

## ORIGINAL RESEARCH

# A game theoretic approach to wireless body area networks interference control

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## Abstract

In this paper we consider a scenario where there are two wireless body area networks (WBANs) interfere with each other from a game theoretic perspective. In particular, we envision two WBANs playing a potential game to enhance their performance by decreasing interference to each other. Decreasing interference extends the sensors' batteries life time and reduces the number of re-transmissions. We derive the required conditions for the game to be a potential game and its associated the Nash equilibrium (NE). Specifically, we formulate a game where each WBAN has three strategies. Depending on the payoff of each strategy, the game can be designed to achieve a desired NE. Furthermore, we employ a learning algorithm to achieve that NE. In particular, we employ the Fictitious play (FP) learning algorithm as a distributed algorithm that WBANs can use to approach the NE. The simulation results show that the NE is mainly a function of the power cost parameter and a reliability factor that we set depending on each WBAN setting (patient). However, the power cost factor is more dominant than the reliability factor according to the linear cost function formulation that we use throughout this work.

## KEYWORDS

biomedical communication, body area networks, body sensor networks, decision theory, game theory, interference suppression, network coding, power control

## 1 | INTRODUCTION

One of the major challenges for WBANs (wireless body area networks) is the power consumption that represents a critical issue [1–3]. Although there are previous studies that addressed the interference procedures of WBANs, the power control problem is still a challenge for WBSNs [1]. The main motivation for addressing the power control problem is to control the undesired interference that degrades the quality of service.

In health applications, many devices are grouped in WBANs that can be categorised according to their functionality into three main classes. The first class is a wireless biosensor node that measures specific parameters collected from the human body [4]. The second type is a relay node used to represent an intermediate node in a WBAN, where it is responsible for forwarding biomedical data that it receives to other nodes in the WBAN until the data reaches the sink node.

The third type is the sink node that is responsible for the collection of all the information received from sensor nodes; also, it is called a personal device (wireless) [5].

The need for energy consumption of any biomedical sensor node stems from one of the following situations (states): sensing and processing, communication (receiving, transmitting), and sleeping state [6]. The energy consumption for transmission and reception in the body sensor network is presented in ref. [7] where the authors concluded that direct transmission is the least efficient strategy to adopt in the WBAN scenario. Generally speaking, there are two types of WBAN propagation models. The first is the line of sight (LOS) propagation, which applies along the front of the torso, and the second is the non-line of sight (NLOS) propagation that is considered around the torso [8]. Also, in the human body, the distances of biomedical sensor nodes have a direct impact on the energy consumption for WBANs.

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In addition, the topology of the WBAN affects the energy usage [1].

Relay nodes are added to a wireless body sensor network to reduce the transmission and reception powers of biomedical sensor nodes, to improve the network lifetime, to enhance the reliability, and to aggregate the bio-packets from sensor nodes to the sink nodes [7, 9]. In ref. [10], the authors proposed a relaying scheme to improve the energy efficiency and to enhance the lifetime of the body area network, as well as to reduce the energy consumption for the nodes that are closest to the sink node.

On the other hand, other authors employed the network coding scheme in the relay node which is used to enhance energy efficiency [11]. Also, researchers suggest that their proposed network coding scheme improves the energy efficiency [12]. Network coding has the added benefits of achieving less error rates and lower transmission power levels [13].

The results in ref. [14] show that in addition to node's energy saving, network coding can provide better channel utilisation and enhance the reliability through the extension to the IEEE 802.15.4 standard, and the proposed scheme employed the fixed traffic in a Markov chain [15]. Adaptive Algorithm to Optimise the Dynamics (AAOD) of IEEE 802.15.4 is presented to reduce energy consumption [14]. In WBSNs, the bio-sensor node receives the biomedical data from a patient to be processed and then communicated to a doctor or nurses to make a decision [16].

This study contributes a novel optimisation formulation through a potential game theory framework. The paper is organised as follows. Section 2 reviews the related work. Section 3 presents the proposed design for the system model. Section 4 shows and discusses the simulation results. Finally, conclusions are drawn in Section 5.

## 2 | RELATED WORK

The most related work to ours, from a game theoretic perspective, is [17] although the perspective of that work is different, but it provides the basis for ours. In ref. [17], players employ cooperate control through creating a potential function to optimise. Potential games are proposed in ref. [18] and have been introduced to be applied in wireless networks in the formulation given in ref. [19]. In ref. [20], the author provides more details on these games. Some recent applications for potential games in wireless networks can be seen, for example, in refs. [21, 22], with works with more detailed analysis provided in refs. [23, 24]. However, to the best of our knowledge, there is no work that involves applying potential games in WBANs modelling. On the other hand, a classical non-cooperative game has found applications in wireless networks interference management and resource allocation such as the work in refs. [25–31]. The seminal work in ref. [25] proposes using game theoretical models to reduce interference through pricing and power control. The work in ref. [26] describes, using the S-modular game for down-link, a power

control algorithm to reduce interference in Femtocell networks. The work by Sung and Fu [27] uses a non-cooperative game model to reduce interference in a non-orthogonal multiple access scenario. The authors in ref. [28] analyse two possible games played among wireless sensor network nodes either non-cooperatively or through a potential game formulation to avoid multiple transmission of the same data and increases the network sensing-communications efficiency. The authors in ref. [29] propose a new evolutionary game model played on a graph where the number of connected players can affect the game outcomes. The work in ref. [30] uses game theory to address the MAC layer access control by strategically adjusting the access probability of the CSMA/CA contention period. The authors in ref. [32, 33] show that a better interference control, by penalising high transmission powers, can be achieved by using a convex pricing cost function rather than a linear cost function. The work in ref. [31] deals with learning the game strategies that allow the coordinator node to choose the access time and transmit power for the WBAN nodes based on their states that are defined as functions of the signal-to-interference plus noise ratio, the transmission priority, the battery level, and the sensors' transmission delay. The work in ref. [34] uses stochastic geometry to model the position of the interference with a game theoretic power control algorithm. In WBANs, the interference management problem has been dealt with in many papers such as [35–39]. Wang and Cai in ref. [35] analysed the problem of co-existing WBANs in a dense environment through geometrical probability and quantified the minimum distance to achieve a desired signal-to-interference plus noise ratio. Zhang et al in ref. [37] formulated the power control problem as a semidefinite programme where there is a controller that has the ability to assign the power levels to different WBANs. Markov decision processes are recently proposed in ref. [38] to find the optimal transmission strategies as a compromise between the network delivery time and the packet delivery ratio while minimising the cost of transmission and the energy needed by using a single-hop or multi-hop transmission with power control. A recent direction in WBANs research is optimising both the transmission power through deep learning and enhancing the age of information (AoI) metric [39]. In particular, power control is used to enhance the signal-to-noise ratio where the assignment of the power levels is accomplished according to link prediction by deep learning; then, an AoI-aware scheduling algorithm is executed to strike a balance between energy efficiency and data freshness. The work in ref. [1, 2] proves the necessity of using relay nodes and network coding nodes to achieve high reliability for the information transmitted within the same WBAN. The results are derived according to each node's consumption, and the decision of the node strategies is assigned by the node coordinator. In this work, we take into account most of the above efforts and formulate a game that models a WBAN scenario where there are some strategies for each WBAN coordinator to choose from according to its interference level cost from another WBAN where both WBANs are assumed to play a potential game. We summarise our contributions as follows:

- We propose, for the first time, a potential game that models WBANs interaction with multiple strategies.
- We analyse the game equilibrium strategies and give their existence conditions.
- Our game model is general enough to be applied to an extended set of strategies. In our work, we consider three strategies that are stemmed from practical needs as has been shown in Refs. [1, 2].
- We propose a learning algorithm to achieve the solution of the game.
- We consider a scenario where the WBANs work in a hostile environment with high interference and simulate the network behaviour in it.

