Derivation of the field due to a magnetic dipole without use of the vector potential

J.E. Proctor and H.T. Gould, Materials & Physics Research Group, University of Salford, Manchester M5 4WT, United Kingdom

Manuscript published in The Physics Teacher (https://doi.org/10.1119/5.0077127).

Introduction

The mathematical form of the magnetic field due to a current loop, and the fact that it is identical to the electric field due to an electric dipole in the far-field, are fundamental to our understanding of electromagnetism. Whilst undergraduate level electromagnetism textbooks usually derive the electric field from an electric dipole, few derive the magnetic field from a current loop. Most simply state it without proof, or do the derivation for simpler cases such as the on-axis field. Those that do the derivation [1] utilize the magnetic vector potential, a relatively advanced concept that most undergraduate students would not encounter until their final year of study, if at all. Here a simple derivation to obtain the magnetic field due to a current loop in the far-field approximation is presented. The derivation begins from the Biot-Savart law and does not require the vector potential. The problem is set up so that only a single integration is necessary (from angle $\alpha = 0$ to $\alpha = 2\pi$ around the current loop) and the result is compared to that for the electric field surrounding an electric dipole.

Derivation

Our mathematical setup for the problem is shown in figure 1. We consider a current loop of radius a centered on the origin in the xy plane carrying a current in the anticlockwise direction looking down from the positive z direction. The vector $\boldsymbol{a}$ leads from the origin to an arbitrary point on the current loop and the angle α is the angle between $\boldsymbol{a}$ and the x-axis. The vector $d\boldsymbol{L}$ is the section of the current loop swept out when α increases to $(\alpha + d\alpha)$.

The vectors $\boldsymbol{a}$ and $d\boldsymbol{L}$ are therefore defined as follows:

$$\boldsymbol{a} = \begin{pmatrix} a\cos\alpha \\ a\sin\alpha \\ 0 \end{pmatrix}$$

$$d\boldsymbol{L} = a\,d\alpha \begin{pmatrix} -\sin\alpha \\ +\cos\alpha \\ 0 \end{pmatrix} \tag{1}$$

We are going to obtain the magnetic field at point P, at location $\boldsymbol{r}_0$ from the origin. We write the cartesian components of $\boldsymbol{r}_0$ in terms of the angle θ between $\boldsymbol{r}_0$ and the z axis. This is in preparation to compare our results to those for the electric field around an electric dipole, which is most easily obtained in spherical polar co-ordinates. Since the current loop has cylindrical symmetry about the z axis the same will be true of the magnetic field so we can set up $\boldsymbol{r}_0$ in the xz plane with no loss of generality. In this case:

$$\boldsymbol{r}_0 = \begin{pmatrix} r_0 \sin\theta \\ 0 \\ r_0 \cos\theta \end{pmatrix}$$

(2)

The magnetic field at P due to section $d\boldsymbol{L}$ of the current loop is given by the Biot-Savart law:

$$d\boldsymbol{B} = \frac{\mu_0 I d\boldsymbol{L} \times \boldsymbol{r}}{4\pi r^3}$$

(3)

Which will be integrated around the entire current loop as follows:

$$\boldsymbol{B} = \int \frac{\mu_0 I d\boldsymbol{L} \times \boldsymbol{r}}{4\pi r^3}$$

(4)

Here, $\boldsymbol{r}$ is the vector leading directly from the location of $d\boldsymbol{L}$ to P. It therefore does have a y-component, but can at least be written in terms of $\boldsymbol{r}_0$ and $\boldsymbol{a}$:

$$\boldsymbol{r} = -\boldsymbol{a} + \boldsymbol{r}_0 = \begin{pmatrix} -a\cos\alpha + r_0 \sin\theta \\ -a \sin\alpha \\ r_0 \cos\theta \end{pmatrix}$$

(5)

