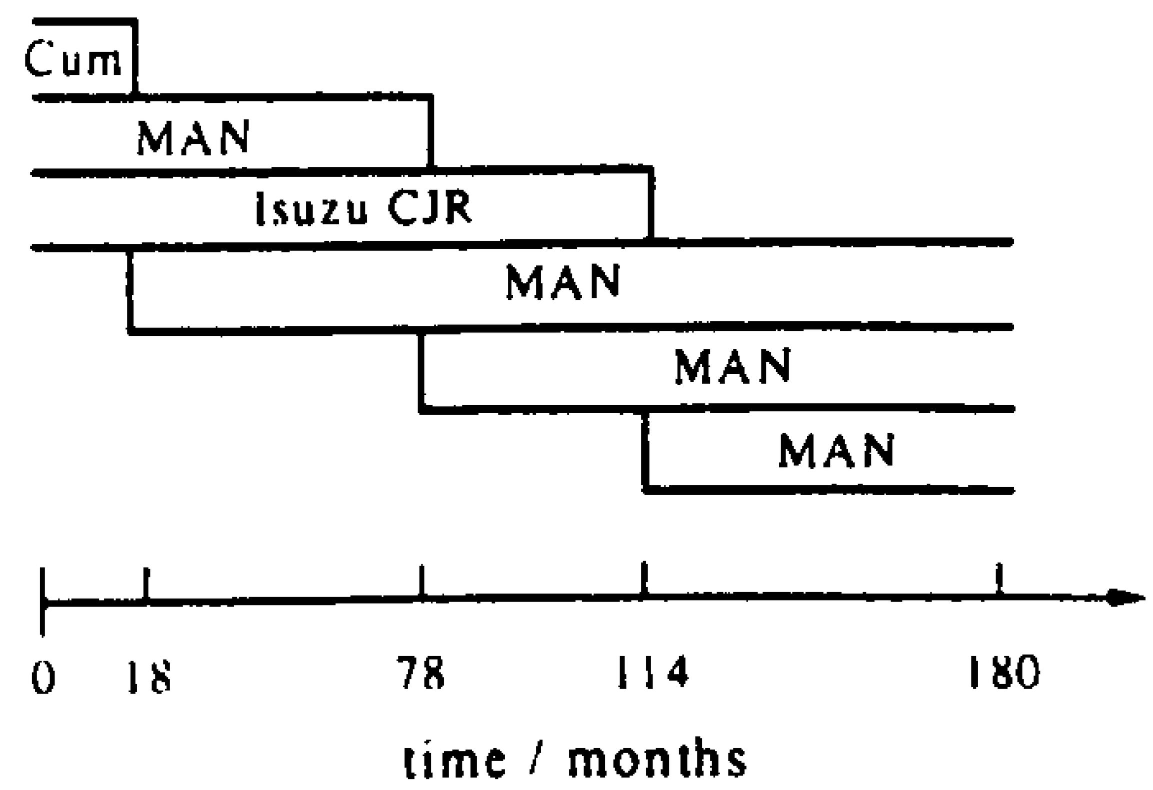


Figure 6.14. Schematic diagram of "optimum" policy for h =180: Cummins-MAN-Isuzu CJR: 4 cycles, L_1=18, L_2=60, L_3=36, L_4=66.

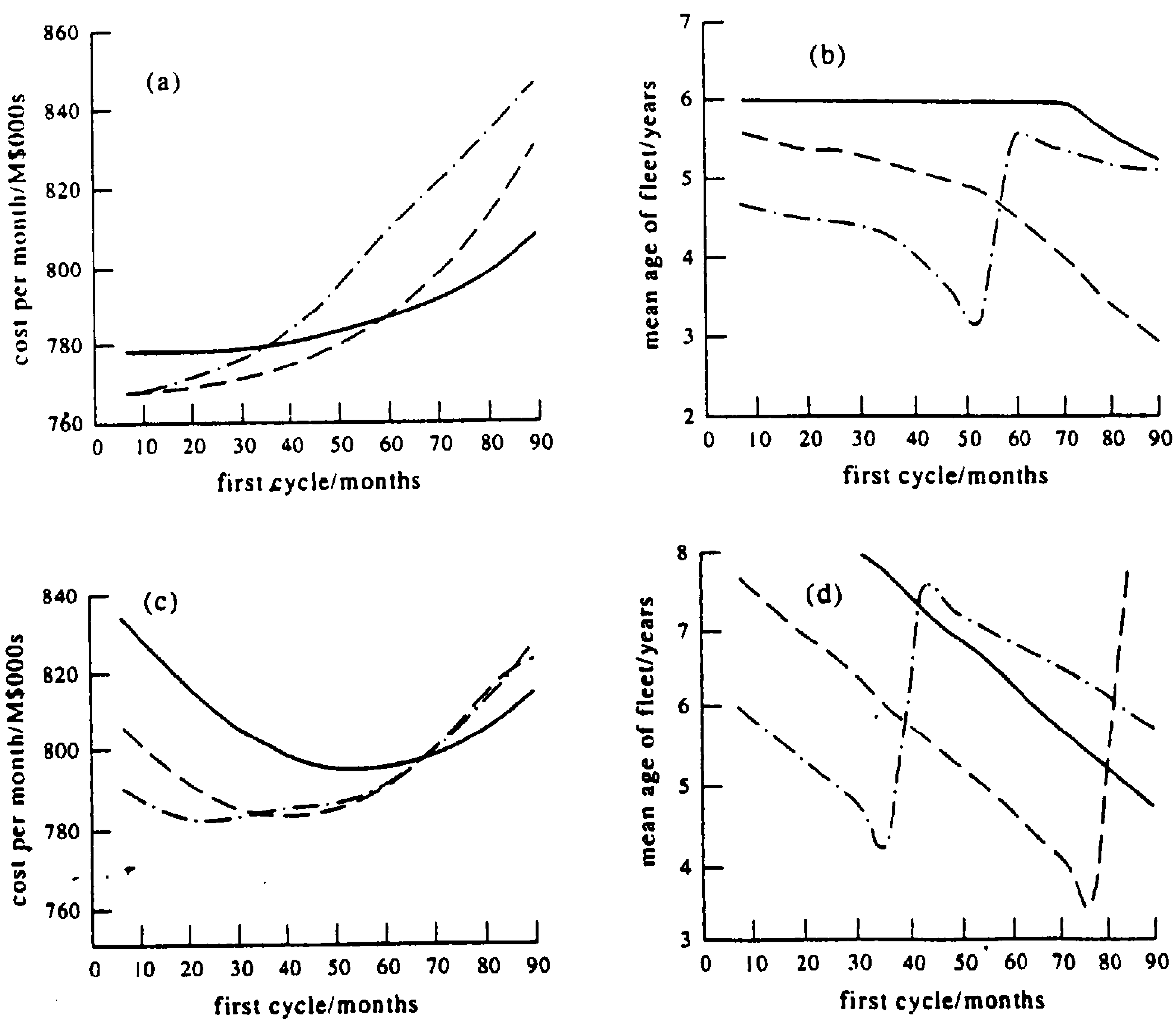
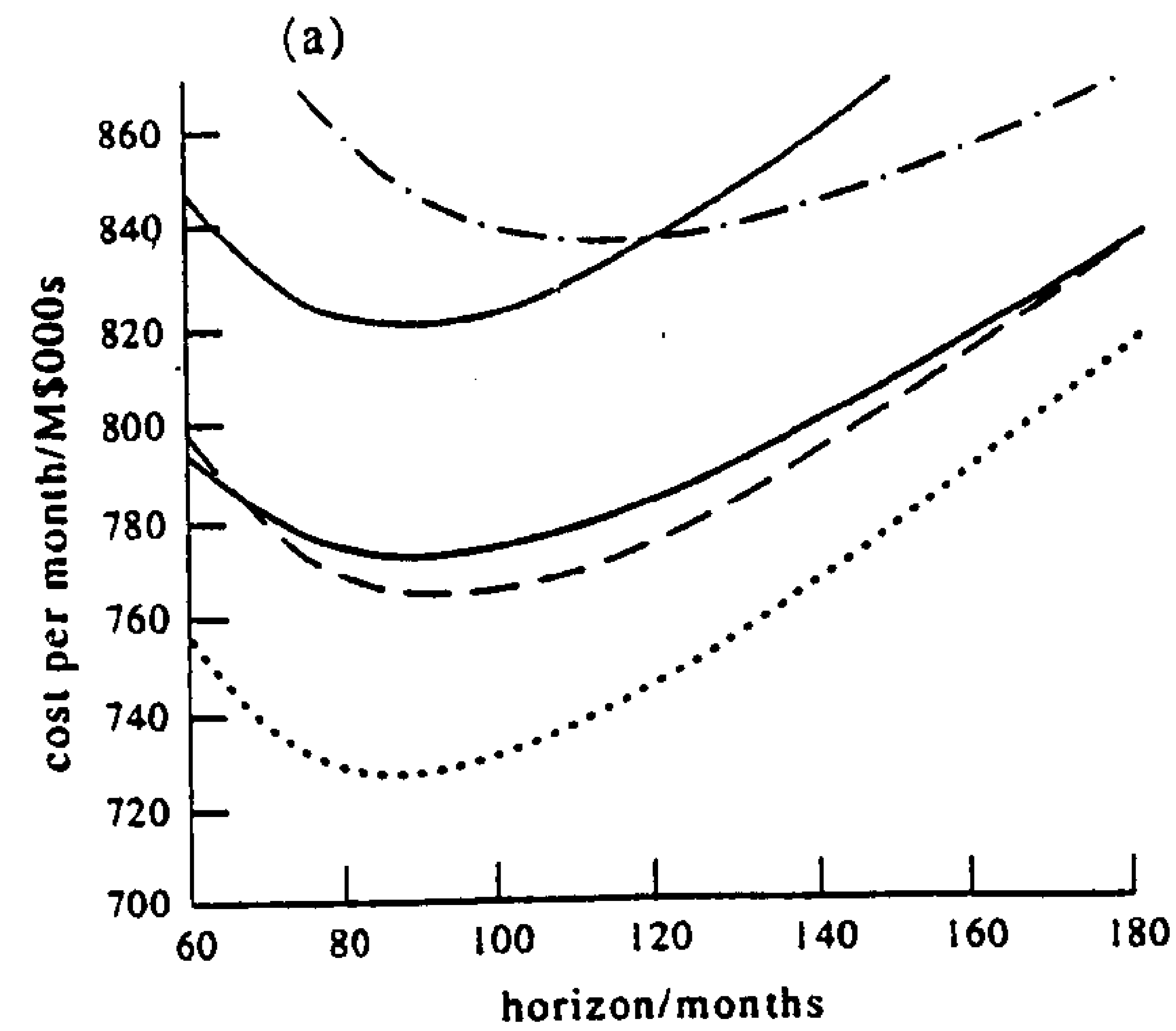


Figure 6.15. (a) cost of equivalent rent (M$000s per month for whole fleet), $C^*_{tdc}(h, L_1)$ vs L_1, Schedule CMI; (b) mean age of fleet at time h for constrained optimal policy vs L_1, schedule MCI; (c) $C^*_{tdc}(h, L_1)$ vs L_1, schedule MCI; (d) mean age of fleet at time h vs L_1, schedule MCI. Penalty cost, p =M$2000s; annual discount rate, υ=0.97. __ . __ h=120; _ _ _ h = 150; _____ h=180.