### 3 | SYSTEM MODEL

In this paper, we envision a scenario of two WBANs and each of them is coordinated by a node that is called a coordinator or sink as shown in Figure 1. Each one of these WBANs is responsible for coordinating transmission within a network of implanted devices such as EMG sensors, body temperature sensor, ECG sensor etc. Furthermore, we follow the WBAN literature assumption that there is no interference between sensors (nodes) within the same WBAN because of the existence of the coordinator node that is responsible for coordinating transmission between different nodes and equipped with a more powerful battery. Furthermore, the coordinator node can choose to coordinate the transmission of its network sensors using different strategies. In other words, the coordinator does not optimise the power control to reduce the unwanted interference to other WBANs. Specifically, we assume

that the coordinator can assign each node either to transmit with a high power level, a low power level and a relay that forwards the transmission or choose a low power level and network-coded transmission. The choice of the coordinator node, strategically, depends on the interference from the ambient that is assumed here to be another WBAN that has the same options for its sensors. However, each WBAN can set its sensor transmission strategy to be the highest power level, but this behaviour drains the transmitting node's battery and increases the interference to the other WBAN which may respond in a similar fashion and sets its sensor to transmit at the highest power level. As a result of this conflict, we strategically model the interaction between these two WBANs using a game theoretic model that takes in consideration all these contradictions. Each WBAN is considered as a player, and each transmission decision is a strategy for that player. Formally, any non-cooperative game  $\mathcal{G}$  is defined by  $\mathcal{G}(\mathbb{N}, \mathbb{S}, \mathbb{U})$  where  $\mathbb{N}$  is a set of players,  $\mathbb{S}$  is a set of strategies used by each player, and  $\mathbb{U}$  is a set of payoff functions [40]. Each player in the game  $\mathcal{G}$  has to choose a strategy from the set  $\mathbb{S}$  and each strategy has a payoff calculated using a payoff function, interchangeably called the utility function from the set  $\mathbb{U}$ . We define the payoff matrices whose entries are derived from the payoff function of each player as  $A$  for the first player (WBAN-1) and  $B$  for the second player (WBAN-2). Let  $\mathbb{N} = \{P1, P2\}$ , where WBAN-1 is  $P1$  and WBAN-2 is  $P2$ . In this case, the matrix dimensions are both  $3 \times 3$  since there are three strategies for each player. The set of strategies for the  $i$ th player that is  $\mathbb{S}$  consists of  $\{s_1^i, s_2^i, s_3^i\}_{i \in \{1,2\} \in \mathbb{N}}$ . Any game is usually described by its normal form as given in Table 1. This form shows the payoff for each player uses a specific strategy. For example, the entry  $a_{11}$  is the payoff of WBAN-1 when it

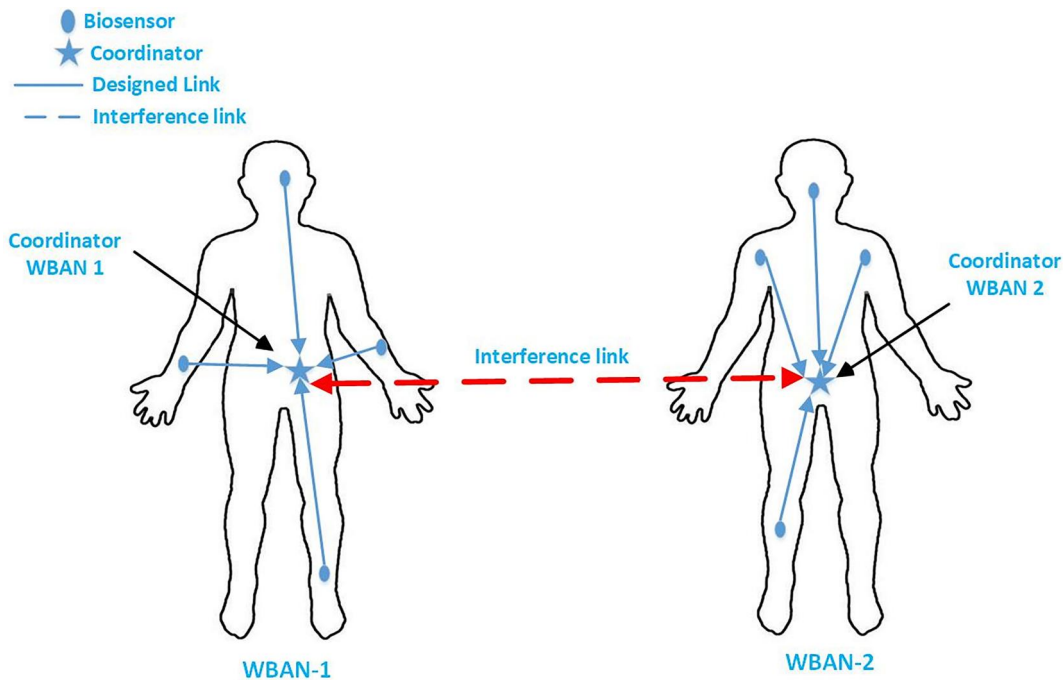


FIGURE 1 The envisioned two-WBAN scenario.

**TABLE 1** Normal form for the WBAN game.

	$s_1^2$ w.p. $x_1$	$s_2^2$ w.p. $x_2$	$s_3^2$ w.p. $x_3$
$s_1^1$ w.p. $y_1$	$a_{11}, b_{11}$	$a_{12}, b_{21}$	$a_{13}, b_{31}$
$s_2^1$ w.p. $y_2$	$a_{21}, b_{12}$	$a_{22}, b_{22}$	$a_{23}, b_{32}$
$s_3^1$ w.p. $y_3$	$a_{13}, b_{31}$	$a_{32}, b_{23}$	$a_{33}, b_{33}$

plays  $s_1^1$  against WBAN-2 playing  $s_2^1$ . In a similar fashion, we can interpret the rest of the entries. In the next section, we go over the reasoning and assumptions used to derive the payoff matrices  $A$  and  $B$  whose entries are used in Table 1. Finally, the choice of strategies is generally done by optimising the probability of choosing them or finding the optimal distribution of choosing them. For the first (second) player, the probability of choosing strategy  $s_i^1 \in \mathbb{S}$  ( $s_i^2 \in \mathbb{S}$ ) is  $y_i$  ( $x_i$ ), respectively. We denote the probability vector for the row player as  $\mathbf{y} = \{y_j\}_{j=1}^3$  and for the column player as  $\mathbf{x} = \{x_i\}_{i=1}^3$ .

### 3.1 | Deriving players utilities

The WBAN envisioned scenario is that the coordinator node has both the flexibility and the ability to control the inter-network transmission without causing interference that degrades the service of the neighbouring WBAN and saves the energy of its sensors. WLOG, we assume that there are two transmission power levels  $\{P_L, P_H\}$ ,  $P_L < P_H$ . Each WBAN has the ability to employ a relaying node and a network coding node with the ability of transmission power control. The advantage of the relaying node is to reduce the transmission power, that is, to use  $P_L$  but there will be some delay in receiving the transmitted information compared with a node that transmits directly using the high power level  $P_H$ . Furthermore, the coordinator can ask its sensors to use the network coding alternative where now there is an added robustness but more delay and processing cost given that the low power level is used. As a result, each WBAN can use any of the above strategies, meaning that the strategy set  $\mathbb{S} = \{s_1^i, s_2^i, s_3^i\}_{i \in \{1,2\}} \in \mathbb{N}$  where  $s_1^i$  is the use of  $P_H$  ( $s_1^i = \{P_H\}$ ),  $s_2^i$  is the use of  $P_L$  and a relay node ( $s_2^i = \{P_L, Relay\}$ ), and  $s_3^i$  is the use of  $P_L$  and network coding node ( $s_3^i = \{P_L, NC\}$ ). The payoff of each strategy is given by the payoff functions given below in Equations (1)–(3).

$$u^i(s_1^i, s_k^j) = \frac{h_i P_H}{\sigma^2 + h_{ij} P_j} - c_i P_H, \quad \{i, j\} \in \mathbb{N}, \quad \{s_1^i, s_k^j\} \in \mathbb{S} \quad (1)$$

