Chapter 5

The leading order equivalence of Oseen's and Imai's representations in the far-field wake for steady two-dimensional flow

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Abstract. *Consider the far-field behind a body in a steady, two-dimensional uniform flow field. In the far-field the Oseen linearisation is valid, and in the far-field wake Imai's asymptotic expansion is applicable. The fundamental solution Green's function of the Oseen equation which represents a point force is called the Oseenlet. The drag and lift Oseenlets are given in [1], and from this representation we determine the corresponding Oseenlet vorticity and Oseenlet stream function. Imai [2] gives the velocity, vorticity and stream function in the far-field wake behind a body in terms of an asymptotic expansion. We shall show that the first terms in Imai's expansion are the same as the drag and lift Oseenlets when approximated in the wake. This demonstrates that Imai's and Oseen's treatments are the same to leading order, and from this we infer that the next order terms in Imai's expansion will correspond to the approximation of the next order terms in Oseen's linearisation. Future work will be to use Imai's result to infer the next order terms in the Oseen linearisation.*

5.1 Introduction

The ultimate objective is to find an approximation to the point force in Navier-Stokes flow, which we call the NSlet. This is a usefull solution for fluid problems for example, for modelling

oil flow exterior to a pipe. We may also find it can be used as a Green's function in the Green's integral representation in the same way as Eulerlets in Euler flow [7]. We start by investigating obtaining the NSlet from the terms in the Oseen linearisation and how these terms are linked to those in Imai's far-field approximation.
Oseen obtained the Oseenlet velocity in the early 1900's. The drag and lift Oseenlets are given in [1]. Moreover, in [1] Chadwick employed two methods, first the velocity was decomposed into potential and wake velocity and the second method used the Oseenlets. He further revealed that the Oseen velocity in the far field cannot be modelled using the Lamb -Goldstein method [10]-[11] rather by expanding each Oseenlets in a Taylor series. Thus, the velocity and pressure expansions in the far-field are obtained.
Moreover, Chadwick [4] also studied an experimental verification of an Oseen flow slender body theory by showing that the resulting lift equation had close agreement with experiment. It was further revealed that in the far-field region, the Navier–Stokes flow is expected to be approximated closely by Oseen flow. A comparison was made between the experiment, inviscid flow slender body theory, Chadwick's Oseen based flow theory and Jorgensen's extension [8] to viscous crossflow theory for slender bodies with elliptical cross-section. It was observed that the experiments follow a gradually increasing straight line variation which closely follows Oseen theory. On the other hand, Imai [2] gives the vorticity and stream function in the far-field wake behind a body in terms of an asymptotic expansion. In the present paper, from the drag and lift Oseenlet given in [1], we determine the corresponding Oseenlet vorticity and Oseenlet stream function. We also obtain the velocity, vorticity and stream function from Imai [2]. We then approximate both in the far-field and demonstrate that the Oseenlet velocity, vorticity and stream function are the same to leading order as Imai's first approximation.
Vorticity is defined as the curl of the velocity, $\omega = \nabla \times u$ where ∇ is the differential operator and u is the velocity. It is the local rotation of the fluid. It is an important derived variable in fluid dynamics that plays both mathematical and physical roles in understanding fluid dynamics problems. However, for the 2-D flows, the vorticity vector has only one non-zero component (in the x_3-direction) where the Cartesian coordinates are given as (x_1, x_2, x_3). In addition, stream function ψ also plays an important role in understanding fluid flow and therefore, it is important to understand its concept and derivation [6]. It is related to the velocity using the Cauchy-Riemann equations [5].