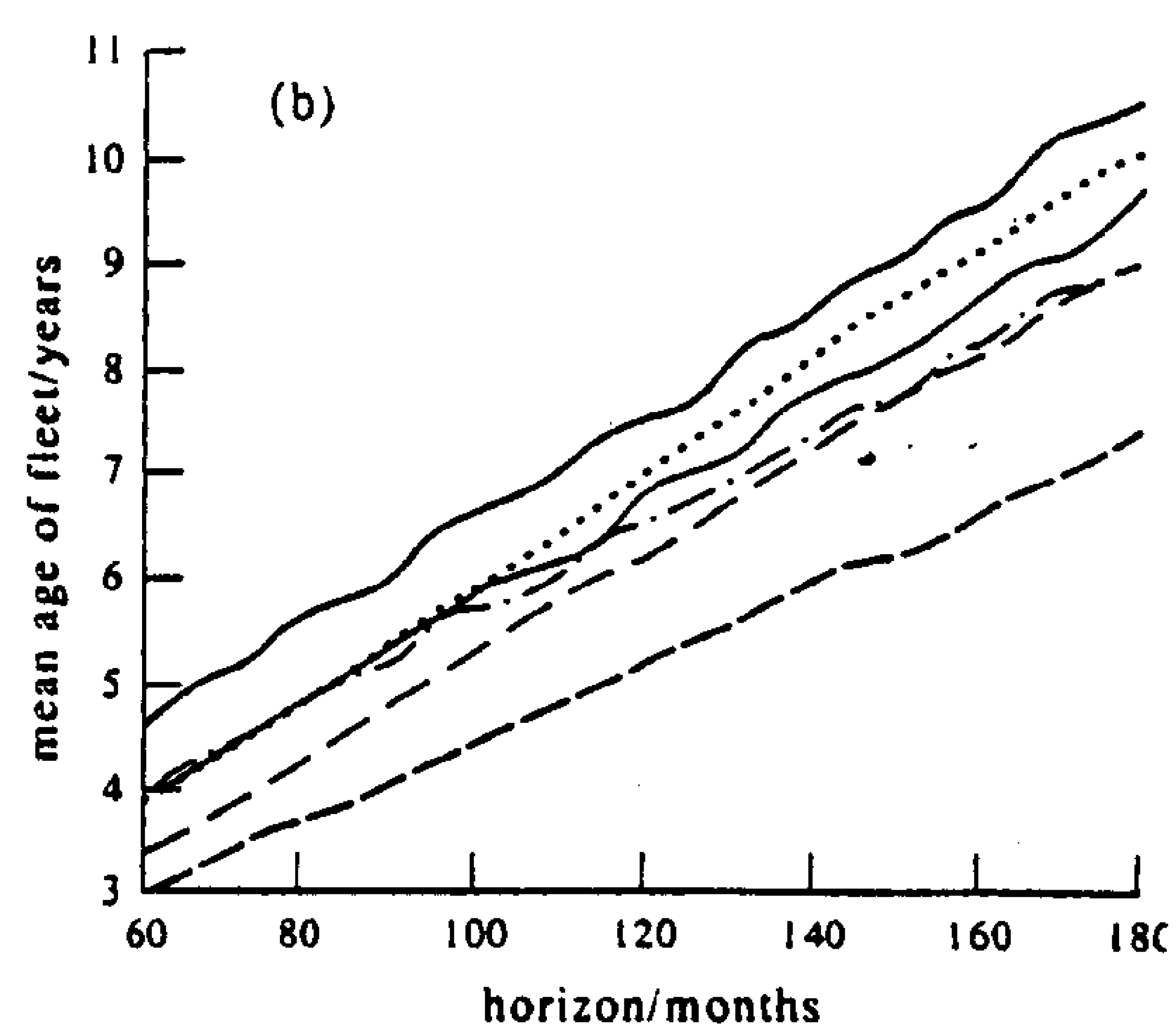


Figure 6.16. Two-cycle model, for all schedules: (a) cost of equivalent rent (M$000s per month for whole fleet), $C^*_{tdc}(h, N=2)$ vs h ; (b) mean age of fleet at time h for optimal policy vs h . Penalty cost, p =M$2000s; annual discount rate, υ =0.97. .... CIM: _ _ _ CMI; _____ ICM: _ _ _ IMC; _____ MCI; __ . __ MIC .

6.6. Discussion

For the fleet as a whole it is difficult to determine optimal replacement policy as two sub-fleets are partially retired, and usage levels are unknown. The problem is made more difficult because it is likely that the maintenance of these sub-fleets is less thorough than that for the newer sub-fleets. Under simple usage assumptions, the optimum policy is to replace the Mitsubishi and Isuzu CSA sub-fleets immediately.

For the particular sub-problem relating to the Cummins, Isuzu CJR and MAN, it appears that the optimum replacement schedule depends on the length of the horizon. Also, the end-of-horizon effect, as represented by the mean age of the fleet, also varies with the replacement schedule. The choice of "optimal" policy is therefore not straightforward. Over a fifteen year planning horizon, there is little to choose between the three schedules Cummins-Isuzu CJR-MAN, Cummins-MAN-Isuzu CJR and MAN-Cummins-Isuzu CJR, both in terms of cost and age. For a planning horizon of less than 13 years, the optimal policy appears to be unstable.

It is interesting to note that an alternative model with two cycles and variable planning horizon (P.A. Scarf & O. Bouamra, 1995) would have indicated that the schedule Cummins-Isuzu CJR is clearly optimal. The schedule Cummins-MAN leads to relatively high costs over a short time scale, but that the schedule Cummins-MAN-Isuzu CJR has relatively low costs over a longer horizon. Constraining the number of replacements can therefore lead to the exclusion, on a cost basis, of some schedules which might otherwise be sensible. The drawback of lifting this restriction, by constraining the horizon length and allowing the number of replacements to vary, is the necessity of choosing a suitable horizon length. A poorly chosen horizon length can lead to a poor replacement schedule.

Once the appropriate schedule has been determined, the effect of varying the time to first replacement is relatively small. For the Cummins-MAN-Isuzu CJR schedule, over a 15 year horizon, a delayed replacement of 12 months, from the optimal of 18 months, leads to an increased cost of M$2800 per month for the whole fleet. The effect of increasing penalty cost is simply to bring forward the replacement of sub-fleet. It is generally observed that optimal policy is insensitive to changes in discounting (Kobbacy & Nicol, 1994) and we do not investigate this further here.

CHAPTER 7

CHAPTER 7

Modelling the Challenger Problem with a Fixed Planning Horizon

7.1. Introduction

Replacement analysis involves the use of mathematical models and analysis to consider future provision of a service currently being provided by some existing asset, traditionally called the "defender" (Fraser & Posey, 1989). Alternative equipment, traditionally called the "challenger", could replace the defender. The "challenger problem" concerns which equipment should be bought to replace the existing equipment, the defender or the challenger. The challenger itself may be chosen from different available challengers. Implicitly, the challenger represents the most economic and effective equipment currently available (Jones & Tanchoco, 1987). Although it is not general, the challenger problem underlines the effect of technological obsolescence on the capital replacement policy (Tanchoco & Leung, 1987). We believe that distinguishing between the defender and challenger(s) requires the fixing of the horizon over which replacement policy is modelled. Suppose that we have two possible choices of action. The first choice is to replace the defender with the defender and the second choice is to replace the defender with the challenger. The question arises: " can we differentiate between the two choices of action for replacement?". The answer is that we cannot differentiate between the two choices for replacement if the "costs" of the optimal policies in each case extend over different periods of time. We argue that if we have a number of choices we can compare the optimal policy of each of them only if the "costs" extend over the same period of time. For example, suppose we have two possible actions A and B and that the results from A indicate that the cost will be £1200 a year for the next five years (it is not known what will happen after), and the results from B indicate that the cost will be £1000 a year for the next ten years. Thus, it is difficult to decide between the two possible actions because, although the cost incurred through the first action is less than that through the second action, the duration of the second action is longer. For this reason we argue that the planning horizon should be fixed globally. In this chapter we illustrate the fixed planning horizon model and compare it with the variable planning horizon modelling approach.

7.2. The model and criteria of the challenger problem

We now consider the challenger problem using a fixed planning horizon model. In our study, we apply the fixed planning horizon model with at most two replacements. The model is as described earlier in studying the behaviour of optimal policy. However here we distinguish between the replacement with the defender ($j = 1$) and replacement with the challenger ($j = 2$) so that the total cost over $[0, h]$ is

$$C_j(x) = \begin{cases} \int_0^x M_1(t+\tau)dt + \int_0^{h-x} M_j(t)dt + 2R, & 0 \le x < h, \\ \int_0^h M_1(t+\tau)dt + R, & x = h, \end{cases} \quad (j = 1,2). \quad (7.1)$$

Here M_j is the maintenance cost per unit time of the defender ($j = 1$), challenger ($j = 2$) at age t; τ is the age of the current equipment; R_j is the purchase price new of the defender/challenger; x is the time of the first replacement and h is the length of the planning horizon. In order to distinguish between different available challengers, the model given in equation (7.1) can be modified:

$$C_j(x) = \begin{cases} \int_0^x M_1(t+\tau)dt + \int_0^{h-x} M_j(t)dt + 2R_j, & 0 \le x < h, \\ \int_0^h M_1(t+\tau)dt + R_j, & x = h, \end{cases} \quad (j = 1,2,\ldots,n+1). \quad (7.2)$$

Where M_1 is the maintenance cost per unit time of the (defender); M_j ($j = 2,\ldots,n+1$) is the maintenance cost per unit time of the different challengers; τ is the age of the current equipment; R_j(j =,2,...,$n+1$) is the purchase price new of the different challengers; x and h are as above and n is the number of possible challengers. The replacement problem is then to determine the optimal policy

$$\{j^*, x_{j^*}, C^*\},$$

where j^* indexes that policy such that C_{j^*} is the smallest among $C_{1^*},\ldots,C_{n^*}$; where $C_{j^*} = \min_x C_j(x)$ for all j; $x_{j^*} = \{x : C_j(x) = C_{j^*}\}$ and

$$C^* = \min\{C_{1^*},\ldots\ldots,C_{n^*}\}.$$

We regard the total cost C^* and cost per unit time C^*/h as equivalent.

With discounting our decision criteria is the total cost over $[0, h]$:

$$C_j(x) = \begin{cases} \int_0^x M_1(t+\tau)\upsilon^t dt + \upsilon^x \left[\int_0^{h-x} M_j(t)\upsilon^t dt + R_j \right] + R_j \upsilon^h, & 0 \le x < h, \\ \int_0^h M_1(t+\tau)\upsilon^t dt + R_j \upsilon^h, & x = h, \end{cases} \quad (7.3)$$

with M_j; R_j; τ; x; h and n are as above and υ is the discounting factor.

The discounted cost per unit time over a planning horizon of length h is given as

$$C_j^u(x) = \frac{C_j^d(x)}{h}, \quad (7.4)$$

and the optimal policy with discounted cost per unit time as a decision criterion is written as

$$(j^{u^*}, x_{j^{u^*}}, C_j^{u^*}).$$

Also with discounting the equivalent rent is

$$C_j^r(x) = \frac{C_j^d(x)}{\sum_{j=1}^{h} \upsilon^j}, \quad (7.5)$$

the optimal policy with equivalent rent as a decision criterion is written as

$$\{j^{r,*}, x_{j^{r,*}}, C^{r,*}\}.$$

7.3. The application

The model described above is applied to study the replacement of the light van Ford A0609 (defender) which performs function A with the light van Dodge S56 as challenger. Also we consider the replacement of the heavy Van Ford T100 (defender) which performs function B (different from A) with the heavy van Bedford CF250 as challenger. It is required to determine whether to buy the defender or challenger at the next replacement. The maintenance cost data is described in Table 7.1 and Table 7.2 as follows

Table 7.1. Maintenance cost for the defender Ford A0609 and the challenger Dodge S56.

Year	Average maint. cost Ford A0609	Average maint. cost Dodge S56
1	£ 167	£ 393
2	353	545
3	759	544
4	622	
5	782	
6	969	
7	1565	
8	2287	

Table 7.2. Maintenance cost for the defender Ford T100 and the challenger Bedford CF250.

Year	Average maint. cost Ford T100	Average maint. cost Bedford CF250
1	£ 163	£ 222
2	245	198
3	434	
4	553	
5	687	
6	828	
7	1029	
8	1240	

The purchase prices for Ford A0609, Dodge S56, Ford T100, Bedford CF250 are £9910, £11776, £6150 and £6215 respectively.