$h_i$  is the channel coefficient between the sensor node and the coordinator.  $\sigma^2$  is the noise at the coordinator node and will be assumed to have an equal value for both WBANs because we are not trying to estimate or eliminate it in this work.  $h_{ij}$  is the channel coefficient between the coordinator node of the  $i$ th WBAN and the  $j$ th WBAN that uses a power transmission

level  $P_j \in \{P_L, P_H\}$ . The final term in Equation (1) is the cost term of the  $i$ th player that is  $c_i$  and it means that the more power used, the more the sensors' batteries are drained. The cost function in this problem is assumed to be linear in the power level used. However, other cost functions are also used, such as quadratic cost, and this term can be designed by the WBAN administrator. It is clear that, for example, if both WBANs use the strategy  $s_1$  or the higher power level to increase the numerator of Equation (1), then this creates more interference to each other that is reflected by the  $h_{ij} P_j$  term at the denominator, and more cost is incurred on the transmitting node. Next, we define the payoff of the second strategy  $s_2^i$  that means using a relay node and a lower power level in Equation (2).

$$u^i(s_2^i, s_k^j) = \frac{h_i P_L + h_i^R P_R}{\sigma^2 + h_{ij} P_j} - c_i P_L - c_{di}, \quad \{i, j\} \in \mathbb{N}, \quad \{s_2^i, s_k^j\} \in \mathbb{S} \quad (2)$$

Note that the term  $c_{di}$  means that using the relay node can lead to a delay cost.  $h_i^R$  is the channel coefficient between the relay node and the coordinator node.  $P_R < P_H$  is the relay transmitting power. Finally, the payoff function of  $s_3^i$  that is the strategy of using  $P_L$  and network coding to provide link reliability and robustness is given by Equation (3).

$$u^i(s_3^i, s_k^j) = \frac{h_i P_L + h_i^{NC} P_{NC}}{\rho(\sigma^2 + h_{ij} P_j)} - c_i P_L - c_{NCi}, \quad \{i, j\} \in \mathbb{N}, \quad \{s_3^i, s_k^j\} \in \mathbb{S} \quad (3)$$

where  $h_i^{NC}$  is the channel coefficient between the network coding node and the coordinator node,  $P_{NC} < P_H$  is the network coding node power level used, and  $c_{NCi}$  is the cost of using this strategy since it adds more delay. Furthermore, we set  $c_{NCi} = \frac{c_{di}}{\rho}$   $\rho \in [0, 1]$  is a factor that reflects the added benefits from using the network coding by the added robustness to the transmission link that is reflected by increasing the numerator of the signal to the interference plus noise power ratio (SINR) in Equation (3) but with the cost of increased delay and processing that is reflected by the last term of Equation (3). Based on the equations given in Equations (1)–(3), each WBAN has to solve the following optimisation problem.

$$\begin{aligned} & \max_{s_k^j \in \mathbb{S}} && u^i(s_k^j, s_l^j), \\ & \text{subject to} && 0 \leq \sum_{i=1}^3 x_i \leq 1, \quad 0 \leq \sum_{j=1}^3 y_j \leq 1, \\ & && 0 \leq \{x_i\}_{i=1}^3 \leq 1, \quad 0 \leq \{y_j\}_{j=1}^3 \leq 1. \end{aligned} \quad (4)$$



### 3.2 | Potential game formulation

In this paper, we assume that the WBANs are playing in order to maximise their sum utilities,  $\max \sum_{i=1}^N u^i(s_k^i, s_l^i)$  where  $u^i(s_k^i, s_l^i)$  is defined in Equations (1)–(3). In other words, players have the same interest in optimising a collective utility function known as the potential function. This leads to the concept of potential games proposed in Ref. [18] and specifically in the field of wireless communications in Ref. [19]. Potential games represent a type of many-player games, and they have a special and attractive property by characterising a multi-player game by using a single function called the potential function. Furthermore, the solution of these games is guaranteed to exist and can be found easily [20]. Some examples of potential functions are the sum rate function, the sum of utilities, and any function that satisfies the potential function conditions given in Ref. [18]. For bimatrix games such as the game in this work, formulating the potential function is less straight forward and the conditions are more involved. We follow the conditions given in Ref. [20] to formulate the potential function. We emphasise that, for any game, the potential function is not unique, but it should satisfy the potential function conditions. For a bimatrix game with two players WBAN-1 ( $P1$ ) and WBAN-2 ( $P2$ ), the payoff of Player  $P1$  is  $A = [a_{ij}]_{m \times n}$  and for  $P2$  is  $B = [b_{ij}]_{m \times n}$ . The game is an exact potential game if there exists a potential function  $\Phi = [\phi_{ij}]_{m \times n}$  such that

$$\begin{aligned} a_{ij} - a_{i\hat{j}} &= \phi_{ij} - \phi_{i\hat{j}}, \quad \forall i, \hat{i} \in \{1, \dots, m\}, j \in \{1, \dots, n\} \\ b_{ij} - b_{i\hat{j}} &= \phi_{ij} - \phi_{i\hat{j}}, \quad \forall i \in \{1, \dots, m\}, \hat{j}, j \in \{1, \dots, n\}. \end{aligned} \quad (5)$$

With the optimisation problem given in Equation (4) and given the potential function conditions in Equation (5), we can state our first claim.

*Claim 1* The two-player WBAN game  $\mathcal{G}(\mathbb{N}, \mathbb{S}, \mathbb{U})$  defined in Table 1 has the following potential function (matrix).

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}. \quad (6)$$

Specifically,

$$\Phi = \begin{bmatrix} \phi_{21} + a_{11} - a_{21} & a_{12} - a_{22} & a_{13} - a_{33} \\ \phi_{12} - a_{11} + a_{12} + b_{11} - b_{12} & 0 & a_{23} - a_{33} \\ \phi_{13} - a_{11} + a_{31} + b_{11} - b_{13} & a_{32} - a_{33} & 0 \end{bmatrix}. \quad (7)$$

*Proof* See Appendix A. ■

Equation (7) is the new payoff matrix for both WBAN-1 and WBAN-2 given their strategies. However, the game is now symmetric, meaning that each player is choosing her strategy with the same probability. Let us define the new probability column vector  $\mathbf{z} = \{z_i\}_{i=1}^3$ . That is  $s_i$  is chosen with probability  $z_i$ . The players now wish to maximise the utility function of the potential game that is given in Equation (8).

$$\begin{aligned} \max_{z_i \in \mathbb{Z}} \quad & \mathbf{z}^T \Phi \mathbf{z}, \\ \text{subject to} \quad & 0 \leq \sum_{i=1}^3 z_i \leq 1, \quad 0 \leq \{z_i\}_{i=1}^3 \leq 1, \end{aligned} \quad (8)$$

### 3.3 | Deriving the Nash equilibrium

In solving non-cooperative game theoretic problems, the most common solution used is the Nash equilibrium (NE) [40]. Simply put, NE is a solution of the game that guarantees that no player will get a better payoff by deviating from it. Mathematically, let the NE of a two-player two-strategy game be denoted as  $\{s_1^*, s_2^*\}$ . Then, the NE must satisfy

$$u_1(s_1^*, s_2^*) \geq u_1(s_1, s_2^*), \quad (9)$$

$$u_2(s_1^*, s_2^*) \geq u_2(s_1^*, s_2), \quad (10)$$

*Claim 2* The potential game defined in Equation (7) has a NE in pure strategies. Furthermore, if  $\Phi$  is a positive definite matrix, then the NE is unique.

*Proof* See Appendix B. ■

In the next claim, we derive the general conditions to achieve the NE in pure and mixed strategies.