5.2 Governing equations

The Navier-Stokes equation for an incompressible fluid is given as:

$$\rho\, {u_j}^{\dagger} \frac{\partial {u_i}^{\dagger}}{\partial x_j} = -\frac{\partial p^{\dagger}}{\partial x_i} + \mu \frac{\partial^2 {u_i}^{\dagger}}{\partial x_j^2} \tag{5.1}$$

where ${u_i}^{\dagger}$ is the fluid velocity, $p^{\dagger}$ is pressure, ρ is the density of the fluid and μ is the dynamic coefficient of viscosity. In the far-field, the velocity tends to a uniform stream

$${u_i}^{\dagger} = U\delta_{i1} + u_i + {u_i}^{I} + ...$$

where ${u_i}^{I}$ is the second order linearisation, U is the uniform stream, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & otherwise, \end{cases}$

$U \gg O(u_i) \gg O(u_i{}^I) \gg ...$ and $'O'$ means 'of the order'. Indices $1 \leq i, j \leq 2$ where repeated index implies summation.
However, Oseen equation is obtained by linearising the Navier-Stokes equation. Thus, the far-field Oseen equation is given as

$$\rho\, U \frac{\partial u_i}{\partial x_1} = -\frac{\partial p}{\partial x_i} + \mu\, \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{5.2}$$

giving the solution $\omega = \Re \sum_{n=0}^{\infty} C_n e^{kx_1} K_n\,(kr)\, e^{ni\,\theta}$, where ω is the vorticity, K_n is the modified Bessel function of order n. From this, the far-field Imai's approximation to leading order in modified Bessel function expansion is $\omega = \Re\left(\frac{c}{\zeta}\, e^{-\eta^2}\right) + ...$, where r is the 2-D radius defined by $r = \sqrt{x_i x_i}, \theta$ is the 2-D polar angle $\theta = \cos^{-1}(\frac{x_1}{r})$, $\zeta = \xi + i\eta$, $\xi = (2kr)^{\frac{1}{2}} \cos \frac{1}{2}\theta$, $k = \frac{\rho U}{2\mu}$, $\eta = (2kr)^{\frac{1}{2}} \sin \frac{1}{2}\theta$, c is a complex constant, $\Re$ represents real part and i is the imaginary part.
The drag Oseenlet velocity as given by [1] is

$$u_i{}^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_i}\left(\ln r + \, e^{kx_1} K_0\,(kr)\right) - 2k\, e^{kx_1} K_0(kr)\delta_{i1}\right] \tag{5.3}$$

and the lift Oseenlet is

$$u_i{}^{(2)} = \frac{1}{2\pi\rho U}\varepsilon_{ij3}\frac{\partial}{\partial x_j}\left(\ln r + e^{kx_1} K_0(kr)\right) \tag{5.4}$$

where ε_{ijk} is called the Levi-Civita symbol. It is a tensor of rank three defined by

$$\varepsilon_{ijk} = \begin{cases} 0, & \text{if if any two labels are the same} \\ 1, & \text{if if i,j,k is an even permutation of 1,2,3} \\ -1, & \text{if if i,j,k is an odd permutation of 1,2,3 .} \end{cases}$$

Meanwhile, Imai's approximation as given in [2] is

$$\psi = \frac{\Gamma}{2\pi} \log r - \frac{m}{2}\left(erf\eta - \frac{\theta}{\pi}\right) \tag{5.5}$$

where $m = \Gamma = \frac{1}{\rho}$ for unit force and $k(r - x_1) = \eta^2 \approx k\frac{x_2^2}{2x_1}$ in the wake.

5.3 Vorticity Evaluation

5.3.1 The drag Oseenlet vorticity

From (5.3), it follows that

$$u_1{}^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_1}\left(\ln r + \, e^{kx_1} K_0(kr)\right) - 2k\, e^{kx_1} K_0\,(kr)\right] \tag{5.6}$$

and

$$u_2{}^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_2}\left(\ln r + \, e^{kx_1} K_0\,(kr)\right)\right] \tag{5.7}$$