7.3.1. Modelling the maintenance cost data

The data given in Table 7.1 and Table 7.2 are the maintenance costs presented by Christer (1988). A power law function was fitted resulting in the form $164t^{1.1}$ for Ford A0609 and $144t^{0.99}$ for Ford T100. The available data for Dodge S56 and Bedford CF250 are too few to establish a model for them. Since the behaviour of

Dodge S56 and Bedford CF250 data is close to that of Ford A0609 and Ford T100 respectively then we proceed as follows: use the same value of β obtained for Ford A0609 and Ford T100 respectively and different values α_1 and α_2 based on the ratio $R_1/R_2 = \alpha_1/\alpha_2$. Here R_1 is the purchase price of the first equipment; R_2 is the purchase price of the second equipment; α_1 is the maintenance cost per unit time factor of the first equipment and α_2 is that of the second equipment. Applying this method to both cases Ford A0609-Dodge S56 and Ford T100-Bedford CF250 led to the forms $195t^{1.1}$ and $155t^{0.99}$ for Dodge S56 and Bedford CF250 respectively.

7.4. Application of the fixed horizon model

We consider three decision criteria: the first of them is cost per unit time criterion.

7.4.1. Results for cost per unit time criterion

For numerical calculations the discrete form of the cost model described by equation (7.1) is used:

$$C_j(x) = \begin{cases} \sum_{t=1}^{x} M_1(t+\tau) + \sum_{t=1}^{h-x} M_j(t) + 2R_j, & 0 \le x < h, \\ \sum_{t=1}^{h} M_1(t+\tau) + R_j, & x = h. \end{cases} \qquad (7.6)$$

Results for the Ford A0609-Dodge S56 challenger problem with fixed planning horizons of length 10 years and 15 years are shown in Table 7.3. We present the cost per year for each year x over the planning horizon h.

From Table 7.3, over a planning horizon of length 10 years, the results of replacing the defender with the defender indicate that the minimum value $C^*(x)$

is £2466.2 occurs at $x^* = h = 10$. Also over the same horizon the results of replacing the defender with the challenger show that when performing only one replacement at the end of the horizon the cost incurred $C^*(x)$ is £2652.8. This occurs at $x^* = h = 10$. Therefore, for a planning horizon of 10 years it is not optimal to replace before the end of the horizon. Therefore, there is no need to distinguish between the defender and challenger. Over a longer horizon of length 15 years the results show that it is optimal to perform two replacements, replacing the defender Ford A0609 after 6 years and buying a new defender. Notice the value of x^* is affected by the length of the planning horizon that when h is larger x^* is smaller and vice versa.

Other results considering different ages of the current equipment are shown in Table 7.4. From Table 7.4 the results show that over all the planning horizon lengths and for all current ages of the defender, the total cost per year for replacing the defender with defender is always less than that of replacing the defender with challenger.

7.4.2. Results for the discounted cost per unit time criterion

For numerical calculations the continuous form (7.3) is converted to the discrete form:

$$C_i^d(x) = \begin{cases} \sum_{t=1}^{x} M_1(t+\tau)\upsilon^t + \upsilon^x \left[\sum_{t=1}^{h-x} M_j(t)\upsilon^t + R_j \right] + R_j\upsilon^h, & 0 \le x < h, \\ \sum_{t=1}^{h} M_1(t+\tau)\upsilon^t + R_j\upsilon^h, & x = h. \end{cases} \tag{7.7}$$

Here υ is the discount factor used to transform future costs to their present values. The discounted cost per unit time is defined as

$$C_j^d(x)/h,$$

where $C_j^d(x)$ is as given in equation (7.7).

The results of the two replacement scenarios of replacing the defender Ford A0609 are shown in Table 7.5 with planning horizons of length 10 years and 15 years using discounting factor $\upsilon = 0.95$. As concluded from Table 7.4, the results in Table 7.5 show that over all the planning horizon lengths and for all current ages of the defender Ford A0609 the discounted cost per year for replacing the defender with defender is always less than that of replacing the defender with challenger (see Figures 7.1 and 7.2).

Table 7.3. The total cost per year for replacing the defender Ford A0609 (2 years old) with the defender and the challenger Dodge S56 over planning horizons h=10 and h=15; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Time	**New vehicle=defender**		**New vehicle=challenger**	
x	**h=10**	**h=15**	**h=10**	**h=15**
1	2847.1	2747.3	3374.6	3258.8
2	2732.8	2588.8	3226.1	3062.1
3	2664.1	2461.6	3127.8	2899.8
4	2641.1	2366.1	3079.6	2772.3
5	2664.1	2302.3	3081.6	2679.7
6	2732.8	**2270.4***	3133.6	2622.0
7	2847.1	2270.5	3235.3	**2599.3***
8	3006.6	2302.3	3386.1	2611.4
9	3210.5	2366.1	3585.1	2658.4
10	**2466.2***	2461.6	**2652.8***	2739.9
11		2588.8		2856.0
12		2747.3		3006.1
13		2936.8		3189.8
14		3156.7		3406.5
15		2745.4		2869.8

Table 7.4. The total cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons h=10 and h=15; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	New vehicle=defender				New vehicle=challenger			
	h=10		h=15		h=10		h=15	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	2466.2	10	2270.4	6	2768.9	10	2719.8	7
4	2904.4	3	2564.7	5	3244.3	10	2441.9	6
6	3020.6	2	2722.8	4	3522.1	3	3114.0	5
8	3092.0	1	1878.5	3	3625.7	2	3273.7	4

Table 7.5. The discounted cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons h=10 and h=15 using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	New vehicle=defender				New vehicle=challenger			
	h=10		h=15		h=10		h=15	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	1731.3	10	1473.4	7	1843.0	10	1660.1	8
4	2091.4	10	1634.9	6	2203.1	10	1842.3	7
6	2267.0	2	1772.2	4	2574.1	10	2004.8	5
8	2347.8	1	1878.5	3	2727.7	2	2140.2	4

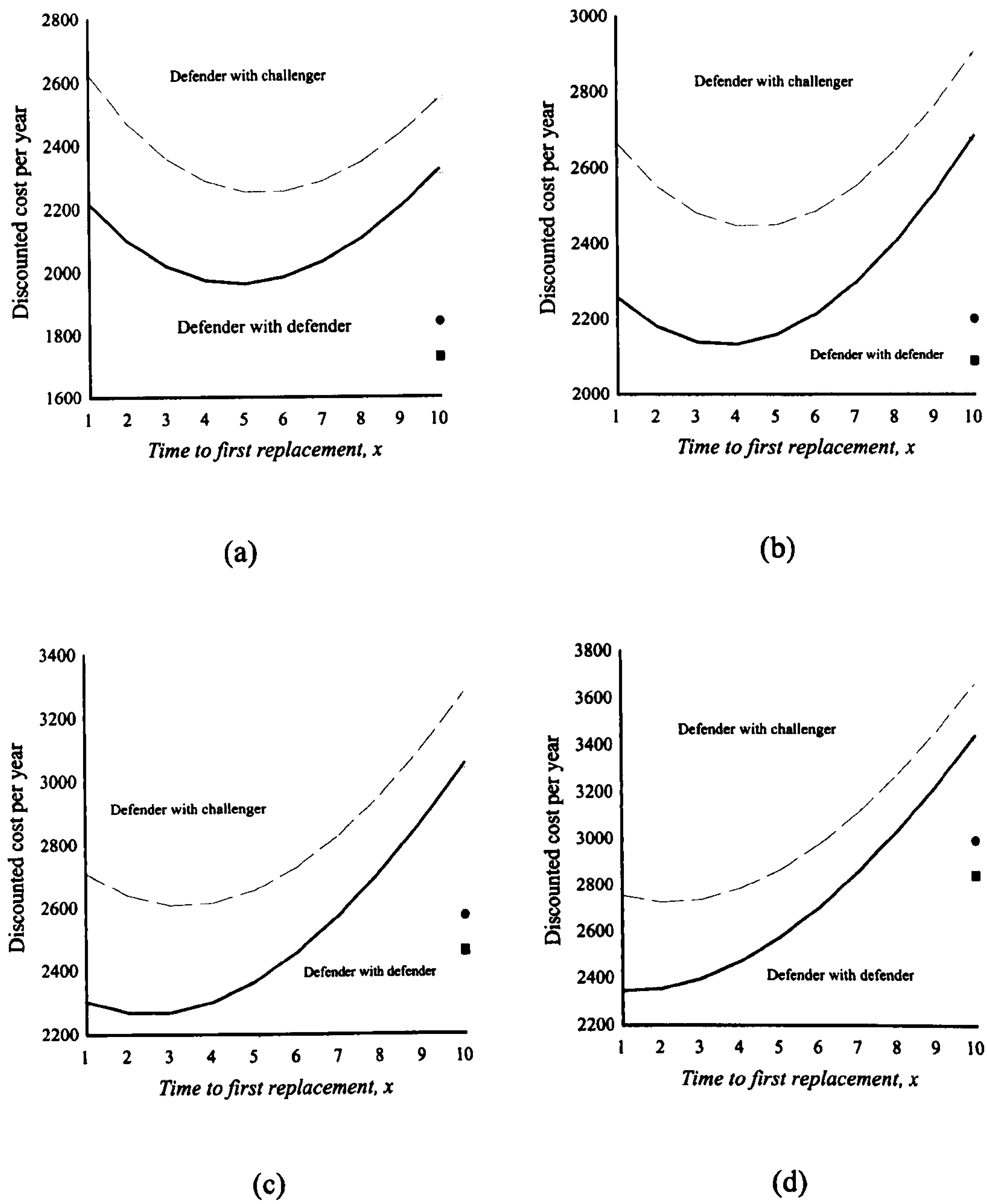


Figure 7.1. The discounted cost per year for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over fixed planning horizon $h = 10$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; ----○●, defender with challenger).