*Claim 3* The two-player WBAN game  $\mathcal{G}(\mathbb{N}, \mathbb{S}, \mathbb{U})$  defined in Table 1 that has the potential function given in Equation (7) has the following NE:

- The  $s_1 = P_H$  strategy,  $z_1^* = 1$ , if  $\phi_{11} > \max\{\phi_{21}, \phi_{31}\}$ ,  $\phi_{12} > \max\{0, \phi_{32}\}$ , and  $\phi_{13} > \max\{0, \phi_{23}\}$ .
- The  $s_2 = \{P_L, Relay\}$  strategy,  $z_2^* = 1$ , if  $\phi_{21} > \max\{\phi_{11}, \phi_{31}\}$ ,  $\max\{\phi_{12}, \phi_{32}\} < 0$ , and  $\phi_{23} > \max\{0, \phi_{13}\}$ .
- The  $s_3 = \{P_L, NC\}$  strategy,  $z_3^* = 1$ , if  $\phi_{31} > \max\{\phi_{21}, \phi_{11}\}$ ,  $\phi_{32} > \max\{0, \phi_{12}\}$ , and  $\max\{\phi_{13}, \phi_{23}\} < 0$ .
- The game has a mixed strategy  $\{z_i^*\}_{i=1}^3$  if the following conditions hold,  $z_1^* = \frac{\alpha_1 \beta_2 - \phi_{13}}{\alpha_2 - \beta_1 \beta_2}$ ,  $z_2^* = \alpha_1 + \beta_1 z_1^*$ , and  $z_3^* = 1 - z_1^* - z_2^*$ , where  $\alpha_1 = \frac{\phi_{23}}{\phi_{23} + \phi_{32}}$ ,  $\beta_1 = \frac{\phi_{21} - \phi_{23} - \phi_{31}}{\phi_{23} + \phi_{32}}$ ,  $\alpha_2 = \phi_{11} - \phi_{13} - \phi_{31}$ , and  $\beta_2 = \phi_{32} + \phi_{13} - \phi_{12}$  given that  $0 \leq \{z_i^*\}_{i=1}^3 \leq 1$  is satisfied.

*Proof* See Appendix C. ■

It can be seen from Claim 3 that finding an explicit expression for the NE is not an easy task. The game gets more complicated as the number of strategies increases. However, potential games are known to have their NE equivalent to the solution that satisfies the Karush–Kuhn–Tucker (KKT) conditions. As we will show in the next section that it is not always necessary to use these conditions, and the NE can be learnt using an algorithm called Fictitious Play (FP) [41] where players update their choices of strategies independently according to the following updating rule:

$$\begin{aligned} \rho_i^{(t+1)} &= \rho_i^{(t)} + \frac{1}{k} \left( \nu_i^{(t)} - \rho_i^{(t)} \right), \quad i \in \mathbb{N}, \\ \nu_i^{(t)} &= \left[ \nu_i^{(t)}(u(s_1, \mathbb{S})), \nu_i^{(t)}(u(s_2, \mathbb{S})), \dots, \nu_i^{(t)}(u(s_{|S|}, \mathbb{S})) \right], \end{aligned} \quad (11)$$

where  $t$  is the iteration number, and  $\nu_i^{(t)}(u(s_l, \mathbb{S}))$  is defined in Equation (12).

$$\begin{aligned} \nu_i^{(t)}(s_l) &= \begin{cases} 1, & \text{if } u^{(t)}(s_l, \mathbb{S}) = \operatorname{argmax}_{s \in \mathbb{S}} u_i(s_l, \rho_{-i}^{(t-1)}) \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (12)$$

Equation (12) means that at the  $t$ th iteration, the  $i$ th player chooses the strategy  $s_l$  that maximises her payoff, and this assigns a value of 1 to  $\nu(s)$  and 0 for the other strategies. In the next iteration, strategies with higher probabilities are chosen until convergence to the NE.

It must be noticed that all the derivations and the results in this paper are for two interacting WBANs. However, one might ask how the game will proceed if there are more than two players. Aside from the obvious answer that most of the games in the literature deal with only two players, potential games are attractive because they find the solution of  $N$ -player games which is one of the few cases that these games have a solution. The potential function of the game proposed here is somewhat difficult to build if there are a large number of players. However, there are several remedies. For example, each of the two WBANs can form a potential function; then, the new potential functions can be combined, and the resultant game still be a potential game [20]. Another approach is that each pair of the WBANs forms a potential function and treats the transmission from other WBANs as interference. Another hierarchical approach would be to make each WBAN pair form a potential function and then a higher potential function can be build where players may adjust only the transmission powers. Adjusting only the transmission powers is the approach used in most WBAN interference control literature. Finally, building a potential function that involves only the transmission power control is interesting, but we believe that our work here is more general and can be extended (with some simplifications).

## 4 | SIMULATION RESULTS

In this section, we simulate our game according to the parameters given in Ref. [2]. In particular, we select the  $P_H$ ,  $P_L$ ,  $P_R$ , and  $P_{NC}$  values according to the average energy consumption levels recorded from [2]. We consider several cases for the cost functions values and the game parameters. In particular, in the first two cases, we assume that the two WBANs have the same cost parameters and then we consider different costs. In the third case, we investigate the effect of an increased interference by other WBANs (or a hostile environment scenario) that are not involved in the game and as a result can be considered as an added interference. We model the added interference as noise added to the noise term  $\sigma^2$ . A final note about the simulation is that the initial probabilities for choosing the strategies are selected uniformly, 0.33, 0.33, 1 – 0.33 because changing the initial probabilities can change the NE of the game. We fix this choice in all our simulations to not clutter the results with many figures. Also, there is no justification for preferring one strategy over the others in this work.

**Case 1 Identical Costs Players:** In this case, we assume that both WBANs have devices that work on the same power level (which is a valid assumption that we employ since manufacturers follow the same standards) and have the same cost functions because the costs are set by the WBAN coordinator. Let  $P_H = 1.832$ ,  $P_{NC} = 0.244$ ,  $P_R = 0.2773$ , and  $P_L = 0.5 \times (P_{NC} + P_R) = 0.2607$  with the appropriate units (no particular units are fixed since they do not affect the outcome of this game formulation.). It should be mentioned that the values themselves are not important as their ratios. This will be clear in choosing the cost parameters. In particular, we choose the cost parameters as  $\frac{c}{c_{di}} = 4$ ,  $i \in \mathcal{N}$ . We assume the channel attenuation coefficient between the communicating node and its WBAN-1 coordinator to be  $h_1 = 0.9$ , the channel between the cross channel between the two WBANs as  $h_{12} = 0.7$  for the interference link from WBAN-2 to WBAN-1 and  $h_{21}$  for the interference link in the other direction, between the communicating node and its WBAN-2 coordinator to be  $h_2 = 0.75$ , the channel between the relay node and its coordinator as  $h_1^R = h_2^R = 0.8$  and between the network coding node and its coordinator as  $h_1^{NC} = h_2^{NC} = 0.6$ . Also the channel between each node and the relay node or the network coding node is assumed to be equal to  $h_i$ ,  $i \in \mathbb{N}$ . Usually it should be stronger, but we try to keep a tractable analysis. Note that we assumed almost equal channel coefficients except for the direct transmission link because the channel coefficient depends on the patient herself. Its effect is more prominent in the direct path than in the path between intermediate nodes (network coding node or the relay node). WLOG, we assume them equal in our simulations unless it is mentioned otherwise. The rest of the parameters are mentioned with each figure. Figure 2 shows the evolution of the players' strategies to  $s_3 = NC$  with the corresponding payoffs as shown in Figure 3 where  $\rho$  in eq = u3 is chosen to be 0.2 and  $\frac{c}{N_i} = 5$ ,  $i \in \mathbb{N}$ . In the second figure, Figure 4, we change the value of the transmission power cost to  $c = 1.5$  meaning that we do not severely penalise high

