Partial Elliptical Two-Regime Speed-Flow Traffic Model Based on the Highway Capacity Manual

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Abstract
There have been several attempts to describe traffic flow behaviour by modelling the relationship between the main variables describing traffic such as speed, flow and density. Some of these models are based on simplistic assumptions and therefore, they are far from being accurate in representing the whole range of traffic conditions (e.g. from free flowing to congested situations). This paper describes a speed-flow traffic model based on a two-regime linear speed-density relationship. The proposed model gives a mathematical representation for the likely speed-flow relationship based on published data from the Highway Capacity Manual. The model is robust and simple to use in describing this relationship for different traffic and roadway conditions. It can be applied in modelling traffic behaviour and used in estimating delays when dealing with stable as well as unstable traffic flow conditions ranging from free-flow to stationary queues. The model is also recommended for use in text books when describing speed-flow-density relationships.

Key words: traffic; speed; flow; density; modelling

1 Introduction

The main variables that form the underpinnings of traffic analysis are speed, flow and density (Mannering *et al.*, 2005). Although there are a number of published theoretical and analytical speed-density relationships, most Traffic Engineering text books refer to Greenshields model which was developed in 1934 when describing such relationships (for example, see Fricker and Whitford (2004), Mannering *et al*. (2005), O'Flaherty (1997), Salter (1986), Salter and Hounsell (1996), and Wright and Dixon (2004)). This is because the model by Greenshields (1934) proposed a simplistic approach by assuming a linear form of speed-density relationship. The derived flow-density relationship gives a symmetrical parabola which has been used later on by Lighthill and Whitham (1955) in describing and explaining what is known as the shockwave phenomenon in traffic streams when traffic density increases suddenly.

Other forms of speed-density relationships are provided elsewhere (see for example, Drew (1965), Duncan (1979) and Pipes (1967)), while Drake *et al.* (1967) refer to a multi-regime linear relationships (i.e. two-regime and three-regime linear speed-density relationships). From experimental observations, Kerner (1999) showed that there are at least two phenomena of what was called “self-organisation without bottlenecks” in real traffic flow.

Normally, it is difficult to obtain accurate measurements of traffic density directly from sites. Hall *et al.* (1986) used occupancy (spot-density) instead of density in a study on flow-density relationships. Speed and flow values are much more accessible than density and are easier to obtain from site observations. Logically, speed and flow (rather than density) should be used as input values to those models representing traffic behaviour in evaluating the performance of traffic schemes and in estimating traffic delays for cost/benefit analysis.

Therefore, this paper uses parameter relating to speed and flow (such as free speed, maximum flow and speed at maximum flow) which can be directly measured from site in proposing a traffic model for speed-flow relationship. The Highway Capacity Manual (2000) is used as the basis for the data needed for this paper to form the proposed analytical model. This model is recommended for use in describing traffic once its parameters are obtained from site.

2 Capacity and Level of Service (LOS)

According to Wright and Dixon (2004), the Highway Capacity Manual (2000) describes traffic operational conditions using a qualitative measure called Level of Service (LOS). There are several Levels of Service ranging from A to F with varying density range measured in pc/mi/ln (passenger car per mile per lane) as shown in Table 1.

Table 1 Levels of service and density range (adapted from Highway Capacity Manual, 2000)

Level of Service	**Description**	**Density range pc/mi/ln *(pc/km/ln)***	**Average density pc/mi/ln *(pc/km/ln)***
A	This represents free-flow, low flows, high speeds, and low density with little or no delay. Drivers are free to choose their speeds and lanes. **(Stable condition)**	0-11 *(0-7)*	5.5 *(3.4)*
B	Operating speeds begin to be restricted by traffic conditions. Drivers are able to reasonably maintain their desired speed and lane of operation. **(Stable condition)**	11-18 *(7-11)*	14.5 *(9.1)*
C	Most drivers are more restricted by the higher traffic flows and have less freedom to select their own speeds, as well as reduced ability to change lanes or pass. **(Stable condition)**	18-26 *(11-16)*	22 *(13.8)*
D	There is little freedom to manoeuvre with lower comfort and convenience but these conditions may be tolerated for short periods. **(Approaching unstable conditions)**	26-35 *(16-22)*	30.5 *(19.1)*
E	Momentary stop-start conditions may prevail and queues start forming and operations are at or near capacity of the road. **(Unstable conditions)**	35-45 *(22-28)*	**40** ***(25.0)***
F	This represents forced flow operation where speeds are low and flows are below capacity with existing queues approaching traffic jam with complete stand still. **(Unstable conditions)**	>45 *(>28)*	Varies

Table 1 illustrates these levels of service (LOS) and gives the density range associated with each of them as described by the Highway Capacity Manual (2000). The stable and unstable traffic conditions associated with these levels are also identified. Capacity of a given section of roadway can simply be defined as the maximum number of vehicles which can pass a given point in one hour under the prevailing roadway and traffic conditions. Thus, there are a whole range of factors which influence capacity, some of which are related to road geometry while others are related to general drivers' behaviour, environmental conditions and the presence of traffic control devices.