Utilizing equations 1, 2 and 5, we now have all the parameters required for the integration of equation (4) expressed in terms of constants, the angle α and it's differential $d\alpha$. All that remains is to use the far-field approximation to make the integral analytically solvable. We require the exact expression for $\boldsymbol{r}$ in the numerator of the Biot-Savart law, but can simplify the $1/r^3$ term using a binomial expansion. Obtaining r^2 directly from equation (5) using Pythagoras with no approximations leads to:

$$r^2 = r_0^2 + a^2 - 2ar_0 \sin\theta\cos\alpha$$

(6)

At this stage it is reasonable to approximate that $r_0^2 \gg a^2$ in the equation above, allowing us to express r^2, then $1/r^3$, in a form suitable for a binomial expansion:

$$\frac{1}{r^3} \approx \frac{1}{r_0^3}\left[1 - \frac{2a}{r_0}\sin\theta\cos\alpha\right]^{-\frac{3}{2}}$$

(7)

The $r_0^2 \gg a^2$ approximation above has already led to the loss of some second-order terms so we perform the expansion to first order and obtain:

$$\frac{1}{r^3} \approx \frac{1}{r_0^3}\left[1 + \frac{3a}{r_0}\sin\theta\cos\alpha\right]$$

(8)

Next, we need to evaluate $d\boldsymbol{L} \times \boldsymbol{r}$ using equations 1 and 5:

$$d\boldsymbol{L}\times\boldsymbol{r}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ -ad\alpha\sin\alpha & +ad\alpha\cos\alpha & 0\\ -a\cos\alpha+r_0\sin\theta & -a\sin\alpha & r_0\cos\theta\end{vmatrix}$$

$$d\boldsymbol{L}\times\boldsymbol{r}=\begin{pmatrix}ar_0\cos\alpha\cos\theta\\ ar_0\sin\alpha\cos\theta\\ a^2-ar_0\cos\alpha\sin\theta\end{pmatrix}d\alpha$$

(9)

Combining this with our expression for $1/r^3$ from equation 8 we can write out all terms in the integral (4) that depend on α:

$$\frac{d\boldsymbol{L}\times\boldsymbol{r}}{r^3}\approx\frac{1}{r_0^3}\left[1+\frac{3a}{r_0}\sin\theta\cos\alpha\right]\begin{pmatrix}ar_0\cos\alpha\cos\theta\\ ar_0\sin\alpha\cos\theta\\ a^2-ar_0\cos\alpha\sin\theta\end{pmatrix}d\alpha$$

(10)

Multiplying out the brackets leads to:

$$\frac{d\boldsymbol{L}\times\boldsymbol{r}}{r^3}\approx\frac{1}{r_0^3}\begin{pmatrix}ar_0\cos\alpha\cos\theta+3a^2\sin\theta\cos\theta\cos^2\alpha\\ ar_0\sin\alpha\cos\theta+3a^2\sin\theta\cos\theta\sin\alpha\cos\alpha\\ a^2-ar_0\sin\theta\cos\alpha-3a^2\sin^2\theta\cos^2\alpha\end{pmatrix}d\alpha$$

(11)

Here, we have excluded $3a^3\sin\theta\cos\alpha/r_0$ term from the z-component since the approximations made leading to equation (8) have already led to the exclusion of other terms to this order. In any case, all terms dependent on $\cos\alpha$ or $\sin\alpha$ will disappear when we integrate α from 0 to 2π due to the symmetry of these functions. This is also the case for the term dependent on the product $\sin\alpha\cos\alpha$, as is evident from applying the trigonometrical identity $\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha$. Removing these terms leads to a vastly simplified equation:

$$\frac{d\boldsymbol{L}\times\boldsymbol{r}}{r^3}\approx\frac{a^2}{r_0^3}\begin{pmatrix}3\sin\theta\cos\theta\cos^2\alpha\\ 0\\ 1-3\sin^2\theta\cos^2\alpha\end{pmatrix}d\alpha$$

(12)