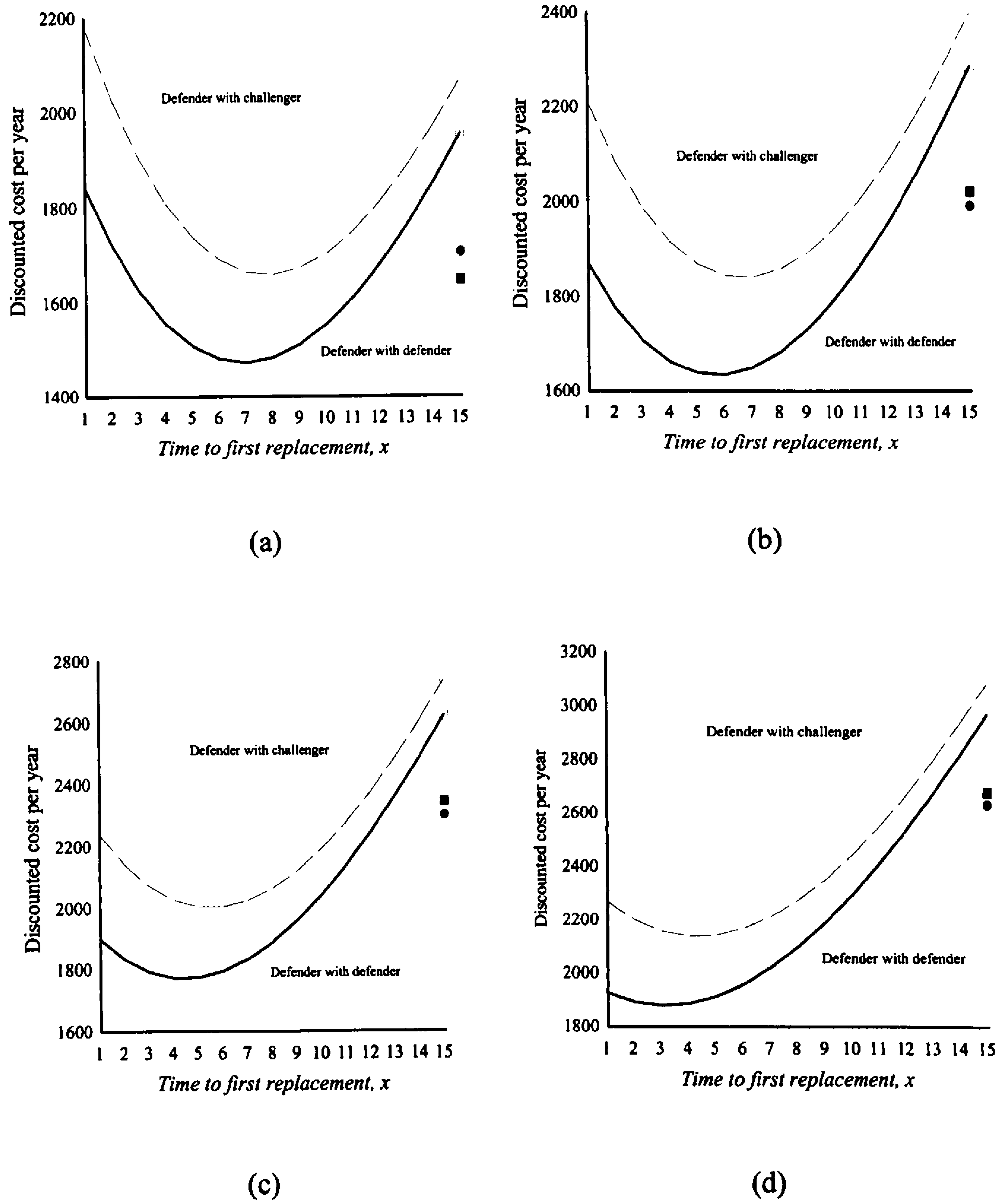


Figure 7.2. The discounted cost per year for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over fixed planning horizon $h = 15$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; ——○●, defender with challenger).

7.4.3. Results for the equivalent rent criterion

The discrete form for the equivalent rent over a fixed planning horizon with at most two replacements is obtained from equations (7.5 and 7.7). The results of the two replacement scenarios of replacing the defender Ford A0609 are shown in Table 7.6, with planning horizons of length 10 years and 15 years using discount factor $\upsilon = 0.95$.

Results from Table 7.6 show that the optimal policy is the same as in the case of the total cost per year and discounted cost per year (see Figures 7.3 and 7.4). Notice that the decisions are the same in Tables 7.5 and 7.6 but the costs are different. This because we are using two different criteria.

Table 7.6. The equivalent rent for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons h=10 and h=15 using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	New vehicle=defender				New vehicle=challenger			
	h=10		h=15		h=10		h=15	
τ	Min. rent	x^*	Min. rent	x^*	Min. rent	x^*	Min. rent	x^*
2	2270.8	10	2167.3	7	2417.4	10	2441.9	8
4	2743.1	10	2404.9	6	2889.7	10	2710.0	7
6	2973.6	2	2598.6	4	3376.4	10	2948.9	5
8	3079.5	1	2763.2	3	3577.8	2	3148.1	4

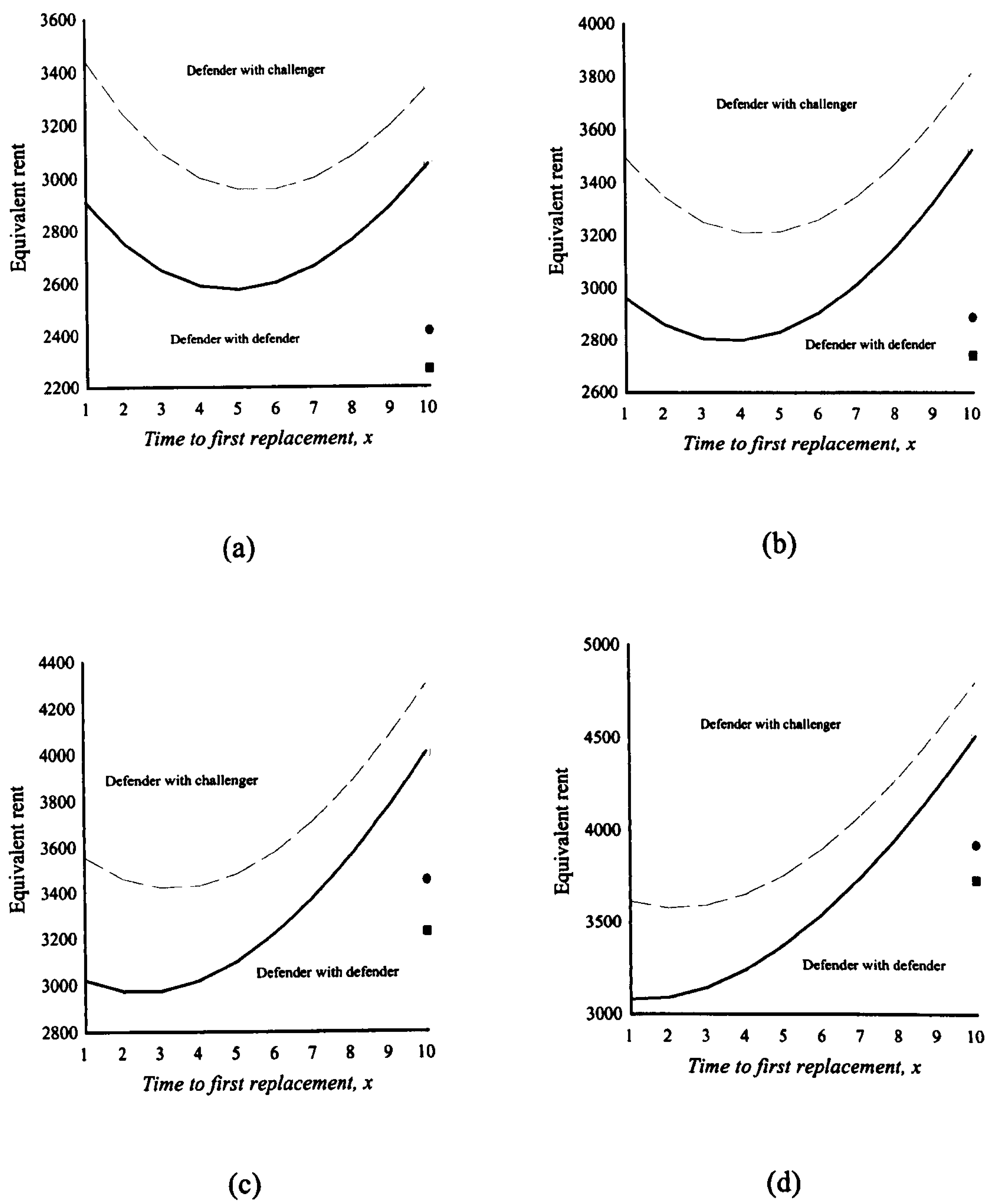


Figure 7.3. The equivalent rent for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over fixed planning horizon $h = 10$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; ——○●, defender with challenger).

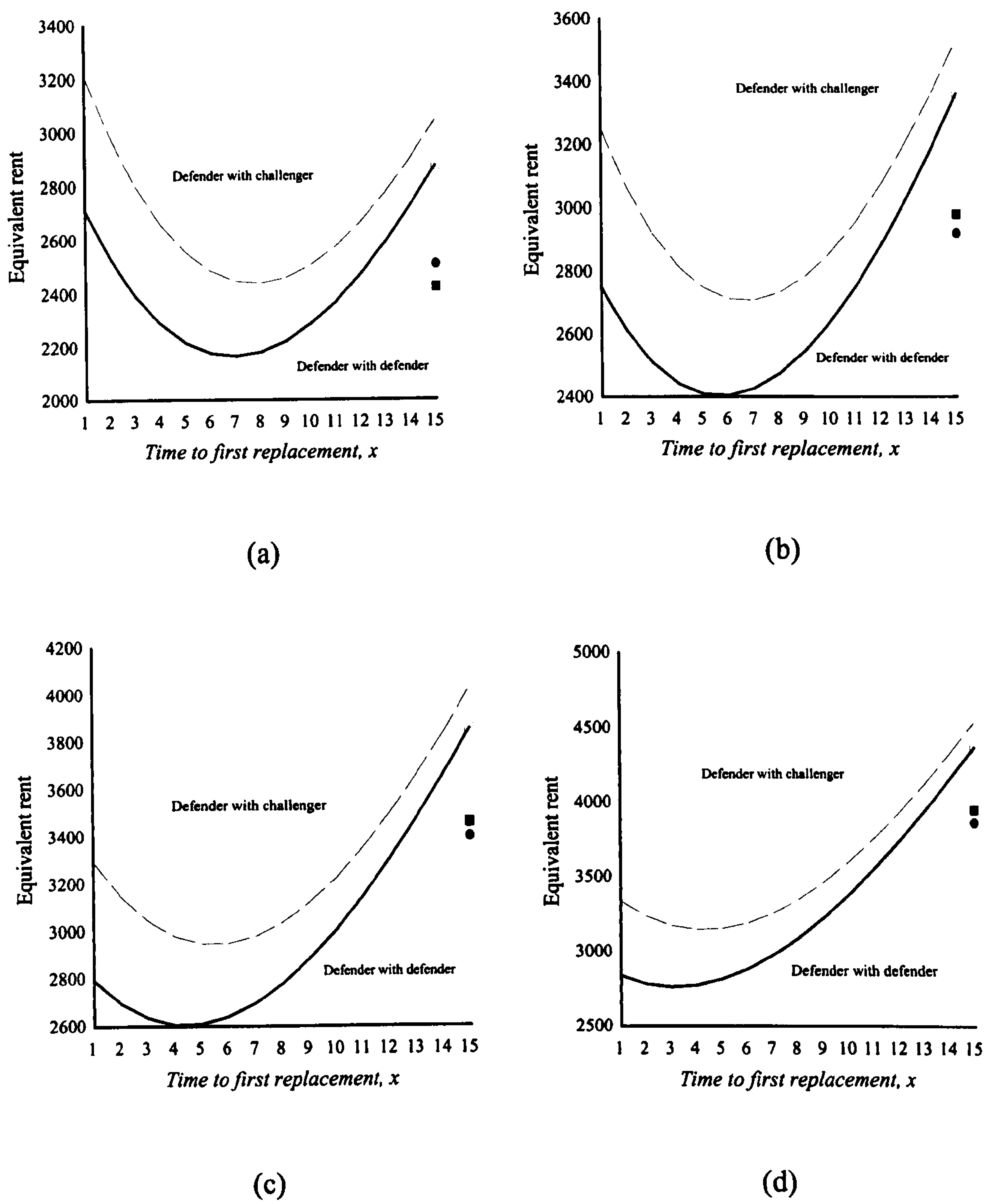


Figure 7.4. The equivalent rent for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over fixed planning horizon $h = 15$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; —○●, defender with challenger).

7.5. Application of the variable planning horizon model

Now, for comparison purposes, we consider the two-cycle replacement model with the variable planning horizon of Christer & Goodbody (1980) applied to the challenger problem.

7.5.1. The cost model and criterion

Over a variable planning horizon with two cycles the total cost is

$$C(K,L)=\int_0^K M_1(\tau+t)dt+\int_0^L M_j(t)dt+2R_j,\ (j=1,2). \tag{7.8}$$

The discrete form of the model is represented as

$$C(K,L)=\sum_{t=1}^{K} M_1(\tau+t)+\sum_{t=1}^{L} M_j(t)+2R_j,\ (j=1,2). \tag{7.9}$$

The cost per unit time is given as

$$C(K,L)/(K+L).$$

Here M_1 is the maintenance cost per unit time of the current equipment (defender); M_j is the maintenance cost per unit time of the challenger; τ is the age of the current equipment; R_j is the purchase price new of the defender $(j=1)$, challenger $(j=2)$; K is the length of the first cycle and L is the length of the second cycle.