Therefore, to derive the vorticity, we use the expression

$$\omega = \left[\nabla \times u_i^{(1)}\right]_3 = \frac{\partial}{\partial x_1}\, u_2^{(1)} - \frac{\partial}{\partial x_2}\, u_1^{(1)} \tag{5.8}$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial x_2}\, u_1^{(1)} &= \frac{1}{2\pi\rho U}\frac{\partial}{\partial x_2}\{\frac{\partial}{\partial x_1}(\ln r + e^{kx_1}K_0(kr)) - 2k\, e^{kx_1}K_0\,(kr)\} \\
\frac{\partial}{\partial x_1}\, u_2^{(1)} &= \frac{1}{2\pi\rho U}\frac{\partial}{\partial x_1}\{\frac{\partial}{\partial x_2}(\ln r + e^{kx_1}K_0(kr))\}.
\end{aligned}$$

However, substituting these into (5.8) and simplifying gives

$$\omega = \frac{1}{\pi\rho U}\{\, e^{kx_1}\frac{\partial}{\partial x_2}\,(K_0(kr))\}\,. \tag{5.9}$$

Therefore, (5.9) is the expression for the Oseenlet vorticity.

5.3.2 Equivalence of Oseen's and Imai's vorticity in far-field

To demonstrate that the Oseenlet vorticity and Imai's vorticity are equivalent, we approximate (5.9) in the far-field and compare with Imai's vorticity where

$$e^{kx_1}\frac{\partial}{\partial x_2}\,(K_0(kr)) \approx -\frac{kx_2}{r}\sqrt{\frac{\pi}{2kr}}\; e^{-k(r-x_1)}.$$

Now substituting into (5.9) we have

$$\omega \approx -\frac{1}{\pi\rho U}\{\frac{k^2 x_2}{r}\sqrt{\frac{\pi}{2kr}}\; e^{-k(r-x_1)}\}. \tag{5.10}$$

Therefore, (5.10) is the far-field Oseenlet vorticty, while Imai's vorticity as given in [2] given is

$$\omega = -\frac{2k^2}{\rho U\sqrt{\pi}}\frac{\eta}{\xi^2+\eta^2}\; e^{-\eta^2}. \tag{5.11}$$

Therefore, putting $k(r-x_1) = \eta^2 \approx k\frac{x_2^2}{2x_1}$, $r = \frac{\eta^2+\xi^2}{2k}$, $\xi = (2kr)^{\frac{1}{2}}\cos\frac{1}{2}\theta$ and $\eta = (2kr)^{\frac{1}{2}}\sin\frac{1}{2}\theta$ into (5.10), it is shown that Oseenlet vorticity is the same as Imai's approximation of vorticity.

5.4 Stream function Evaluation

5.4.1 Drag Oseenlet stream function

From the Cauchy-Riemann equations, the drag Oseenlet velocity (5.3) can be written in terms of the stream function such that

$$\psi = \int_{X_i}^{x_i} \varepsilon_{3ij}\, u_i^{(1)}\, dx_j^{'}$$

where ψ is the stream function, x_j' is the variable of integration and $u_i^{(1)}$ is the drag Oseenlet velocity. The constant X_i is chosen such that $\psi(x_i) = 0$. Integrating along x_1 only gives $\psi = -\int_{X_1}^{x_1} u_2' dx_1'$, this implies that

$$\psi = -\frac{1}{2\pi\rho U}\int_{X_1}^{x_1} \frac{\partial}{\partial x_2}(\ln r' + e^{kx_1'} K_0\left(kr'\right))dx_1' \tag{5.12}$$

where $r' = \sqrt{x_1'^2 + x_2^2}$. However, (5.12) consist of the potential term and the wake term. i.e

$$\psi = -\frac{1}{2\pi\rho U}\left(\psi^{pot} + \psi^{wake}\right)$$

where $\psi^{pot} = \int_{X_1}^{x_1} \frac{\partial}{\partial x_2}(\ln r')dx_1'$ gives the velocity potential and $\psi^{wake} = \int_{X_1}^{x_1} \frac{\partial}{\partial x_2}(e^{kx_1'} K_0\left(kr'\right))dx_1'$ give the wake velocity. To evaluate the potential term, we use complex analysis, letting $z = x_1 + ix_2, z' = x_1' + ix_2$ and $Z = X_1 + ix_2$, see 5.1.