Substituting this into equation 4 leads to:

$$\boldsymbol{B}\approx\int_0^{2\pi}\frac{\mu_0 I a^2}{4\pi r_0^3}\begin{pmatrix}3\sin\theta\cos\theta\cos^2\alpha\\ 0\\ 1-3\sin^2\theta\cos^2\alpha\end{pmatrix}d\alpha$$

(13)

Since $\int_0^{2\pi}\cos^2\alpha\,d\alpha=\pi$, this leads directly to our final result for the magnetic field due to a current loop under the approximation $r_0^2\gg a^2$:

$$\boldsymbol{B}=\frac{\mu_0 I a^2}{4\pi r_0^3}\begin{pmatrix}3\pi\sin\theta\cos\theta\\ 0\\ 2\pi-3\pi\sin^2\theta\end{pmatrix}$$

(14)

If preferred, this can be written in terms of the magnetic dipole moment $m = IA = I\pi a^2$:

$$\boldsymbol{B} = \frac{\mu_0 m}{4\pi r_0^3}\begin{pmatrix} 3\sin\theta\cos\theta \\ 0 \\ 2 - 3\sin^2\theta \end{pmatrix} \tag{15}$$

Comparison to the electric dipole

The simplest form of the derivation for the electric field due to an electric dipole is in spherical polar co-ordinates [2], leading to:

$$\boldsymbol{E} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{p}{r_0^3}\left[(2\cos\theta)\hat{\boldsymbol{r}} + (\sin\theta)\hat{\boldsymbol{\theta}}\right] \tag{16}$$

Where p is the electric dipole moment. Labelling the contents of the square bracket, that describe the θ-dependence of $\boldsymbol{E}$, as $E_r = 2\cos\theta$ and $E_\theta = \sin\theta$, it is straightforward to convert this to cartesian co-ordinates, for locations in the xz plane:

$$\left[E_r\hat{\boldsymbol{r}} + E_\theta\hat{\boldsymbol{\theta}}\right] \rightarrow \begin{pmatrix} E_r\sin\theta + E_\theta\cos\theta \\ 0 \\ E_r\cos\theta - E_\theta\sin\theta \end{pmatrix}$$

$$\boldsymbol{E} = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{p}{r_0^3}\begin{pmatrix} 3\sin\theta\cos\theta \\ 0 \\ 2 - 3\sin^2\theta \end{pmatrix} \tag{17}$$

This is identical to our result (equation 14) for the magnetic field, save for the changes $m \rightarrow p$ and $1/\varepsilon_0 \rightarrow \mu_0$.

Related problems

The determinant in equation 9 is a good place to begin for tackling related problems in electromagnetism. In particular, if we extract the z-component only from the determinant and proceed with $\theta = 0$ but without making the $r_0^2 \gg a^2$ approximation we obtain the correct expression for the magnetic field at all locations along the z-axis. From this, the magnetic field along the axis of a solenoid can be obtained.

Conclusions

The derivation presented here for the magnetic field due to a magnetic dipole arrives at the correct result directly from the Biot-Savart law, without using the vector potential. The only calculus required is the integration of the sine and cosine functions, and the only approximation made is the far-field approximation $r_0^2 \gg a^2$. The proof should generally be suitable for lecture material at second year undergraduate level and can assist students in understanding magnetic phenomena on vastly different length scales, from the magnetic moments of subatomic particles to the earth's magnetic field. When combined with the derivation of the electric field originating from an electric dipole (equation 16) it provides students with a rigorous understanding of how the analogy between the electric and magnetic dipoles arises.

For completeness, we should mention that the magnetic dipole field was recently obtained using a completely different method beginning from the concept of mutual inductance [3].

References

[1] P. Lorrain, D.R. Corson and F. Lorrain, *Electromagnetic Fields and Waves*, Freeman (1988).

[2] W.J. Duffin, *Electricity and Magnetism*, McGraw-Hill (1990).

[3] J. Wang, Phys. Teach. **58**, 590 (2020).

Figure 1. Notation for calculation of the magnetic field at P (in the xz plane) due to a current loop located in the xy plane.