7.5.2. The variable planning horizon model with discounting

The total discounted cost is

$$DC(K,L)=\int_0^K M_1(\tau+t)\upsilon^t dt+\upsilon^K\left[R_j+\int_0^L M_j(t)\upsilon^t dt+R\upsilon^L\right],\quad (j=1,2), \tag{7.10}$$

and in the discrete form becomes

$$DC(K,L)=\sum_{t=1}^{K}M_1(\tau+t)\upsilon^t+\upsilon^K\left[R_j+\sum_{t=1}^{L}M_j(t)\upsilon^t+R\upsilon^L\right],\quad (j=1,2),\ (7.11)$$

with M_1; M_j; τ and R_j as above. υ is the discount factor. We can use two possible decision criteria: the discounted cost per unit time and the equivalent rent. The discounted cost per unit time is defined as

$$DC(K,L)/(K+L),$$

and the equivalent rent is defined as

$$DC(K,L)/\sum_{i=1}^{K+L}\upsilon^i.$$

7.5.3. Results for the variable planning horizon model

The results for the Ford A0609-Dodge S56 defender-challenger problem without discounting ($\upsilon=1$) are shown in Table 7.7. From Table 7.7, for example, the results of replacing the defender Ford A0609 with the defender show that when the current age is 2 years it is optimal to replace after 8 years with minimum cost of £2332.5 per year. Also the results of replacing the defender Ford A0609 with the challenger Dodge S56 show that it is optimal to replace after the cycle of 9 years length with minimum cost of £2646.0 per year. The results show that as the current age increases the length of the horizon ($K+L$) decreases and the cost of replacing the defender with the defender is always less than that of replacing the defender with the challenger but over different planning horizon lengths in each case.

The results of replacing the defender Ford A0609 based on the discounted cost per year criterion are shown in Table 7.8. From Table 7.8 the results of replacing the defender Ford A0609 with the defender show that when the current age is 2 years it is optimal to replace after 17 years with minimum discounted cost of £1065.1 per year. Also the results of replacing the defender Ford A0609 with the challenger Dodge S56 show that it is optimal to replace after the cycle of 20 years length with minimum cost of £1134.1 per year. As in the case of the total cost per year, the results show that as the current age increases the length of the horizon decreases and the horizon length varies from replacing defender with defender to replacing defender with challenger. It is noticed that the discounting factor has a large influence on the optimal policy which leads to a longer planning horizon than that in the case of the total cost per year (see Figure 7.5).

The results of replacing the defender ford A0609 with discounting representing the equivalent rent are shown in Table 7.9. From Table 7.9 the results of replacing the defender Ford A0609 with the defender show that when the current age is 2 years it is optimal to replace after 9 years with minimum rent of £2093.2. Also the results of replacing the defender Ford A0609 with the challenger Dodge S56 show that it is optimal to replace after the cycle of 10 years length with minimum rent of £2321.2. As mentioned above, the length of the planning horizon is affected by the current age and varies from replacing defender with defender to replacing defender with challenger. It is noticed that the discounting factor has an influence on the optimal policy and this influence appears when the current age increases (see Figure 7.6).

Table 7.7. The total cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min.cost	K^*	L	Min.cost	K^*	L
2	2332.5	8	10	2646.0	9	10
4	2527.3	7	11	2866.9	8	10
6	2700.1	6	12	3065.3	7	11
8	2840.7	4	12	3242.9	6	12

Table 7.8. The discounted cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min.cost	K^*	L	Min.cost	K^*	L
2	1065.1	17	20	1134.1	20	20
4	1208.5	15	20	1284.0	20	20
6	1351.2	13	20	1437.2	19	20
8	1489.7	11	20	1591.3	17	20

Table 7.9. The equivalent rent for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $195t^{1.1}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min.rent	K^*	L	Min.rent	K^*	L
2	2087.5	9	11	2319.4	10	11
4	2335.8	7	12	2598.8	9	12
6	2553.0	6	13	2857.2	7	12
8	2732.3	4	14	3080.2	6	13

It should be noted that the results described above are obtained from different criteria. In these tables we are effectively presenting results for different values of the discount factor ($\upsilon = 1$ in Table 7.7). It is for the user to decide whether he discounts costs or not and to what extent he discounts costs; that is; the user should choose the value of the discount factor. We are merely concerned with the effect of discounting or otherwise on optimal policy.

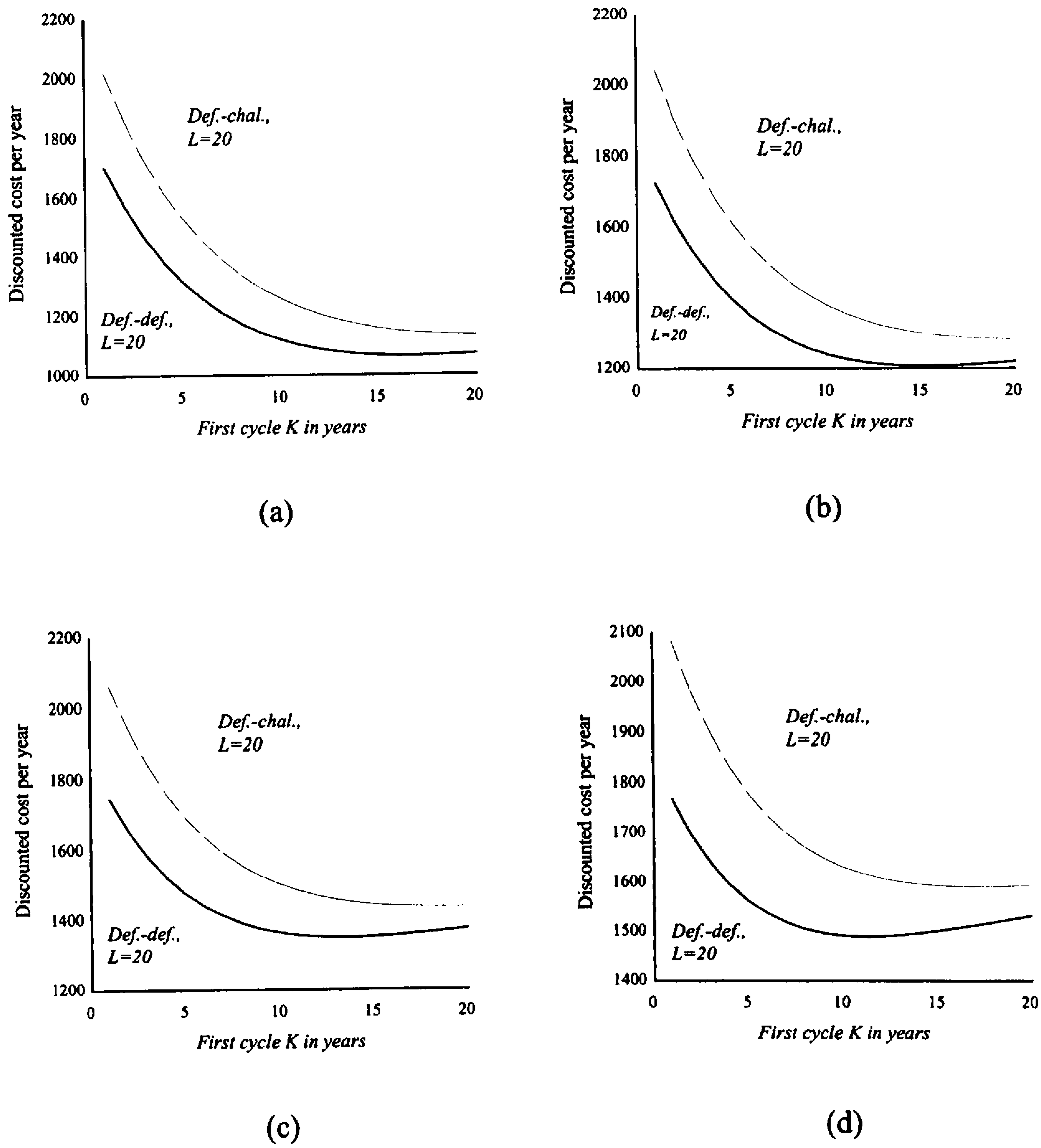


Figure 7.5. The discounted cost per year for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over variable planning horizon of length $K + L$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$.

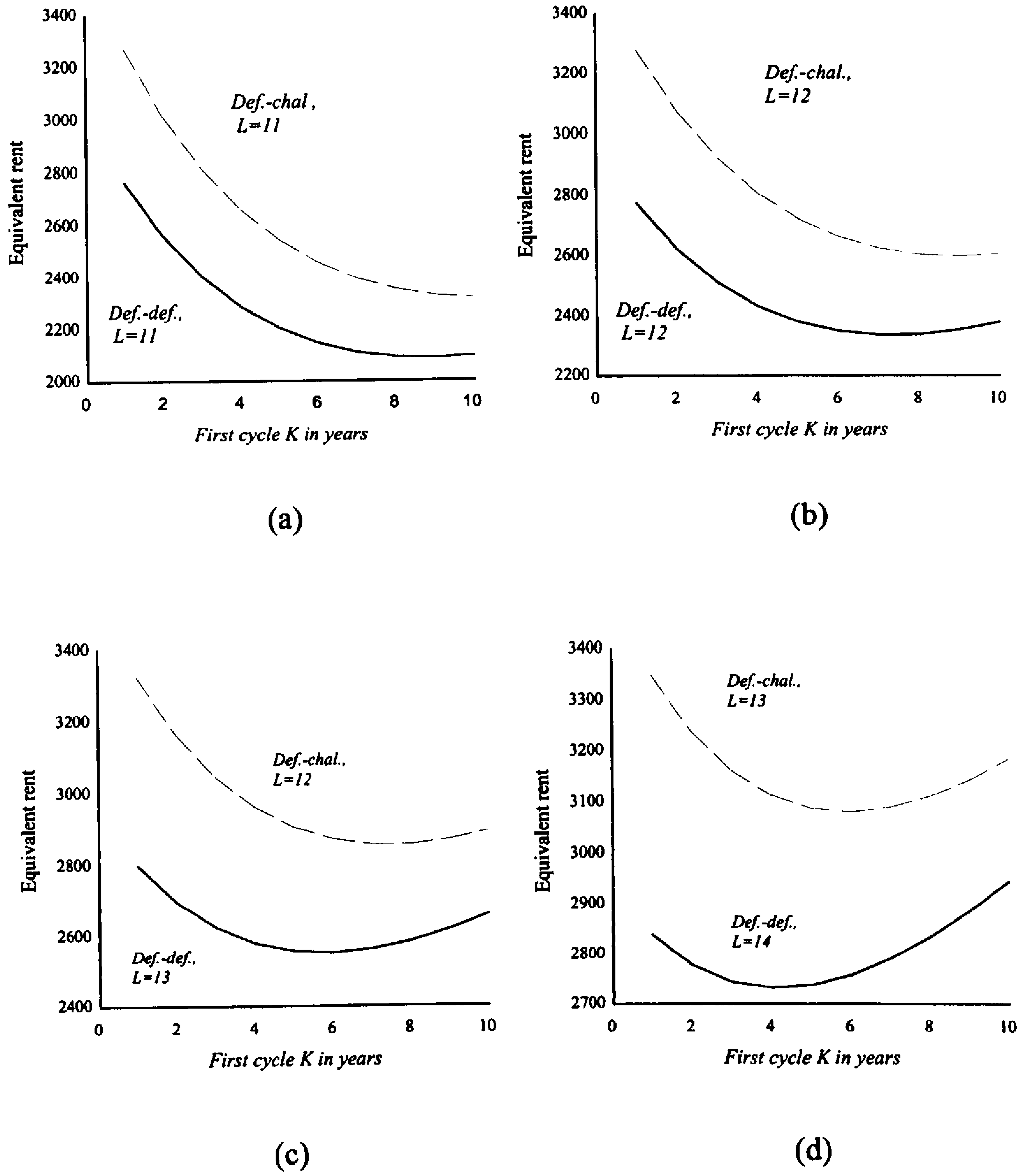


Figure 7.6. The equivalent rent for replacing the defender Ford A0609 (maintenance cost per unit time $164t^{1.1}$ and purchase price $R = £9910$) with the defender (Ford A0609) and the challenger Dodge S56 (maintenance cost per unit time $195t^{1.1}$ and purchase price $R = £11776$) over variable planning horizon of length $K + L$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$.