3 Modelling Traffic

According to Kreyszig (2006), modelling is translating a physical or other problem into a mathematical form using an algebraic equation, a differential equation, a graph or some other mathematical expression. It is one of three phases which might be necessary in problem solving and interpretation of results for practical use.

Different models were used in describing traffic behaviour. These models could be analytical (which uses theoretical considerations based on field data), descriptive (which are mathematical models that applies theoretical principles), deterministic (which are mathematical models that are not subject to randomness) and empirical (that uses statistical analysis of field data in describing the behaviour). Computer simulation or stochastic techniques could be used in the modelling of traffic behaviour.

The model used in this paper is a simple descriptive analytical model which is based on published data from reliable sources such as the Highway Capacity Manual (2000).

4 Speed-flow-density models

In this section, two speed-density relationships are considered in more details, namely, the one- and two-regime linear models. Both of these models are simple to use. However, there are other forms of models of more complex nature.

4.1 One-regime linear speed-density relationship

4.1.1 Speed-density

The one-regime linear relationship between speed and density, as represented by Greenshields (1934), is shown in Figure 1.

This relationship is represented in Equation 1, as follows:

$$v = v_{free} [1 - (k / k_{jam})] \quad \text{....Eq. 1} \quad \text{(linear form)}$$

where,

v is the space mean speed
v_{free} is the free-flow speed (i.e. the speed on a roadway that can be maintained when no other vehicles are present)
k is the density
k_{jam} is the jam density (i.e. the maximum possible density on a roadway).

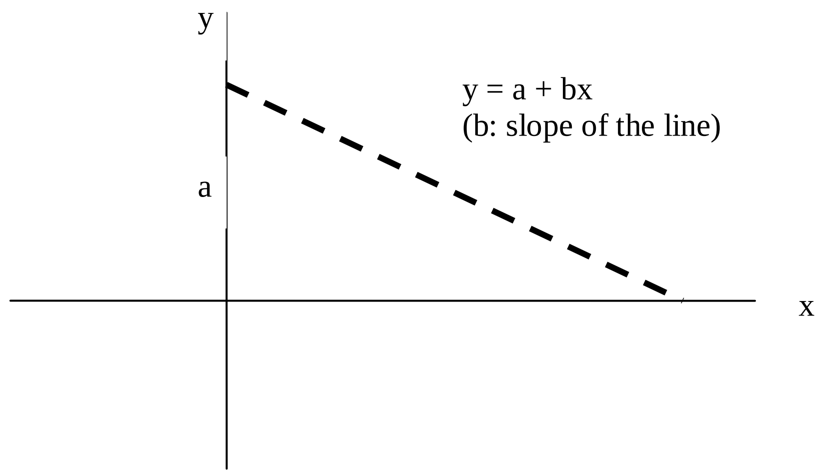

Figure 1 Equation of a line (representing a linear speed-density relationship)

4.1.2 Flow-density

According to Wardrop (1952), flow is the product of space mean speed and density. This is represented by Equation 2.

$q = k\ v$ …. Eq. 2

where,

q is the flow

Therefore, Equation 1 could be rewritten as follows:

$q = k\ [v_{free}\ (1 - (k / k_{jam}))]$

$q = k\ v_{free} - v_{free}\ (k^2 / k_{jam})$ .... Eq. 3 (parabolic function)

This is a parabolic representation of the relationship between flow and density.

