Vector cnoidal waves in spatiotemporal propagation: exact solutions beyond slowly-varying envelopes

J. M. Christian and É. N. McAteer
Joule Physics Laboratory, University of Salford, Greater Manchester M5 4WT, United Kingdom

TALK ABSTRACT

Cnoidal waves are periodic solutions to certain classes of nonlinear partial differential equations (PDEs). They are fundamental excitations in theories of waves and their more familiar localized counterparts—solitons—typically emerge as limits. Here, attention is focused primarily on the context of nonlinear optical waves confined to two-dimensional waveguides.

We consider an extension of the scalar spatiotemporal propagation model with cubic nonlinearity [J. M. Christian *et al.*, Phys. Rev. A vol. 86, art. nos. 023838 and 023839 (2012)] into two-component vector regimes. The model retains the double-z derivative terms, where z denotes the longitudinal space coordinate, and waves are most conveniently described in the laboratory frame of reference rather than in a local-time frame. It captures the spatial dispersion effect present in some novel semiconductor materials, and it also possesses inherent bidirectionality (where both forward- and backward-propagating solutions are supported). Our spatiotemporal formulation is thus rather general in nature, going well beyond the slowly-varying envelope approximation (SVEA) as prescribed so widely in the literature by nonlinear Schrödinger-type equations. Moreover, it has an instructive geometrical representation in the space-time plane that is tightly connected to coordinate transformations akin to those routinely encountered in special relativity.

Despite the complexity of the model—two coupled cubically-nonlinear PDEs that are fully second-order in (longitudinal) space and time coordinates, alongside an eight-dimensional parameter space—it is still possible to derive families of exact analytical vector cnoidal waves. These solutions involve Jacobi elliptic functions of which there are just three types: elliptic cosine (cnoidal, or cn), elliptic sine (snoidal, or sn), and delta amplitude (dnoidal, or dn). There are six fundamental combinations that need to be considered: three composed of similar elements (cn-cn, dn-dn, sn-sn), and three composed of mixed elements (cn-dn, cn-sn, dn-sn).

The presentation will survey our new spatiotemporal vector cnoidal wave solutions (obtained by deploying an ansatz method) and show that their existence is contingent upon two complementary conditions being satisfied simultaneously. Those auxiliary criteria demand that the components have equal velocities and that their overlap in the space-time plane be optimized. The double-z derivatives endow the solutions with a set of corrections to the conventional predictions for propagation constants, velocities, and pulse widths. As required physically and mathematically, such corrections vanish in a multiple limit indicating that predictions made under the 'SVEA + local-time' paradigm are, in essence, special cases. We will conclude by demonstrating the asymptotic recovery of corresponding vector solitons.

Keywords: Cnoidal waves, optical waveguides, solitons, spatial dispersion.