7.6. Optimal policy results for replacing Ford T100

Similarly the Ford T100-Bedford CF250 challenger problem yields the same conclusion obtained in the case of Ford A0609-Dodge S56 challenger problem over fixed planning horizon: replacing the defender with the defender is the optimal policy. For example, the results of discounted cost per year criterion are shown in Table 7.10 and Figures 7.7 and 7.8.

Table 7.10. The discounted cost per year for replacing the defender Ford T100 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Bedford CF250 over planning horizons h=10 and h=15 using discount factor υ=0.95; the maintenance cost per unit time of the defender is $144t^{0.99}$ and the maintenance cost per unit time of the challenger is $155t^{0.99}$.

Current age	New vehicle=defender				New vehicle=challenger			
	h=10		h=15		h=10		h=15	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	1133.5	10	970.3	7	1137.4	10	988.4	7
4	1347.8	10	1069.3	6	1351.7	10	1091.1	6
6	1467.7	2	1151.5	4	1496.1	3	1178.1	5
8	1515.9	1	1213.6	3	1554.8	2	1248.3	4

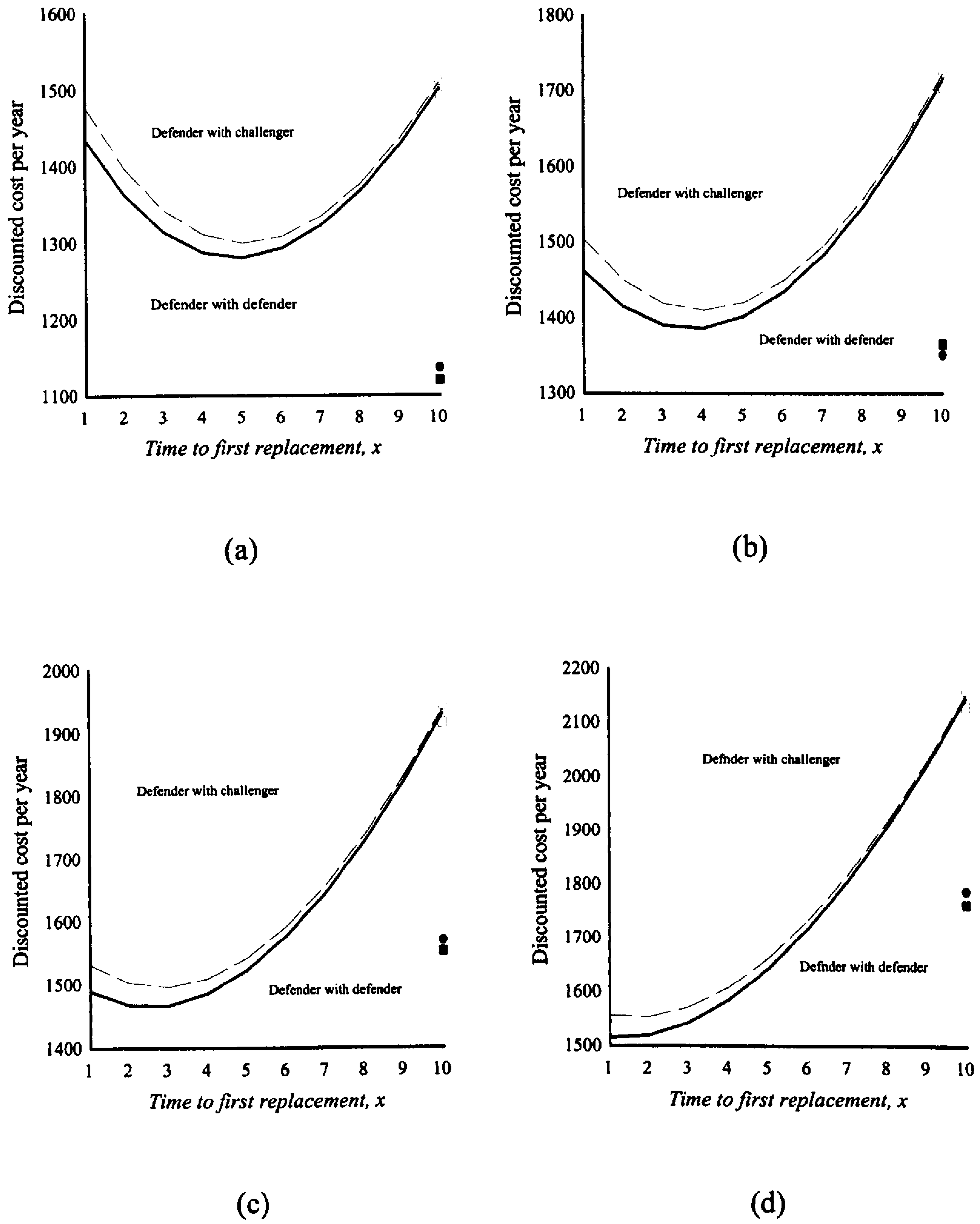


Figure 7.7. The discounted cost per year for replacing the defender Ford T100 (maintenance cost per unit time $144t^{0.99}$ and purchase price $R = £6150$) with the defender (Ford T100) and the challenger Bedford CF250 (maintenance cost per unit time $155t^{0.99}$ and purchase price $R = £6215$) over fixed planning horizon $h = 10$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; ——○●, defender with challenger).

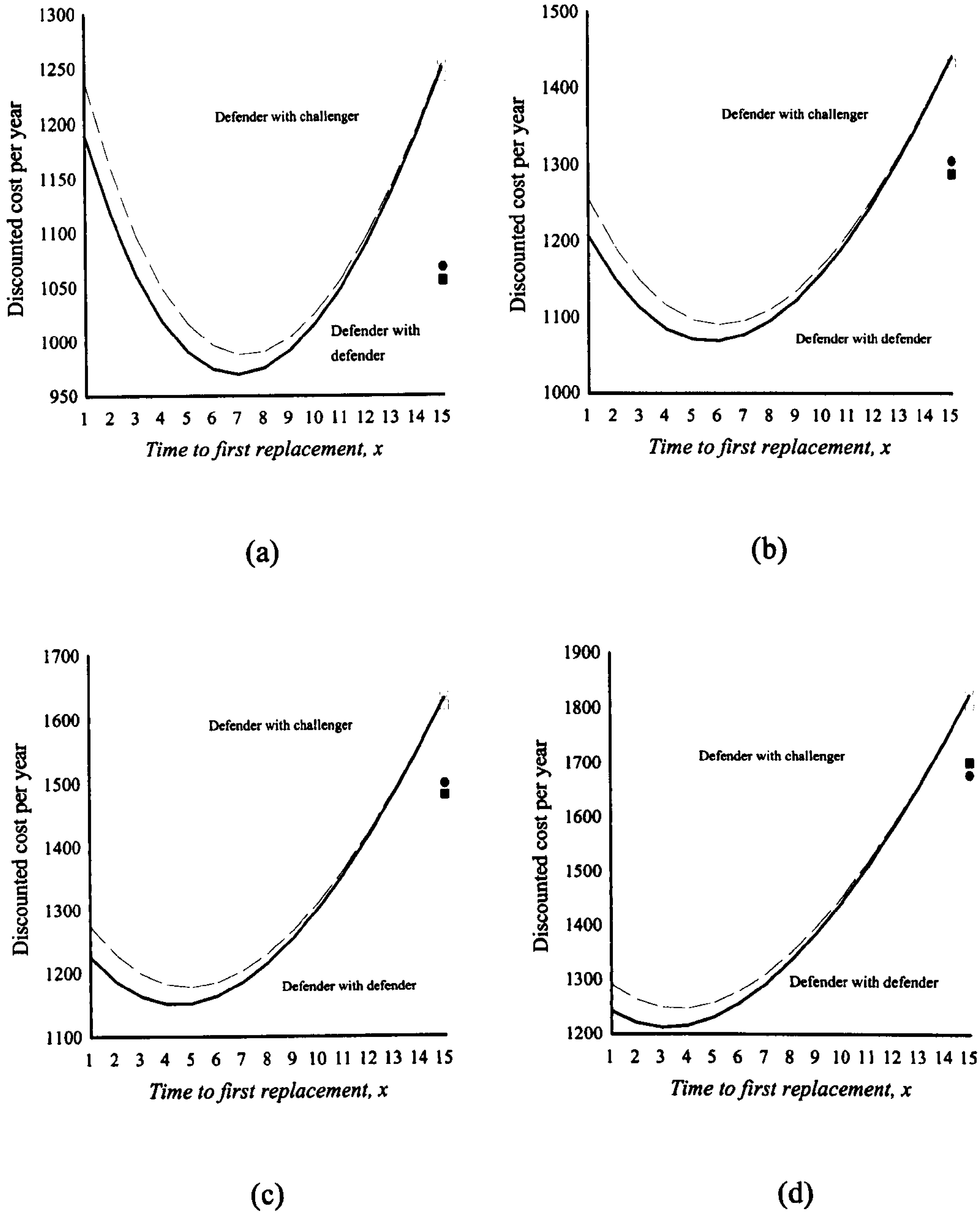


Figure 7.8. The discounted cost per year for replacing the defender Ford T100 (maintenance cost per unit time $144t^{0.99}$ and purchase price $R = £6150$) with the defender (Ford T100) and the challenger Bedford CF250 (maintenance cost per unit time $155t^{0.99}$ and purchase price $R = £6215$) over fixed planning horizon $h = 15$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$. (—□ ■, defender with defender; ----○●, defender with challenger).

For the variable planning horizon model, the results representing the total cost per year are shown in Table 7.11. From Table 7.11 the results of replacing the defender Ford T100 with the defender show that when the current age is 2 years it is optimal to replace after 8 years with minimum cost of £1527.4 per year. Also the results of replacing the defender Ford T100 with the challenger Bedford CF250 show that it is optimal to replace after the cycle of 9 years length with minimum cost of £1565.7 per year. The results show that as the current age increases the length of the first cycle decreases and the critical value of the horizon length is almost constant across replacing defender with defender to replacing defender with challenger.

Table 7.11. The total cost per year for replacing the defender Ford T100 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Bedford CF250 over variable planning horizon with two cycles K and L within the range 1 to 20 years; the maintenance cost per unit time of the defender is $144t^{0.99}$ and the maintenance cost per unit time of the challenger is $155t^{0.99}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min. cost	K^*	L	Min. cost	K^*	L
2	1527.4	8	10	1565.7	9	10
4	1644.6	7	11	1689.8	7	11
6	1746.0	6	12	1796.9	6	11
8	1831.3	4	12	1886.6	5	12

Also for Ford T100-Bedford CF250 results obtained from the discounted cost per year and equivalent rent criteria indicated the same conclusion as in the previous case. That either replacing defender with defender or replacing defender with challenger over variable planning horizon the optimal policy extends over different planning horizon lengths and with different minimum costs. See Table 7.12 and Figure 7.9.

