Dynamics on $\mathbb{C}$ with generalized Newton-Raphson maps: Julia sets, uncertainty dimension, and periodic orbits

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POSTER ABSTRACT

The Newton-Raphson (NR) method is a well-known iterative scheme for approximating the roots of functions. Deployed on the complex plane, $\mathbb{C}$, perhaps its most famous application is to finding the cube roots of –1. One often regards any specific outcome of the root-finding exercise as a secondary consideration, and instead interprets the NR map as a two-dimensional discrete process with nonlinear feedback. In so-doing, an extremely rich spectrum of dynamical phenomena emerges that includes fixed points, periodic points, instabilities, unpredictability, fractal basin boundaries, and the Wada property. Beyond their abstract nature, these concepts underpin modern understandings of physics.

Here, we investigate a direct extension of the standard NR class of maps. The procedure generates a variant that is cubically convergent, but where the computational overhead involves additional function evaluations at each iteration (including an awkward square-root operation). Binomial expansions within that generalized formulation lead to a hierarchy of nonlinear maps on $\mathbb{C}$, the lowest order of which recovers the standard-NR algorithm. Each successively higher-order map can be regarded as a dynamical system in its own right; for example, the next order corresponds to the Schröder algorithm [A. S. Househölder, *Principles of Numerical Analysis* (Dover, New York, 1974)]. It is this hierarchy and the interconnected properties of its constituent maps that are of interest to us.

A systematic survey of results will be presented for a range of the lowest-order maps, with particular emphasis placed on problems involving the Nth roots of –1. New families of fixed points have been found in addition to the expected N roots, and the basins of attraction computed for each map. Furthermore, the uncertainty dimension of the basin boundaries (which are examples of Julia sets) have been estimated and parametrized by system properties. The inclusion of a complex relaxation parameter has facilitated a linear stability analysis of the fixed points. That parameter leaves the fixed points unchanged, but allows the fractal boundaries to be transformed (via stretching and twisting) into patterns with enhanced complexity. Finally, periodic solutions have been identified by deriving—and subsequently solving (with the aid of symbolic computation)—the polynomial equations that prescribe period-2 and period-3 orbits. Accordingly, new formulae may be deduced for predicting the number of period-M points in a given map (where $M = 1, 2, 3, \ldots$). These orbits are demonstrably unstable both analytically and numerically, and they are contained within the Julia set of the corresponding map.