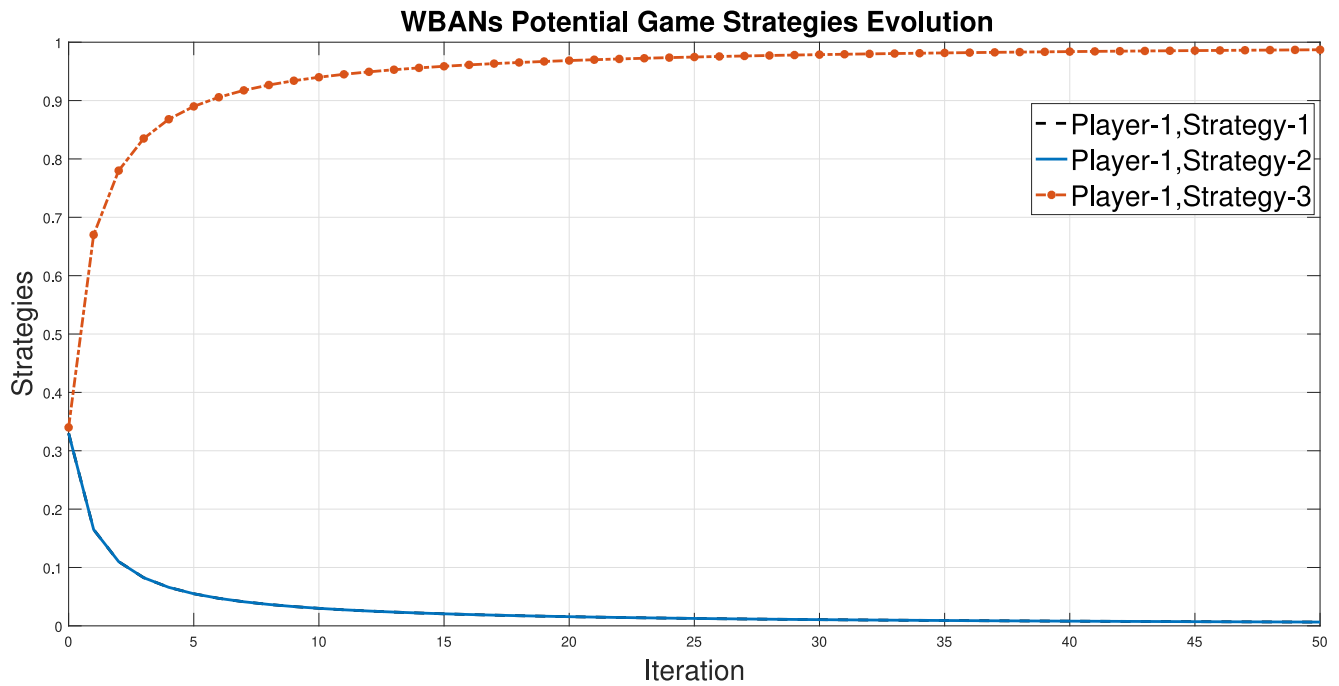


FIGURE 2 Strategy evolution with  $\frac{c}{c_{di}} = 4$ ,  $\frac{c}{N_i} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 3.5$ , and  $\rho = 0.2$ .

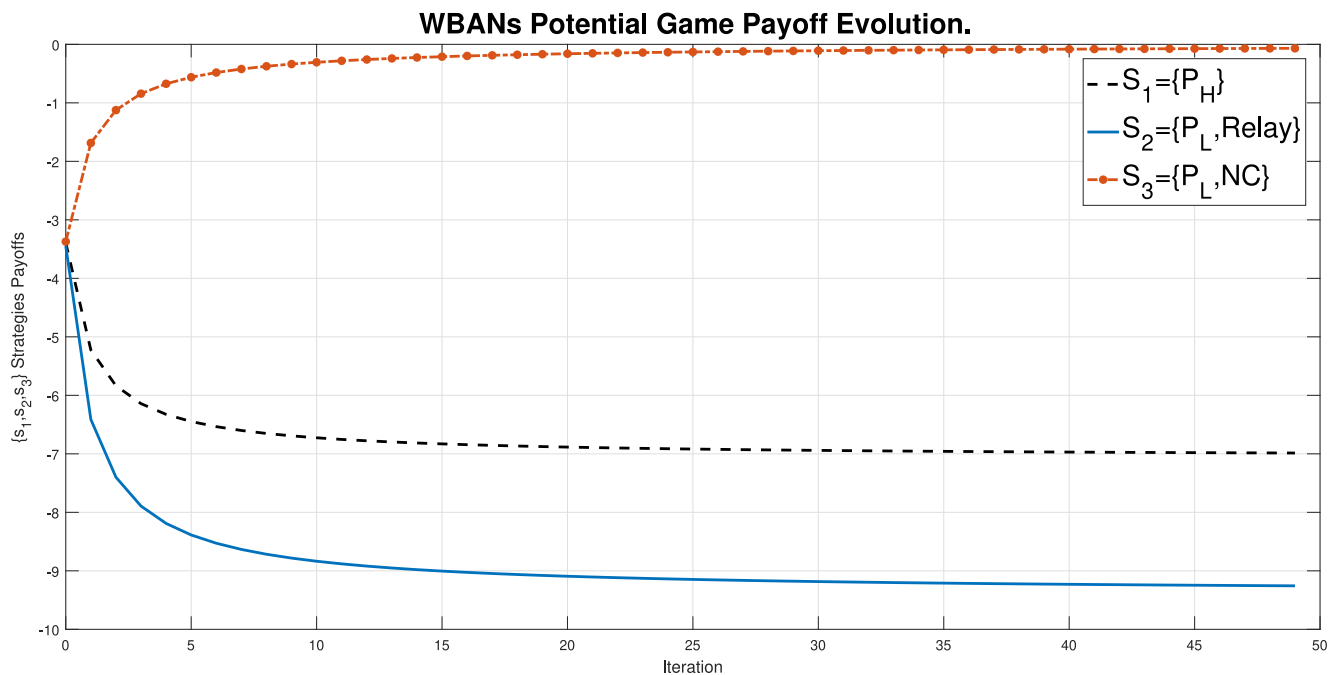


FIGURE 3 Payoff evolution with  $\frac{c}{c_{di}} = 4$ ,  $\frac{c}{N_i} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 3.5$ , and  $\rho = 0.2$ .

transmission powers. We can see from Figures 4 and 5 that the *NC* strategy is not a NE anymore, and the game converges to a mixed NE. The players are more into using  $s_1$  or  $s_2$  rather than the  $s_3 = NC$ .

In the next figures, we study the effect of  $\rho$  on the NE while increasing the cost to be  $c = 3.5$  as in Figures 2 and 3. We can see from Figures 6 and 7 that increasing  $\rho$  that corresponds

to lowering the reliability factor (see Equation 3) stems from using the network coding strategy from 0.2 to 0.25 which is enough to make players refrain from using that strategy. Furthermore, players prefer using the relay node, that corresponds to  $s_2$ , over the other two strategies, but the game still converges to a mixed NE.

We close the discussion of similar WBANs by studying the effect of unwanted interference that stems from other WBANs