Table 7.12. The discounted cost per year for replacing the defender Ford T100 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Bedford CF 250 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor υ=0.95; the maintenance cost per unit time of the defender is $144t^{0.99}$ and the maintenance cost per unit time of the challenger is $155t^{0.99}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min. cost	K^*	L	Min. cost	K^*	L
2	676.8	20	20	687.5	20	20
4	672.2	19	20	773.0	20	20
6	846.9	18	20	858.2	20	20
8	929.7	15	20	943.4	20	20

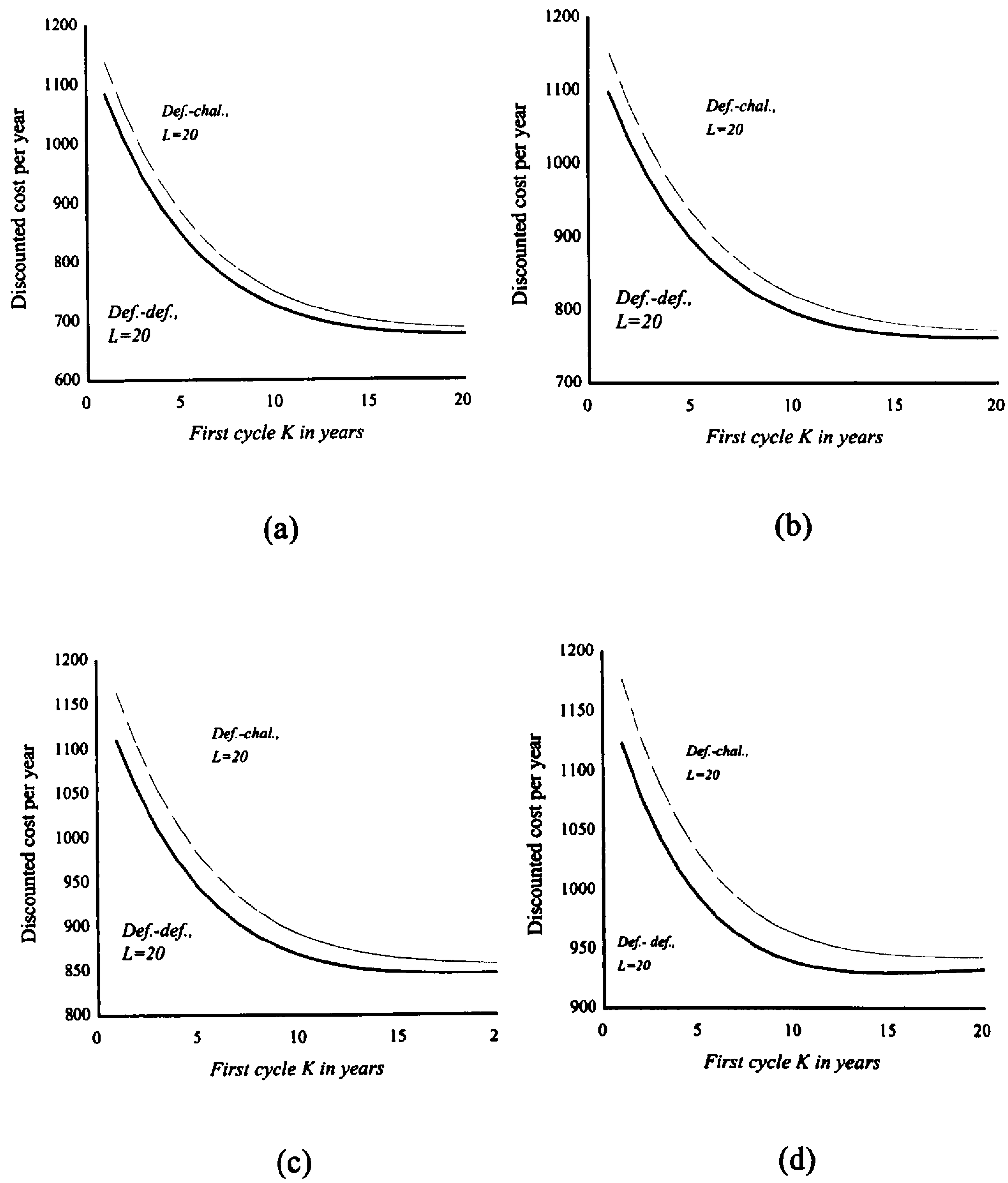


Figure 7.9. The discounted cost per year for replacing the defender Ford T100 (maintenance cost per unit time $144t^{0.99}$ and purchase price $R = £6150$) with the defender (Ford T100) and the challenger Bedford CF250 (maintenance cost per unit time $155t^{0.99}$ and purchase price $R = £6215$) over variable planning horizon of length $K + L$. (a) $\tau = 2$, (b) $\tau = 4$, (c) $\tau = 6$ and (d) $\tau = 8$.

7.7. Results of replacing defender with challenger using different cost parameters for the challenger

Notice that all the previous results have been obtained by using the estimated maintenance cost per unit time parameters for the defender and challenger based on the data given in Table 7.1 and 7.2 and the simplification that $\beta_1 = \beta_2$ and $\alpha_2 = \alpha_1 R_2 / R_1$ (see section 7.3.1). Alternative results can be obtained by estimating independent parameters for the challenger using the data estimated by Christer (1988). These data are given in Tables 7.13 and 7.14.

Table 7.13. Maintenance cost for the defender Ford A0609 and the challenger Dodge S56.

Year	Average maint. cost Ford A0609	Average maint. cost Dodge S56
1	£ 167	£ 393
2	353	545
3	759	544
4	622	409*
5	782	567*
6	969	660*
7	1565	1196*
8	2287	1768*

Table 7.14. Maintenance cost for the defender Ford T100 and the challenger Bedford CF250.

Year	Average maint. cost Ford T100	Average maint. cost Bedford CF250
1	£ 163	£ 222
2	245	198
3	434	445*
4	553	568*
5	687	700*
6	828	844*
7	1029	1057*
8	1240	1274*

The marked data for challenger was obtained by Christer (1988). It is observed from the table that there are only 3 data points for the challenger Dodge S56 and two data points for the challenger Bedford CF250. Here, in this study we consider the challenger data given in Tables 7.13 and 7.14 as a real maintenance cost data. Power law functions were fitted to the challenger data in both cases giving maintenance cost per unit time forms for Dodge S56 and for Bedford CF250. These forms are $322t^{0.5}$ and $162t^{0.9}$ respectively.

An alternative approach might use regression with the defender maintenance cost per unit time function as perior for the challenger.

The results of replacing the defender with defender are given in the previous sections. Therefore, we give results only for replacing the defender with the challenger based on the alternative maintenance cost per unit time function for the challenger. With a fixed planning horizon, the results of replacing the defender Ford A0609 with the challenger Dodge S56 are shown in Tables 7.15-7.17. Results in Tables 7.15-7.17 show that the optimal policy for the Ford A0609-Dodge S56 challenger problem is always buy a new defender Ford A0609 to replace the current defender Ford A0609.

Table 7.15. The total cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons $h=10$ and $h=15$; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	**New vehicle=defender**				**New vehicle=challenger**			
	$h=10$		$h=15$		$h=10$		$h=15$	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	2558.3	10	2380.2	6	2768.9	10	2987.9	9
4	2904.4	3	2564.7	5	3244.3	10	3267.3	8
6	3020.6	2	2722.8	4	3733.6	10	3525.3	8
8	3092.0	1	1878.5	3	4077.3	4	3765.6	7

Table 7.16. The discounted cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons $h=10$ and $h=15$ using discount factor $\upsilon=0.95$; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	**New vehicle=defender**				**New vehicle=challenger**			
	$h=10$		$h=15$		$h=10$		$h=15$	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	1731.3	10	1473.4	7	1843.0	10	1706.6	15
4	2091.4	10	1634.9	6	2203.1	10	2000.0	9
6	2267.0	2	1772.2	4	2574.1	10	2211.5	8
8	2347.8	1	1878.5	3	2954.1	10	2408.9	7

Table 7.17. The equivalent rent for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over planning horizons h=10 and h=15 using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	New vehicle=defender				New vehicle=challenger			
	h=10		h=15		h=10		h=15	
τ	Min. rent	x^*	Min. rent	x^*	Min. rent	x^*	Min. rent	x^*
2	2270.8	10	2167.3	7	2417.4	10	2510.3	15
4	2743.1	10	2404.9	6	2889.7	10	2942.0	9
6	2973.6	2	2598.6	4	3376.4	10	3253.0	8
8	3079.5	1	2763.2	3	3874.8	10	3543.3	7

With a variable planning horizon the results of replacing the defender Ford A0609 with the challenger Dodge S56 are shown in Tables 7.18-7.20. From Table 7.18 the cost of replacing the defender with the challenger is always less than that of replacing the defender with the defender but over different horizon length in each case. These results differ from the previous results in Table 7.7 because of the difference in the value of the rate of increasing maintenance cost per unit time β, which is less than that in the case of Table 7.7. Similarly, the results from Tables 7.19 and 7.20 are affected by the discount factor; the optimal horizon lengths are different from those in Table 7.18.

Table 7.18. The total cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min. cost	K^*	L	Min. cost	K^*	L
2	2332.5	8	10	2012.5	7	20
4	2527.1	7	11	2119.9	5	20
6	2700.1	6	12	2206.2	4	20
8	2840.7	4	12	2263.5	2	20

Table 7.19. The discounted cost per year for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min. cost	K^*	L	Min. cost	K^*	L
2	1065.1	17	20	935.9	12	20
4	1208.5	15	20	1060.5	10	20
6	1351.2	13	20	1177.7	8	20
8	1489.7	11	20	1281.2	6	20

Table 7.20. The equivalent rent for replacing the defender Ford A0609 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Dodge S56 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor υ=0.95; the maintenance cost per unit time of the defender is $164t^{1.1}$ and the maintenance cost per unit time of the challenger is $322t^{0.5}$.

Current age	New vehicle=defender			New vehicle=challenger		
τ	Min. rent	K^*	L	Min. rent	K^*	L
2	2087.5	9	11	1856.8	7	20
4	2335.8	7	12	2032.9	6	20
6	2553.0	6	13	2173.8	4	20
8	2732.3	4	14	2269.9	2	20

Considering different cost parameters the optimal policy for Ford T100-Bedford CF250 ($162t^{0.9}$) over fixed planning horizon is the same as in the case of Ford T100-Bedford CF250 ($155t^{0.99}$). For example, the total cost per year is shown in Table (7.21).

Over variable planning horizon, the analysis of Ford T100-Bedford CF250 ($162t^{0.9}$) challenger problem showed that replacing the defender with the challenger gives different results from that obtained in the case Ford T100-Bedford ($155t^{0.99}$). On the other hand the results also showed that the main finding (that the optimal decision occurs over different planning horizon lengths) is true in both cases. For example, the results of equivalent rent are shown in Table (7.22).

Table 7.21. The total cost per year for replacing the defender Ford T100 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Bedford CF250 over planning horizons $h=10$ and $h=15$; the maintenance cost per unit time of the defender is $144t^{0.99}$ and the maintenance cost per unit time of the challenger is $162t^{0.9}$.

Current age	**New vehicle=defender**				**New vehicle=challenger**			
	$h=10$		$h=15$		$h=10$		$h=15$	
τ	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*	Min. cost	x^*
2	1678.1	10	1559.3	6	1684.6	10	1610.4	7
4	1884.7	3	1671.9	5	1947.2	3	1733.5	6
6	1155.1	2	1765.6	4	2031.7	2	1838.9	5
8	1997.2	1	1840.4	3	2087.8	2	1926.6	4

Table 7.22. The equivalent rent for replacing the defender Ford T100 (2 years old, 4 years old, 6 years old and 8 years old) with the defender and the challenger Bedford CF250 over variable planning horizon with two cycles K and L within the range 1 to 20 years using discount factor $\upsilon=0.95$; the maintenance cost per unit time of the defender is $144t^{0.99}$ and the maintenance cost per unit time of the challenger is $162t^{0.9}$.