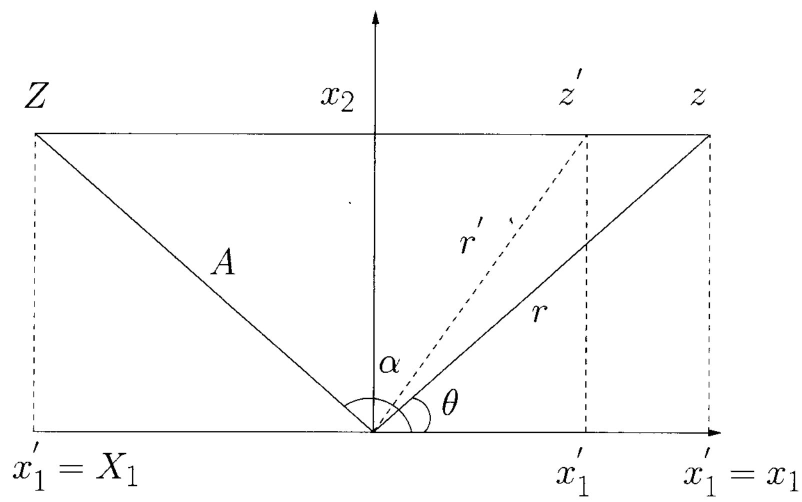

Figure 5.1: Points of integration

Consider $\Psi = \frac{d}{dz}\ln z = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r} = \frac{r\cos\theta - ir\sin\theta}{r^2}$. Therefore, the negative of the imaginary part of Ψ gives $-Im\{\frac{d}{dz}\ln z\} = \frac{x_2}{r^2} = \frac{\partial}{\partial x_2}\ln r$. So, $\psi^{pot} = -Im\{\int \frac{\partial}{\partial z'}\ln z' dz\} = -Im\{\left[\ln z'\right]_Z^z\} = -Im\{\ln z - \ln Z\} = -Im\{\ln r + i\theta - \ln A - i\alpha\}$ $= -\theta + \alpha$. As ψ^{pot} is a potential, we can ignore the constant α, and without loss of generality we can let

$$\psi^{pot} = -\theta. \tag{5.13}$$

The wake term from (5.12) can be determined from $\psi^{wake} = -\frac{1}{2\pi\rho U}\int_{X_1}^{x_1} \frac{\partial}{\partial x_2}(\ e^{kx_1'} K_0\left(kr'\right))dx_1'$, but $\frac{\partial}{\partial x_2} K_0\left(kr\right) = -\frac{kx_2}{r} K_1(kr) \approx -\frac{kx_2}{r}\sqrt{\frac{\pi}{2kr}}\, e^{-kr}$ from Abramowitz and Stegun [9]. Therefore, the approximation to the wake term is obtained as

$$\psi^{wake} \approx -\frac{1}{2\pi\rho U}\int_{X_1}^{x_1} \{\frac{kx_2}{r'}\sqrt{\frac{\pi}{2kr'}}\, e^{-k(r'-x_1')}\}dx_1'. \tag{5.14}$$

Finally, the expression for the Oseenlet stream function is obtained by adding (5.13) and (5.14)

$$\psi = \frac{1}{2\pi\rho U}\{\theta - \int_{X_1}^{x_1} \{\frac{kx_2}{r'}\sqrt{\frac{\pi}{2kr'}}\, e^{-k(r'-x_1')}\}dx_1'\}. \tag{5.15}$$