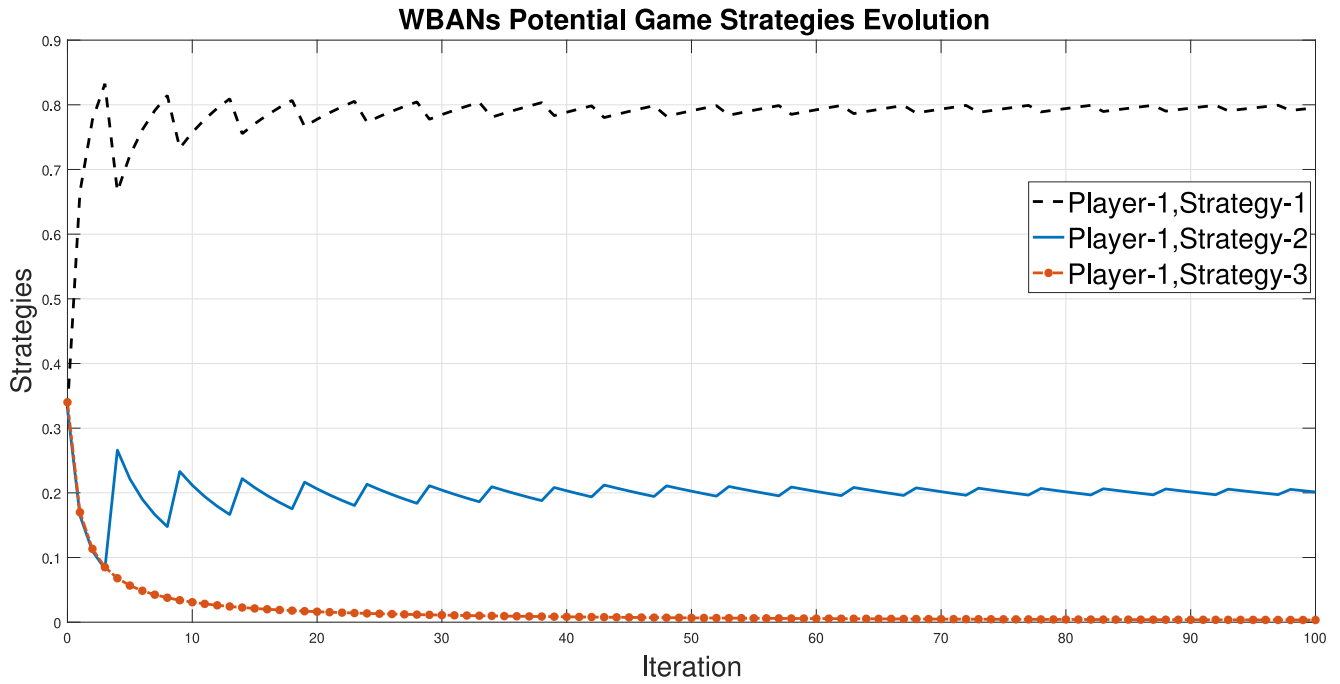


FIGURE 4 Strategy evolution with  $\frac{c}{c_{th}} = 4$ ,  $\frac{c}{N_{th}} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 1.5$ , and  $\rho = 0.2$ .

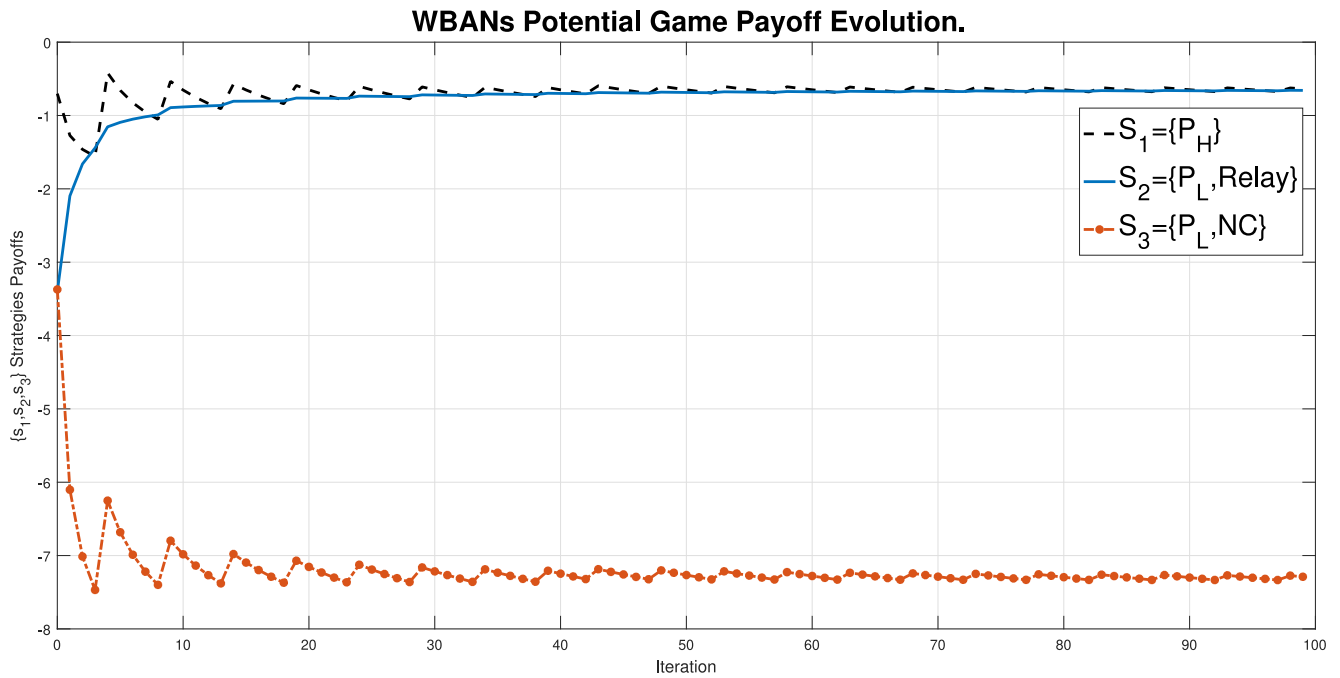


FIGURE 5 Payoff evolution with  $\frac{c}{c_{th}} = 4$ ,  $\frac{c}{N_{th}} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 1.5$ , and  $\rho = 0.2$ .

that are not participated in the game, from the ambient devices, and maybe from a malicious device (a hostile environment scenario). We assume that the WBAN coordinator nodes deal with it as a high level noise. We run the simulation for the parameters:  $c \in \{1.5, 3.5\}$ ,  $\rho \in \{0.2, 0.25\}$ , and expose the players to interference at the level of high interference assumed to be  $b_1 P_H$  and low to medium interference at the level of  $b_1 P_L$ . The game

converged to a pure NE that corresponds to choosing  $s_3 = NC$  as shown in Figure 8. To conclude this section, players tend to choose either the higher power level or lower power level with an aiding relay when there is no interference. However, any increase of uncoordinated interference motivates the players to resort to the safest option that is using network coding although there are extra costs of processing and delay.