Current age	**New vehicle=defender**			**New vehicle=challenger**		
τ	Min.rent	K^*	L	Min.rent	K^*	L
2	1373.8	9	11	1375.3	9	12
4	1525.0	8	12	1523.8	8	13
6	1653.7	6	13	1649.5	6	14
8	1759.6	4	14	1752.0	4	15

7.8. Comparison between the replacement over fixed and variable planning horizons

We now want to compare the two modelling approaches (fixed horizon, variable horizon) in the context of these challenger problems described above. For example, from Table 7.8 over the variable planning horizon, replacing the defender Ford A0609 with the defender leads to minimum discounted cost of £1489.7 per year when $\tau = 8$ over a horizon of length 31 years and replacing the defender Ford A0609 with the challenger Dodge S56 leads to minimum discounted cost of £1591.3 per year over a horizon of length 37 years. Therefore, it is not straightforward to choose between replacing the defender with the defender and replacing the defender with the challenger because each replacement occurs over different horizon length. Another example from Table 7.9 is that the minimum rent of replacing the defender Ford A0609 with the defender is £2335.8 over a horizon of length 19 years and the minimum rent of replacing the defender Ford A0609 with the challenger Dodge S56 is £2598.8 over a horizon of length 21 years. Thus, the minimum rent occurs over two different lengths of planning horizons making it difficult to decide between these choices (replacing the defender with the defender and replacing the defender with the challenger). In another example from Table 7.22 the equivalent rent for replacing the defender Ford T100 with the defender is £1653.7 over a horizon of length 19 years and the equivalent rent for replacing the defender with the challenger Bedford CF250 is £1649.5 over a horizon of length 20 years. Therefore with the variable planning horizon model, an extra step is required to determine the optimum policy.

Over the fixed planning horizon, applying the discounted cost per unit time criterion, the discounted cost per year of replacing the defender Ford A0609 with the defender is £1731.3 over a horizon of length 10 years and the discounted cost per year of replacing the defender Ford A0609 with the challenger Dodge S56 is 1843.0 over a horizon of length 10 years as well. Therefore, we can choose replacing the defender with the defender because it gives the minimum of the two minima over the same horizon length. Similar results are found when applying the equivalent rent or the discounted cost criterion.

From the previous results of replacement over fixed and variable planning horizons it has been noticed that fixing the planning horizon length enables the decision-maker to choose between different replacement choices. Therefore, it is easy to choose whether to replace the defender with the defender or replace the defender with the challenger over the same period of time. On the other hand, replacement over the variable planning horizon indicates that different replacement choices cannot necessarily be found over the same horizon length, so that the optimal value of the horizon length depends on the choice of vehicle to be replaced. Consequently, it is difficult to compare different replacement policies. The drawback with the fixed planning horizon approach is that h has to be specified. But since this has to be done in advance it should not bias the replacement decision. In this case, the manager and the decision maker can work together to choose the appropriate length of the horizon. In strategic planning the length of the horizon is typically specified in advance.

7.9. The Challenger Problem: a Dynamic programming approach

The dynamic programming approach for representing equipment replacement has been discussed in some depth earlier in chapter 4 (section 4.10). In this part we also discuss how to use DP to represent the challenger problem.

As we mentioned earlier in this chapter the challenger problem is a replacement problem with the addition of choosing between two equipment (the defender and the challenger) to replace the current equipment. Therefore we have two alternative equipment types with replacement costs $R_j (j = 1,2)$. We have the maintenance cost per period for equipment of age m periods as $M_j(m)\,(j = 1,2)$ and we aim to minimise the total cost.

We consider the dynamic programming approach for n time periods remaining in the horizon h that is divided into equally spaced time periods of length Δh. The current equipment is of age m time periods. In this case the maintenance cost per period, as it is represented in chapter 4 for non-like-with-like replacement, is

$$M_j(m) = \frac{\alpha_j}{\beta_j + 1}\left[(m\Delta h)^{\beta_j+1} - ((m-1)\Delta h)^{\beta_j+1}\right].$$

The dynamic programming approach gives us a set of recurrence relations which solves the problem of minimising the total cost over the total time of N periods. These recurrence relations are

$$\underset{j=1,2}{V_j(n,m)} = \min\begin{bmatrix} \text{(K)}: \ M_j(m+1) + V_j(n-1, m+1) \\ \text{(R)}: \min\limits_{j=1,2}\left[M_j(1) + V_j(n-1,1) + R_j\right] \end{bmatrix}. \tag{7.12}$$

Notice that the second part of the function means replace with type 1 or type 2 according to which of them gives the minimum.

We also require the relation

$$V_j(0,m) = R_j. \tag{7.13}$$

At a particular epoch, the optimal policy is that decision (keep the current equipment or replace with the defender or replace with challenger) which minimises $V(n,m)$. Thus over the complete horizon, optimal policy will consist of a sequence of decisions that are keep 1 (the defender) or replace with the defender (type1) or the challenger (type 2). This will imply an optimal number of replacements N^* over the planning horizon and optimal ages at replacement (for each replacement epoch), $x_1^*,\dots\dots,x_{N^*}^*$.

In general the optimal policy for the challenger problem is $(N^*,(\delta_1^*,x_1^*),(\delta_2^*,x_2^*),\dots\dots,(\delta_n^*,x_n^*))$ where $\delta_i = 1,2\,(i=1,2,\dots\dots,n)$. But we would only ever expect that if the first replacement is with the challenger (defender), then all subsequent replacements will be with the challenger (defender). In this case optimal policy is $(N^*,\delta^*,x_1^*,\dots\dots,x_{N^*}^*)$ where

$$\delta = \begin{cases} 1: & \text{replace with defender at all replacements,} \\ 2: & \text{replace with challenger at all replacements.} \end{cases} \tag{7.14}$$

7.10. Discussion

The work presented in this chapter describes the challenger problem which has an important part to play in capital replacement. The challenger problem is described, for two examples, with a fixed planning horizon and with a variable planning horizon. Results using discounted cost, cost per unit time and equivalent rent criteria were obtained when applying two different approaches for the maintenance cost data. Results obtained from the fixed planning horizon model showed that replacing the defender with the defender is the optimal policy. Results obtained from variable planning horizon model indicate also that replacing the defender with the defender is optimal, but these results are obtained over different planning horizon lengths. Note that the results over the variable planning horizon show that the discounted cost changes dramatically with the discount factor. On the contrary the results from the equivalent rent are straightforward and easily interpreted. Therefore, this suggests that the equivalent rent criterion is the most suitable for studying the replacement problem in general.

Our aim of this study was to compare between the results over the fixed planning horizon and variable planning horizon. This study shows that the fixed planning horizon model is a valid approach to study the challenger problem, and has advantages over the variable planning horizon model.

For complete view on the challenger problem we describe a dynamic programming approach. In this we allow any number of replacements over the planning horizon. Of course in implementing this approach suppose that we are able to quantify costs incurred beyond the second replacement.

CHAPTER 8

CHAPTER 8

Conclusion

This thesis is concerned with fixed planning horizon models and their importance for decision-making in capital replacement. We consider variable planning horizon models for comparison. Decision-support is the basic outcome of a replacement modelling study. The results obtained with straightforward interpretation provide support from the modeller to the manager/operator in order to decide which equipment to replace and when.

We review capital replacement as a part of strategic planning of capital expenditure. In particular, the fixed planning horizon model is introduced and described.

We begin our study with the behaviour of optimal policy for simple capital replacement models for like-with-like replacement and non-like-with-like replacement. It is sometimes difficult to find tractable expressions for the decision variables in our model. The difficulty appears in the case of non-like-with-like replacement. For this study in chapter 4 we used a first order approximation and a second order approximation to find a useful approximate value for the decision variable under appropriate circumstances. The advantage of these approximations is that tractable expressions for decision variables can be obtained. Insight has been gained into the behaviour of optimal policy for none-like-with-like numerically, using many possible combinations of the cost parameters. The results are illustrated graphically. This investigation was done without considering the discount factor, which has little effect on the optimal policy (Kobbacy & Nicol, 1994).

We present some theoretical results for the behaviour of optimal policy for the mixed fleet. The replacement models in this case consider a mixed fleet with each subfleet consists of single item. Extensions to this case are discussed briefly.These models allow the determination of which subfleet to replace first and when. The case study presented in chapter 6 illustrates an application of the fixed planning horizon models for the mixed fleet problem and compares different scenarios over the same planning horizon.

The challenger problem is concerned with which equipment to be bought at replacement and when. We contend that the fixed planning horizon model is the most appropriate model for the challenger problem. The variable planning horizon model can be used for this study but in some circumstances different replacement scenarios (defender-defender or challenger-challenger) will extend over different planning horizons and choices between these scenarios will not be straightforward as a consequence.

We also suggest that the rent criterion is the most sensible criterion among those studied. The rent criterion can be thought of as a discounted cost per unit time in which time is also "discounted".

We study the behaviour of optimal policy for a replacement model with a fixed planning horizon to decide if this model is a sensible model for use in practice. We are interested in how cost parameters and the control variables affect optimal policy for this replacement model. In particular we are interested in the effect of fixing the planning horizon. We compare our models with other models that currently used, such as the variable horizon model (Christer & Goodbody, 1980) and economic life model (Eilon et al., 1966). In certain circumstances these models are mathematically equivalent and we establish this; it is their interpretation that is different. There are circumstances under which the variable horizon two-cycle models are degenerate when $\beta = 0$ (equipment for which maintenance costs do not increase e.g. electrical equipment). Here, sensible results about optimal policy in the case of non-like-with-like, using a fixed planning horizon model, can still be obtained.

In order to investigate the restriction of at most two replacements in the fixed horizon economic life model, we have considered a dynamic programming approach. We describe how this approach can be used to determine optimal policy, for like-with-like replacement, non-like-with-like replacement and the challenger problem. Implementation for the like-with-like replacement using the dynamic programming approach found that generally speaking at most two replacements was optimal over the fixed planning horizon for reasonable values of the model parameters.

Throughout all this work many points can be extended for further work. For example, the behavior of optimal policy taking account of discount factor can be studied in more detail. We could also extend our work to consider a numerical investigation of the mixed fleet replacement problem over fixed planning horizon. Bayesian regression can be applied to the challenger problem studied in this thesis by using defender maintenance data as prior for information for challenger. A fully subjective approach has been taken in the capital replacement context, for example, (Apeland & Scarf, 2001). Also one can investigate numerically, the effect of resale value for non-like-with-like, mixed fleet replacement and the challenger problem. Another suggestion for future work is investigating the fleet size effect on the optimal policy relating to replacement problems with many items in the fleet. Lastly, but not least a computational investigation can be done using the dynamic programming approach for non-like-with-like replacement and the challenger problem over fixed planning horizon.

APPENDICES

Appendix 1

```
C:  This program determines the optimal policy for like-with-like replacement
C:  over a fixed planning horizon using dynamic programming approach.
    DIMENSION V1(50,50),V2(50,50),IX(50,50),CM(50),V(50,50)
    OPEN(10,FILE='Cost.DAT')
    OPEN(11,FILE='Decision.DAT')
    PRINT *,'ENTER THE PARAMETERS'
    READ *,A,B,R,DH,N,M
    DO 20 I=1,N
    DO 30 J=0,M-1
    V(0,J+1)=R
    V(I,M)=V(I,0)+R
    IX(I,M)=0
    CM(J+1)=A*((((J+1)*DH)**(B+1))-((J*DH)**(B+1)))/(B+1)
    V1(I,J)=CM(J+1)+V(I-1,J+1)
    V2(I,J)=CM(1)+V(I-1,1)+R
      IF(V1(I,J).LT.V2(I,J))THEN
      V(I,J)=V1(I,J)
C:    The decision is keep the equpment.
      IX(I,J)=1
      ELSEIF(V1(I,J).GT.V2(I,J))THEN
```

```
    V(I,J)=V2(I,J)
C:  The decision is replace the equipment.
    IX(I,J)=0
    ELSE
    V(I,J)=V2(I,J)
C:  The decision is keep or replace the equipment.
    IX(I,J)=2
    ENDIF
   PRINT*,V(I,J)
   WRITE(10,*) V(I,J)
   PRINT*, IX(I,J)
   WRITE(11,*) IX(I,J)
30  CONTINUE
20  CONTINUE
   CLOSE(10)
   CLOSE(11)
   STOP
   END
```

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