4.1.3 Capacity

Capacity (i.e. maximum flow) is of interest to practitioners such as traffic engineers and transport planners when designing roads and modelling traffic behaviour. In order to find the optimum density (k_{cap}) when flow is maximum (i.e. at capacity (q_{cap})), Equation 3 is used to find the maximum point on the curve by differentiation and setting the terms to zero as follows:

$dq/dk = 0$ (from Eq. 3)

$dq/dk = v_{free} - 2k\ v_{free} / k_{jam} = 0$, then

$k_{cap} = 0.5k_{jam}$ (i.e. density at capacity) …. Eq. 4

Similarly, optimum speed (v_{cap}) at maximum flow (q_{cap}) from Equations 1 and 4:

$v_{cap} = v_{free}\ [1 - (k_{cap} / k_{jam})] = v_{free}\ [1 - (0.5k_{jam} / k_{jam})] = 0.5v_{free}$ … Eq. 5

To calculate maximum flow (i.e. capacity):

$$q_{cap} = k_{cap}\, v_{cap} = (0.5\, k_{jam})\,(0.5\, v_{free}) = 0.25\, k_{jam}\, v_{free} \quad \text{....Eq. 6}$$

4.1.4 Speed-flow

In order to find the relationship between speed and flow, Equations 1 and 2 are used to form:

$$k = (k_{jam} / v_{free})\,(v_{free} - v), \text{ and } q = k\, v,$$

$$q = v\,(k_{jam} / v_{free})\,(v_{free} - v) = k_{jam}\,(v - v^2 / v_{free}) \quad \text{(parabolic function) ... Eq. 7}$$

Since speed is dependent on flow (rather than the other way round), Equation 7 could be transformed to show speed as the dependent variable and flow is the independent variable. This will result in a symmetrical shape of a partial ellipse. This representation of the speed-flow relationship is often found in most Traffic and Transport Engineering text books as mentioned earlier.

The mathematical representation for an ellipse in the xy-plane with the centre at the origin can be shown as follows (Kreyszig, 2006):

$$x^2/m^2 + y^2/n^2 = 1 \quad \text{.... Eq. 8a}$$

Where

m is the x-intercept and n is the y-intercept as shown in Figure 2.

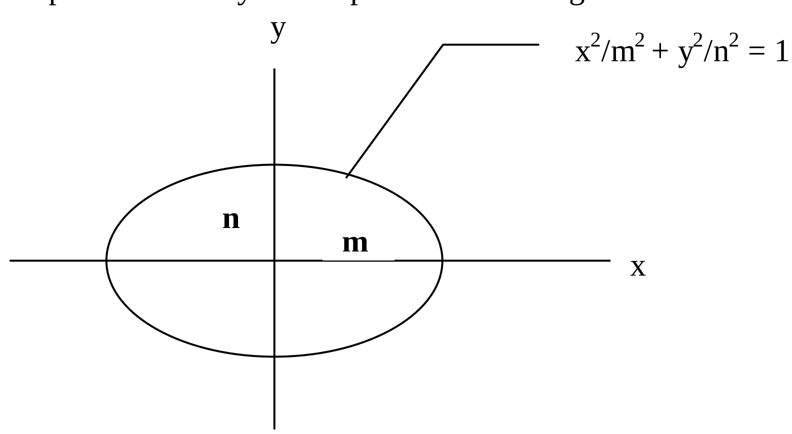

Figure 2 Equation of an ellipse

Therefore,

$$y^2 = n^2\,(1 - x^2/m^2) \quad \text{.... Eq. 8b} \qquad \text{or} \qquad y = \pm\, n\,\sqrt{(1 - x^2/m^2)} \quad \text{.... Eq. 8c}$$

In order to take into consideration that there are no negative speed values (i.e. y values could only be positive), the above equation is shifted up by the value of the y-intercept (i.e. the value of n) to form:

$$y = n \pm n\,\sqrt{(1 - x^2/m^2)} \quad \text{...Eq. 8d}$$

Equation 8d represents both stable and unstable conditions, as described in Table 1, for the one-regime linear speed-density relationship.

4.2 Two-regime linear speed-density relationship

A more realistic approach to the relationship between speed and density could take the form of a two- (or more) regime linear relationship representing both stable and unstable conditions. Figure 3 shows a two-regime linear relationship between speed and density which results in a non-symmetrical parabolic shape for the speed-flow relationship (as shown in Figure 4).