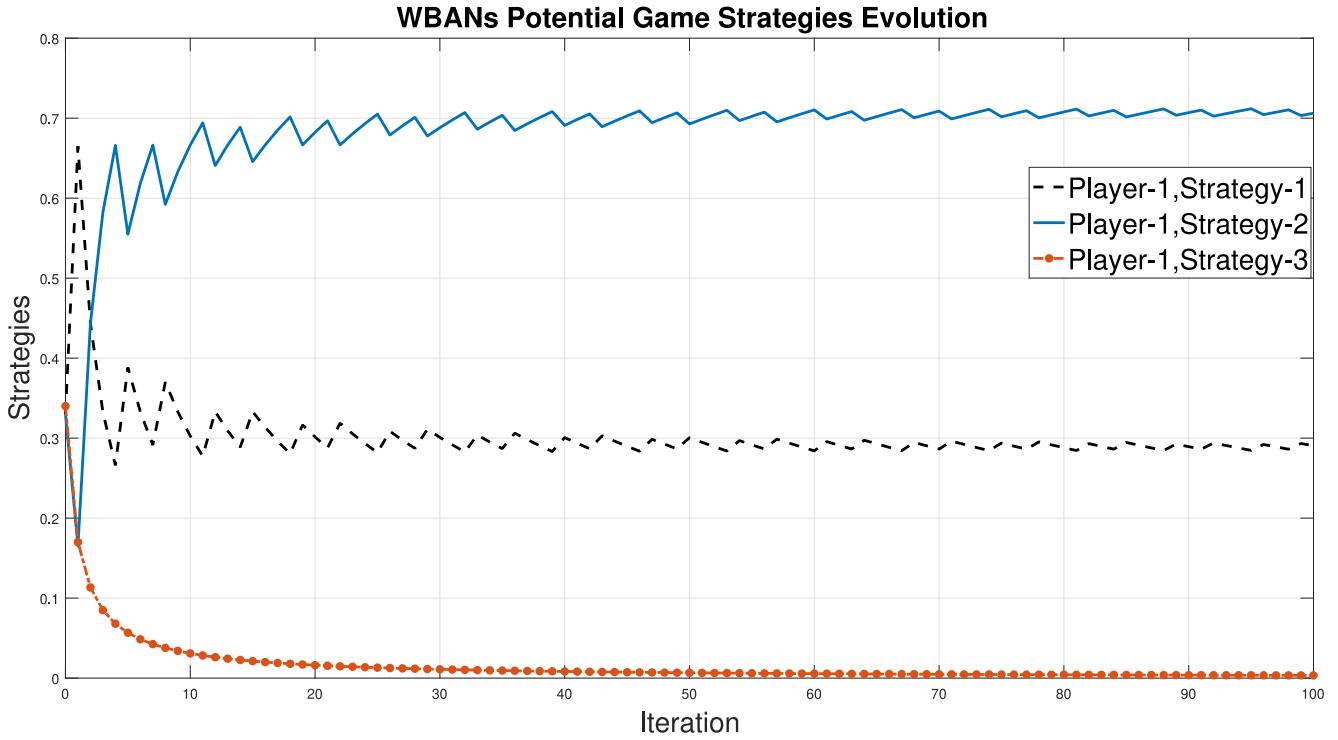


FIGURE 6 Strategy evolution with  $\frac{c}{c_{di}} = 4$ ,  $\frac{c}{N_i} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 3.5$ , and  $\rho = 0.25$ .

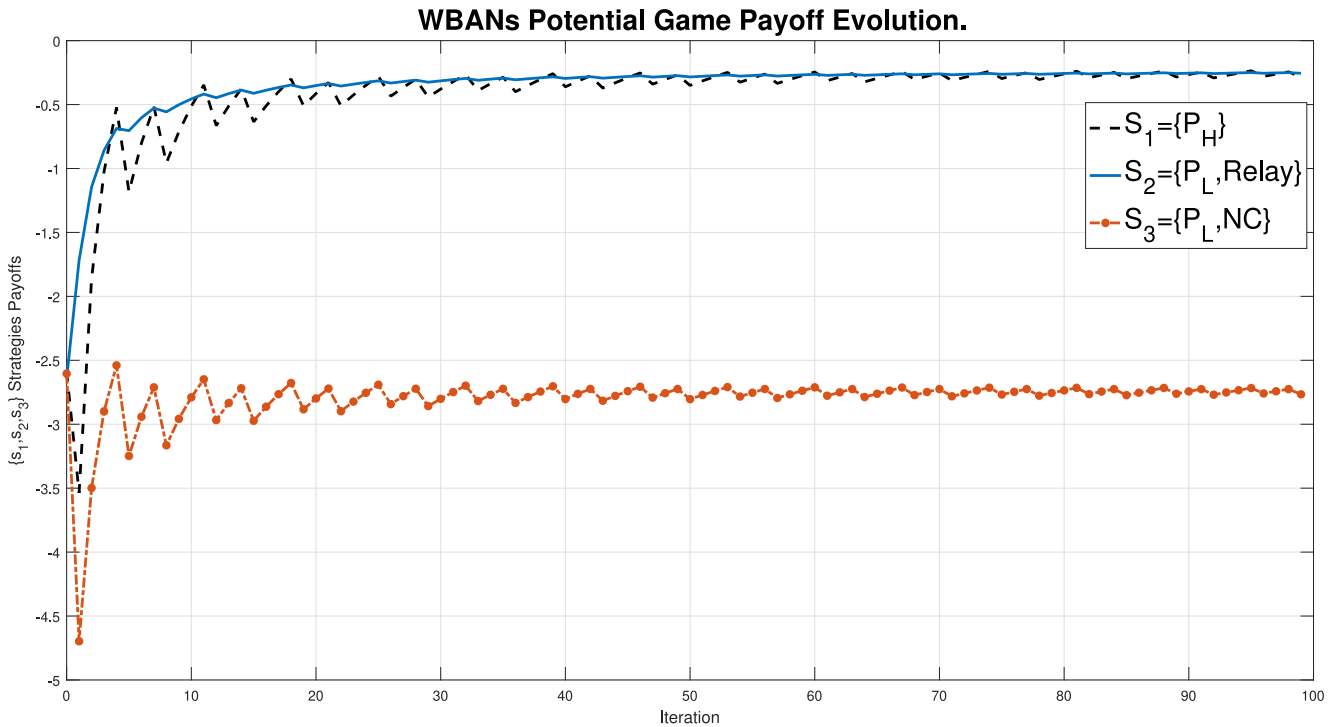


FIGURE 7 Payoff evolution with  $\frac{c}{c_{di}} = 4$ ,  $\frac{c}{N_i} = 5$ ,  $i \in \mathbb{N}$ ,  $c = 3.5$ , and  $\rho = 0.25$ .

**Case 2 Non-Identical Costs Players:** In this scenario, we study the effect of WBANs having different transmission cost values, different relaying delay cost values, and different network coding and delaying costs. We also assume that there is no

extra interference caused by other networks or hostile environments. For space limitation, we collect the results of the simulations in Table 2. The first column of Table 2 has two entries. The first entry shows the cost that is changed as a

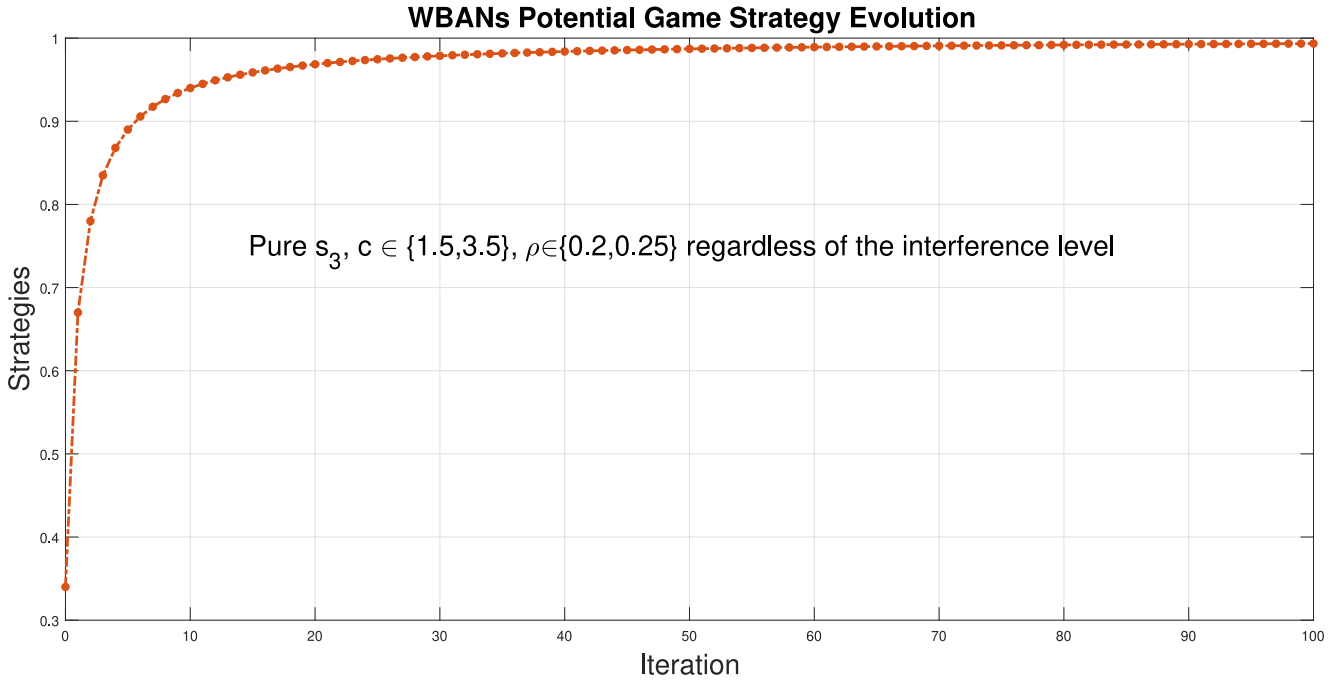


FIGURE 8 Comparing different interference levels, cost levels, and reliability levels on the players' performance.