5.4.2 Equivalence of Oseen's and Imai's stream function in far-field

The Oseenlet stream function is given by (5.15) and that of Imai's stream function [2] is given as

$$\psi = \frac{1}{2\pi\rho U}\{\theta - \pi erf\eta\} \tag{5.16}$$

To show that Imai's stream function and Oseenlet stream function are equivalent, recall that $k(r - x_1) = \eta^2 \approx k\frac{x_2^2}{2x_1}$ and $erf\eta = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta'^2} d\eta'$, so let $\psi_1 = \frac{1}{2\pi\rho U}\int_{X_1}^{x_1} \frac{kx_2}{r'}\sqrt{\frac{\pi}{2kr'}}\, e^{-k(r'-x_1')}dx_1'$, where $\psi = \frac{\theta}{2\pi\rho U} - \psi_1$, implies that $\psi_1 \approx \frac{1}{2\pi\rho U}\frac{2}{\sqrt{\pi}}\int_0^{\eta'} \eta' e^{-\eta'^2}\frac{2}{\eta'}d\eta'$ thus, $\psi_1 = \frac{1}{2\rho U}erf\eta$. Therefore,

$$\psi = \frac{1}{2\pi\rho U}\{\theta - \pi erf\eta\}. \tag{5.17}$$

Thus, equation (5.17) is the Oseenlet stream function and it is shown that it is equivalent to the Imai's stream function given in (5.16).

5.5 Velocity

5.5.1 Drag Oseenlet velocity

Recall that the drag Oseenlet is given in (5.3) as $u_i^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_i}\left(\ln r + e^{kx_1}K_0(kr)\right) - 2k\, e^{kx_1}K_0(kr)\,\delta_{i1}\right]$ where

$$u_1^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_1}\Big(\ln r + e^{kx_1}K_0(kr)\Big) - 2k\, e^{kx_1}K_0(kr)\right] \tag{5.18}$$

and

$$u_2^{(1)} = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_2}\Big(\ln r + e^{kx_1}K_0(kr)\Big)\right]. \tag{5.19}$$

Recall that Imai's first approximation to the stream function is $\psi = \frac{1}{2\pi\rho U}\{\theta - \pi erf\eta\}$ where $m = \frac{1}{\rho U}$. We can now obtain the velocity using the relation $u_i = \varepsilon_{ij3}\frac{\partial\psi}{\partial x_j}$. It follows that $u_1 = \frac{\partial\psi}{\partial x_2}$ and $u_2 = -\frac{\partial\psi}{\partial x_1}$. Therefore, Imai's velocity can be represented as

$$u_i^{(1)} = \varepsilon_{ij3}\frac{\partial}{\partial x_j}\{-\frac{1}{2\pi\rho U}erf\eta + \frac{\theta}{\pi}\} \tag{5.20}$$

so

$$u_1^{(1)} = -\frac{1}{2\pi\rho U}(\pi\frac{\partial}{\partial x_2}erf\eta - \frac{\partial\theta}{\partial x_2}) \tag{5.21}$$

and

$$u_2^{(1)} = \frac{1}{2\pi\rho U}(\pi\frac{\partial}{\partial x_1}erf\eta - \frac{\partial\theta}{\partial x_1}) \tag{5.22}$$

5.5.2 Equivalence of Oseen's and Imai's velocity in the far-field

To show that the Oseen's velocity is equivalent to Imai's velocity, we first show that the first components u_1 from equation (5.18) and (5.21) are equivalent. Similarly, the second components u_2 from equation (5.19) and (5.22) are also equivalent. Now, the first terms give $\frac{\partial(\ln r)}{\partial x_1} = \frac{x_1}{r^2}$ and $\frac{\partial \theta}{\partial x_2} = \frac{x_1}{r^2}$. These are shown to be the same. However, the next terms in the velocity components give, $\frac{\partial}{\partial x_1}\left(e^{kx_1} K_0(kr)\right) - 2k\, e^{kx_1} K_0(kr) \approx \frac{\partial}{\partial x_1}(e^{kx_1} \frac{\sqrt{\pi}}{\sqrt{2kr}} e^{-kr}) - 2k e^{kx_1} \frac{\sqrt{\pi}}{\sqrt{2kr}} e^{-kr}$. But $\frac{\partial}{\partial x_1}(\frac{1}{\sqrt{r}}) = \frac{x_1}{2r^2\sqrt{r}} = O(\frac{1}{r\sqrt{r}})$ which is of lower order as $r \longrightarrow \infty$. Therefore,
$\frac{\partial}{\partial x_1}(e^{kx_1} K_0(kr)) - 2k\, e^{kx_1} K_0(kr) = \{\frac{kx_1}{r}\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)} + k\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)} - 2k\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)}\}$.
Simplifying gives