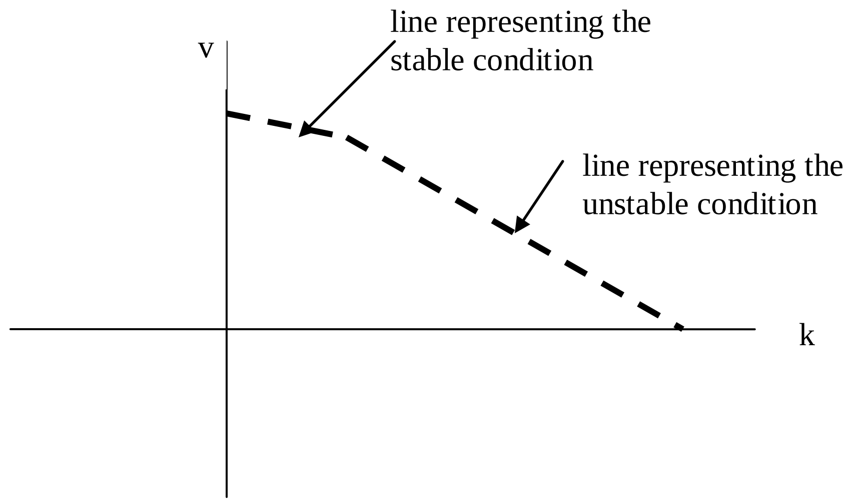

Figure 3 Two-regime linear for the speed-density relationship

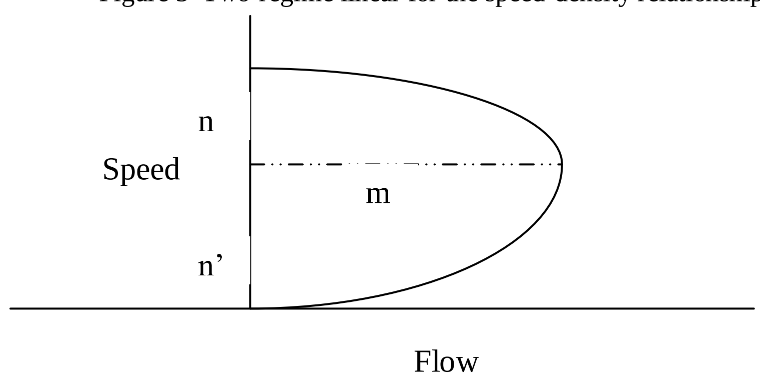

Figure 4 Speed-flow for the two-regime linear speed-density relationship

Making use of Equation 8c, and in order to take into consideration that there are no negative speed values for the speed-flow relationship (as discussed earlier), the equation is shifted up by n', as shown in Figure 4.

Therefore, Equation 8c, for the stable conditions, becomes:

$$y = n' + n \sqrt{(1 - x^2/m^2)} \quad \text{…Eq. 9a}$$

Similarly, for the unstable conditions:

$$y = n' - n' \sqrt{(1 - x^2/m^2)} \quad \text{…Eq. 9b}$$

In Figure 4, the intercept (m) takes only positive values representing capacity (q_{cap}), with (n) is represented by the difference between the free-flow speed (v_{free}) and optimum speed (v_{cap}), while (n') represents the optimum speed (v_{cap}) at capacity. This representation is more realistic than the previously described symmetrical form since it clearly differentiates between the stable (i.e. before capacity is reached) and the unstable traffic conditions (i.e. flows lower than capacity but with relatively low speeds and higher densities). In order to find the values of the intercepts used in these equations, published data from the Highway Capacity Manual (2000) were used.

5 Typical Values from the Highway Capacity Manual

For uninterrupted flow facilities, capacity (as described under level of service E) occurs where average density is in the region of 40 pc/mi/ln (or 25 pc/km/ln) with a maximum value of 45 pc/mi/ln as shown in Table 1. In Table 2 and according to Wright and Dixon (2004), for any known density, the maximum service flow rate (i.e. capacity, column 3) will increase for higher speed roadways (i.e. free-flow speed, column 1).

Fricker and Whitford (2004) stated that capacity varies by free-flow speed and that figures of about 2400 pc/hr/ln were used in design for most rural and suburban freeways with free-flow speed of 70 to 75 mph, whereas capacity values of 2250 pc/hr/ln corresponding to lower free-flow speeds of about 55 mph were often used in design of urban freeways.