$$\frac{\partial}{\partial x_1}(e^{kx_1} K_0(kr)) - 2k\, e^{kx_1} K_0(kr) \approx \frac{-k(r-x_1)}{r}\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)}$$

Recall that $k(r-x_1) = \eta^2 \approx k\frac{x_2^2}{2x_1}$ and $r = \frac{\eta^2+\xi^2}{2k}$, $[\xi = (2kr)^{\frac{1}{2}}\cos\frac{1}{2}\theta, \eta = (2kr)^{\frac{1}{2}}\sin\frac{1}{2}\theta]$ where $\xi^2+\eta^2 = 2kr, \sin^2\frac{1}{2}\theta = O(1)$.Thus,

$$\frac{\partial}{\partial x_1}(e^{kx_1} K_0(kr)) - 2k\, e^{kx_1} K_0(kr) \approx -2k\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)}. \tag{5.23}$$

Therefore, substituting into (5.18) give

$$\begin{aligned}\frac{\partial}{\partial x_1}(e^{kx_1} K_0(kr) - 2k\, e^{kx_1} K_0(kr)) &\approx \frac{1}{2\pi\rho U}(-2k\sqrt{\frac{\pi}{2kr}}\, e^{-k(r-x_1)}) \\ &\approx -\frac{\sqrt{k}}{\rho U\sqrt{2\pi x_1}}\, e^{-\eta^2}.\end{aligned}$$

Therefore,

$${u_1}^{(1)} \approx -\frac{\sqrt{k}}{\rho U\sqrt{2\pi x_1}}\, e^{-\eta^2}. \tag{5.24}$$

This is the second term in the Oseenlet velocity. However, evaluating the second term in Imai's velocity, we have $\pi\frac{\partial}{\partial x_2} erf\eta$, but $\frac{\partial}{\partial \eta}(erf\eta) = \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}$ Therefore, $\pi\frac{\partial}{\partial x_2} erf\eta = \pi\frac{\partial \eta}{\partial x_2}\frac{\partial}{\partial \eta}(erf\eta)$, but $\eta \approx \sqrt{\frac{k}{2x_1}}x_2$, implies that $\frac{d\eta}{dx_2} = \sqrt{\frac{k}{2x_1}}$, therefore

$${u_1}^{(1)} \approx -\frac{1}{2\rho U}\frac{\partial}{\partial x_2} erf\eta = -\frac{\sqrt{k}}{\rho U\sqrt{2\pi x_1}}\, e^{-\eta^2}. \tag{5.25}$$

Therefore, it is shown that the first components of Imai and Oseenlet velocity are equivalent. Now, we can show that the second components are also equivalent. The second components of Oseenlet and Imai's velocity are given in (5.19) and (5.22) respectively. It can be seen that $\frac{\partial(\ln r)}{\partial x_2} = \frac{x_2}{r^2}$, similarly $-\frac{\partial\theta}{\partial x_1} = \frac{x_2}{r^2}$. Meanwhile, the second term in the Oseenlet velocity as given in (5.19) gives $\frac{\partial}{\partial x_2}(e^{kx_1} K_0(kr)) \approx (e^{kx_1}\frac{\sqrt{\pi}}{\sqrt{2kr}} e^{-kr})$, but $\frac{\partial}{\partial x_2}(\frac{1}{\sqrt{r}}) = \frac{x_2}{2r^2\sqrt{r}} = O(\frac{1}{r\sqrt{r}})$ which is of lower order as $r \longrightarrow \infty$ and $\frac{\partial}{\partial x_2}(e^{-kr}) = -k\frac{x_2}{r}e^{-kr} = O(1)$. Therefore,

$$\begin{aligned}\frac{\partial}{\partial x_2}(\, e^{kx_1}K_0(kr)) &\approx \frac{\partial}{\partial x_2}(\, e^{kx_1}\frac{\sqrt{\pi}}{\sqrt{2kr}}e^{-kr}) \\ &\approx [(\, e^{kx_1}\frac{\sqrt{\pi}}{\sqrt{2kr}}e^{-kr})\frac{\partial}{\partial x_2}e^{-kr}](1+O(\frac{1}{r\sqrt{r}})) \\ &\approx (\, e^{kx_1}\frac{\sqrt{\pi}}{\sqrt{2kr}}(\frac{-kx_2}{r}e^{-kr})) \\ &\approx -\frac{\sqrt{\pi}}{x_1}\eta\, e^{-\eta^2}.\end{aligned}$$

Thus,

$$\frac{\partial}{\partial x_2}(\, e^{kx_1}K_0(kr)) \approx -\frac{\sqrt{\pi}}{x_1}\eta\, e^{-\eta^2}. \tag{5.26}$$

Therefore, (5.26) is the second term in the Oseenlet velocity. However, Imai's second term can be evaluated as $\pi\frac{\partial}{\partial x_1}erf\eta$, but $\frac{\partial}{\partial \eta}(erf\eta) = \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}$ Therefore, $\pi\frac{\partial}{\partial x_1}erf\eta = \pi\frac{\partial \eta}{\partial x_1}\frac{\partial}{\partial \eta}(erf\eta)$, but $\eta \approx \sqrt{\frac{k}{2x_1}}x_2$, implies that $\frac{d\eta}{dx_1} = -\frac{1}{2}\frac{\eta}{x_1}$, therefore,

$$\pi\frac{\partial}{\partial x_1}erf\eta = -\frac{\sqrt{\pi}}{x_1}\eta\, e^{-\eta^2}. \tag{5.27}$$

Therefore, it is shown that Oseenlet velocity and Imai's first appoximation of velocity are equivalent.

5.6 Conclusion

We derived the Oseenlet vorticity and stream function, evaluated the far-field approximations and compared with Imai's first approximation of vorticity and stream function. We further obtained the far-field approximation of velocity from Imai's approximation of stream function and compared with the Oseenlet velocity. The results show that the derived Oseenlet vorticity, velocity and stream function are equivalent to the Imai's approximations to leading order.
The future work will be to infer the corresponding second order term in the Oseen linearisation

	Oseenlet	**Imai's approximation**
Velocity	$u_i = \frac{1}{2\pi\rho U}\left[\frac{\partial}{\partial x_i}\left(\ln r + e^{kx_1}K_0\left(kr\right)\right) -2k\, e^{kx_1}K_0\left(kr\right)\delta_{i1}\right]$	$u_i = \varepsilon_{ij3}\frac{\partial}{\partial x_j}\{-\frac{1}{2\pi\rho U}erf\eta + \frac{\theta}{\pi}\}$
Stream function	$\psi = \frac{1}{2\pi\rho U}\{\theta - \int_{X_1}^{x_1}\frac{\partial}{\partial x_2}\, e^{kx_1}K_0\left(kr\right)dx\}$	$\psi = \frac{1}{2\pi\rho U}\{\theta - \pi erf\eta\}$
Vorticity	$\omega = \frac{1}{\pi\rho U}\{\, e^{kx_1}\frac{\partial}{\partial x_2}\left(K_0(kr)\right)\}$	$\omega = -\frac{2k^2}{\rho U\sqrt{\pi}}\frac{\eta}{\xi^2+\eta^2}\, e^{-\eta^2}$

Table 5.1: Comparison between Oseenlet and Imai's first approximation

from Imai's second approximation to the Navier-Stokes equation. This is the second order term in the NSlet expansion, which has not yet been given in the literature to the best knowledge of the authors.

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