Table 2 Typical maximum service flow rates for Level of Service E in pc/hr/ln for different free-flow speeds (Adapted from Wright and Dixon (2004))

	1	2	3	4	5
	Free-flow Speed (v_{free}) in mph *(km/hr)*	**Minimum Speed at Level of Service E (v_{cap}) in mph *(km/hr)***	**Maximum Service Flow Rate (q_{cap}) in pc/hr/ln**	**Speed intercept for stable conditions ($n=v_{free}-v_{cap}$) in mph (km/hr)**	**Speed intercept for unstable conditions ($n'=v_{cap}$) in mph (km/hr)**
rural	**75 *(120)***	**53.3 *(85.3)***	**2400**	21.7 (34.7)	53.3 *(85.3)*
↑	**70 *(112)***	**53.3 *(85.3)***	**2400**	16.7 (26.7)	53.3 *(85.3)*
	65 *(104)*	**52.2 *(83.5)***	**2350**	12.8 (20.5)	52.2 *(83.5)*
↓	**60 *(96)***	**51.1 *(81.8)***	**2300**	8.9 (14.2)	51.1 *(81.8)*
urban	**55 *(88)***	**50.0 *(80.0)***	**2250**	5.0 (8.0)	50.0 *(80.0)*

The capacity values shown in Table 2 (column 3) are typical ones which may be adjusted depending on other factors, such as lane width, lateral clearance, traffic composition, type of drivers (e.g. commuters or unfamiliar users of the road), number of lanes, spacing between interchanges and general terrain.

Table 2 shows the constant values which could be used in formulating the equations representing both stable and unstable conditions of flow for different free-flow speeds (e.g. representing different road types). This could easily be adjusted for use in modelling more realistic speed-flow relationships for different roadway conditions.

Fricker and Whitford (2004) stated that density is the primary determinant of the Level of Service and the speed criterion is the speed at maximum density for that Level. For a given LOS at capacity (i.e. LOS E), the maximum density reached will determine the minimum speed of that level as shown in column 2, Table 2.

In order to find the intercepts, n and n', which were previously described in Equations 9a and 9b, columns 4 and 5 in Table 2 were formed. Column 4 represents the speed intercept (n) in the stable condition and is formed by deducting column 2 from column 1, while column 5 representing the speed intercept for the unstable condition (n') is taken from column 2, since

$$v_{free} = n + n' , \quad v_{cap} = n' , \text{ and } q_{cap} = m$$

Therefore,

$$v = v_{cap} + [(v_{free} - v_{cap}) \sqrt{(1 - (q^2 / q_{cap}^2))}] \quad \text{Eq. 10a (stable condition)}$$

$$v = v_{cap} - [(v_{cap}) \sqrt{(1 - (q^2 / q_{cap}^2))}] \quad \text{Eq. 10b (unstable condition)}$$

The values of these parameters could be obtained from the Highway Capacity Manual (2000) (as shown in Table 2 based on analytical studies for different types of roadway conditions). The above two equations are recommended in practice for use in modelling traffic conditions ranging from free-flowing to stationary queuing. Also, they are recommended for use in Traffic Engineering text books when describing speed-flow-density relationships since they represent both stable and unstable conditions which are likely to occur on site.

6 Typical Values for Jam Density and Free-Flow Speed

Jam density (k_{jam}) could be obtained from Equation 6 as follows:

$$q_{cap} = k_{cap} v_{cap} = (0.5\, k_{jam})\, v_{cap}, \text{ therefore, } k_{jam} = 2\, q_{cap} / v_{cap} \; \ldots.\text{ Eq. 11}$$

Using the values from the Highway Capacity Manual (2000) shown in columns 2 and 3 of Table 2 and Equation 11, the calculated jam density, k_{jam}, for all free-flow speed conditions (i.e. ranging between 55 and 75 mph representing urban to rural conditions, respectively), reveals a figure of about 90 pc/mi/ln (i.e. 56 pc/km/ln). This indicates that when stationary queues are formed, the effect of the type of road (i.e. urban to rural) is negligible and jam density is more or less unchanged.

Using the value of 90 pc/mi/ln or 56 pc/km/ln for jam density obtained from Equation 11 above, the calculated average distance headway (h_d) when stationary is in the region of 18 metres. Obviously the average distance headway could vary depending on traffic composition (affecting the factor used in converting different types of vehicles into equivalent passenger car units).

According to Wright and Dixon (2004), the lengths of some articulated transit buses and semi-trailers are in excess of 60 ft (18.3 m). The higher the number of long vehicles in the traffic stream (such as buses and trucks), the lower is the jam density measured in veh/mi/ln (or veh/km/ln). Thus, the calculated average distance headway (h_d) of 18 metres when stationary is relatively high and some text books (e.g. Fricker and Whitford (2004) and Salter 1986) refer to a more realistic values in the region of 8 metres which corresponds to a jam density of 125 pc/km/ln (equivalent to 200 pc/mi/ln). Leutzbach (1988) suggested a higher figure for jam density of 150 veh/km/ln (i.e. 240 veh/mi/ln) as a rough guideline based on European studies. This results in relatively lower equivalent distance headway (h_d) of 6.7 meters.

Figure 5 shows the flow-density relationship for the two-regime linear speed-density relationship suggested in this paper based on distance headway (h_d) of 8 meters and those obtained from the Highway Capacity Manual (2000). The assumption used in Equation 4 for calculating the optimum density at capacity should therefore be adapted as follows:

$k_{cap} = 0.50\ k_{jam}$ ….. Eq. 12a (based on a maximum density of 45 pc/mi/ln or 28 pc/km/ln as used by the Highway Capacity Manual (2000))

and

$k_{cap} = 0.20\ k_{jam}$ ….. Eq. 12b (based on h_d of 8 meters when stationary)

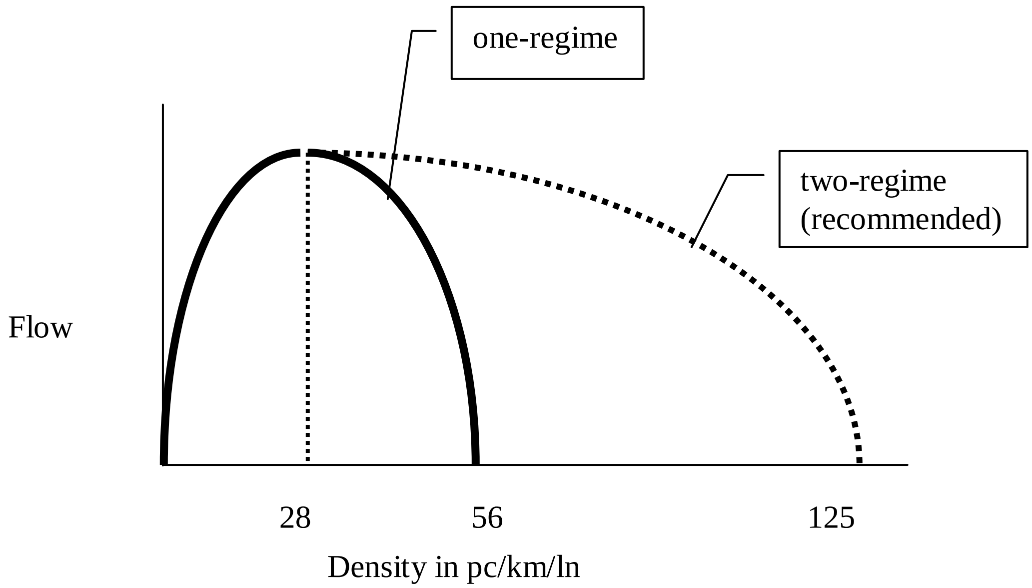

Figure 5 Flow-density relationship for the two-regime linear speed-density relationship

Similarly, one could obtain the relationship between free-flow speed (v_{free}) and speed at capacity (v_{cap}) for different roadway conditions (i.e. rural to urban) using columns 1 and 2 in Table 2 from the Highway Capacity Manual (2000). This will result in v_{free} values ranging between 1.10 and 1.41 times v_{cap} for urban to rural freeways, respectively.

7 Conclusions

The assumption that speed-density relationship is linear is widely used due to its simplistic representation of the behaviour of traffic. A more realistic approach to speed-density models is to use the two-regime linear form to take into account the effect of stable and unstable conditions. This relationship gives a non-symmetrical partial elliptical shape for the speed-flow relationship as shown in Equations 10a and 10b. These equations are recommended for use in relevant Traffic Engineering text books when describing speed-flow-density relationships since they give better representation of traffic behaviour for stable and unstable conditions.

Maximum density of 45 pc/mi/ln (i.e. 28 pc/km/ln) could be used as the boundary value for the stable traffic condition. Jam density values calculated from the Highway Capacity

Manual (2000) are in the region of 90 pc/mi/ln (i.e. 56 pc/km/ln). This yields average distance headways of 18 meters when dealing with stationary queues which is relatively high. However, it is more realistic to obtain jam density in the region of 200 pc/mi/ln (i.e. 125 pc/km/ln) which are nearly double those figures obtained from the Highway Capacity Manual (2000). The parameters used for the proposed partial elliptical model could be tested and validated using data from various sites operating under free, medium and congested traffic conditions.

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