TABLE 2 NE for the non-identical costs players game.

$\{c_i, c_{di}, \text{ or } c_{NC_i}\}, \{c, \rho\}$	Prob. of $\{s_1, s_2, s_3\}$
$c_2 = 0.25c_1, \{3.5, 0.2\}$	$\{0, 0, 1\}$
$c_2 = 0.25c_1, \{3.5, 0.25\}$	$\{0.25, 0.75, 0\}$
$c_2 = 0.25c_1, \{1.5, 0.2\}$	$\{0.78, 0.22, 0\}$
$c_2 = 0.25c_1, \{1.5, 0.25\}$	$\{0.78, 0.22, 0\}$
$c_{d2} = 0.25c_{d1}, \{3.5, 0.2\}$	$\{0, 0, 1\}$
$c_{d2} = 0.25c_{d1}, \{3.5, 0.25\}$	$\{0.25, 0.75, 0\}$
$c_{d2} = 0.25c_{d1}, \{1.5, 0.2\}$	$\{0.78, 0.22, 0\}$
$c_{d2} = 0.25c_{d1}, \{1.5, 0.25\}$	$\{0.78, 0.22, 0\}$
$c_{NC_2} = 0.25c_{NC_1}, \{3.5, 0.2\}$	$\{0, 0, 1\}$
$c_{NC_2} = 0.25c_{NC_1}, \{3.5, 0.25\}$	$\{0.25, 0.75, 0\}$
$c_{NC_2} = 0.25c_{NC_1}, \{1.5, 0.2\}$	$\{0.77, 0.23, 0\}$
$c_{NC_2} = 0.25c_{NC_1}, \{1.5, 0.25\}$	$\{0.79, 0.21, 0\}$

parameter, while the second field depicts the power cost  $c$  and the reliability factor  $\rho$  since all other costs are defined in terms of the power cost as shown in the first case,  $c_{NC_i} = cd_i = 0.2c \forall i \in \mathbb{N}$ . The second column in Table 2 represents the probabilities of choosing the NE strategy from  $\mathcal{S}$ . If there is only one non zero entry, it means that the game has a pure NE, otherwise it means the game converges to a mixed NE. We can see that the game converges to  $s_3 = NC$  as a NE whenever both the power cost and the reliability factor are high. Other than that, the game converges to a mixed NE between  $s_1$  (using the higher power level) and  $s_2$  (using a relay with the low power level). In particular, whenever the power

cost increases ( $c = 3.5$ ), players are more into using  $s_2$ . Otherwise, they prefer to use  $s_1$ .

## 5 | CONCLUSION

In this paper, we investigated, for the first time, the behaviour of two WBANs playing a potential game. Each WBAN has three strategies to choose from where each strategy has its costs and rewards. The game can be designed as a potential game to produce a certain NE. Our results show that players can learn the NE through the FP algorithm. The results show that the NE is more sensitive to the power cost parameter and then to the reliability factor  $\rho$ . Setting the power cost depends on each WBAN parameters such as the consumed energy in its sensors. The reliability factor can be independently set depending on each patient's situation. However, a particular NE can be achieved through a potential game formulation. We derived the NE of the game and formulated the game in mixed strategies as an LP. The conditions for each NE are derived too.

## AUTHOR CONTRIBUTIONS

**Ahmed A. Alabdel Abass:** Conceptualization; data curation; formal analysis; funding acquisition; investigation; methodology; project administration; resources; software; supervision; validation; visualization; writing—original draft; writing—review and editing. **Hisham Alshaheen:** Conceptualization; data curation; formal analysis; resources; software; validation; writing—original draft; writing—review and editing. **Haifa Takruri:** Project administration; writing—original draft; writing—review and editing.

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## CONFLICT OF INTEREST STATEMENT

The authors declare no conflict of interests.

## DATA AVAILABILITY STATEMENT

Data sharing not applicable—no new data generated.

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## REFERENCES

- Alshaheen, H., Takruri-Rizk, H.: Energy saving and reliability for wireless body sensor networks (wbsn). *IEEE Access* 6, 16678–16695 (2018). <https://doi.org/10.1109/access.2018.2817025>
- Alshaheen, H., Rizk, H.T.: Improving the energy efficiency for biosensor nodes in the wbsn bottleneck zone based on a random linear network coding. In: 2017 11th International Symposium on Medical Information and Communication Technology (ISMICT), pp. 59–63. *IEEE* (2017)
- Alshaheen, H., Rizk, H.T.: Improving the energy efficiency for the wbsn bottleneck zone based on random linear network coding. *IET* 8(1), 17–25 (2018). <https://doi.org/10.1049/iet-wss.2017.0056>
- Akyildiz, Y.S.I.F., Su, W., Cayirci, E.: A survey on sensor networks. In: *IEEE*, vol. 40, pp. 102–114 (2002)
- Movassaghi, J.L.D.S.S., Abolhasan, M., Jamalipour, A.: Wireless body area networks: a survey. In: *IEEE*, vol. 16, pp. 1658–1686 (2014)
- Rout, S.M.R.R., Ghosh, S.K.: Enhancement of lifetime using duty cycle and network coding in wireless sensor networks. In: *IEEE*, vol. 12, pp. 656–667 (2013)
- Braem, B., et al.: The need for cooperation and relaying in short-range high path loss sensor networks. In: 2007 International Conference on Sensor Technologies and Applications (SENSORCOMM 2007), pp. 566–571. *IEEE* (2007)
- Reusens, G.V.E., Joseph, W., Martens, L.: On-body measurements and characterization of wireless communication channel for arm and torso of human. In: 4th International Workshop on Wearable and Implantable Body Sensor Networks (BSN 2007), pp. 264–269. *Springer* (2007)
- Ehyaie, M.H.A., Khadivi, P.: Using relay network to increase life time in wireless body area sensor networks. In: 2009 IEEE International Symposium on a World of Wireless, Mobile and Multimedia Networks Workshops. *IEEE* (2009)
- Reusens, E., et al.: Characterization of on-body communication channel and energy efficient topology design for wireless body area networks. *IEEE Trans. Inf. Technol. Biomed.* 13(6), 933–945 (2009). <https://doi.org/10.1109/itb.2009.2033054>
- Kartsakli, L.A.E., Antonopoulos, A., Verikoukis, C.: A cloud-assisted random linear network coding medium access control protocol for healthcare applications. In: *MDPI: Sensors*, vol. 14, pp. 4806–4830 (2014)
- Shi, M.M.X., Lucani, D.: On-body measurements and characterization of wireless communication channel for arm and torso of human. In: International Conference on Research in Networking: NETWORKING 2011: NETWORKING 2011 Workshops, pp. 119–128. *Springer* (2011)
- Yokota, A.M.K., Morita, H.: An xor encoding for wireless body area networks. In: *BodyNets '13: Proceedings of the 8th International Conference on Body Area Networks*, pp. 240–243. *ACM Digital Library* (2013)
- Hurtado-lópez, J., Casilari, E.: An Adaptive Algorithm to Optimize the Dynamics of Ieee 802.15.4 Networks
- Yokota, A.M.K., Morita, H.: Adaptive sleeping periods in ieee 802.15.4 for efficient energy savings: Markov-based theoretical analysis. In: 2011 IEEE International Conference on Communications (ICC). *IEEE* (2011)
- Hurtado-lópez, J., Casilari, E.: High snr analysis of inter-body interference in body area networks. In: 2017 International Conference on Communication, Computing and Digital Systems (C-CODE). *IEEE* (2017)
- Marden, J.R., Arslan, G., Shamma, J.S.: Cooperative control and potential games. *IEEE Trans. Syst. Man. Cybern. B Cybern. Part B (Cybernetics)* 39(6), 1393–1407 (2009). <https://doi.org/10.1109/tsmcb.2009.2017273>
- Monderer, D., Shapley, L.S.: Potential games. *Game. Econ. Behav.* 14(1), 124–143 (1996). <https://doi.org/10.1006/game.1996.0044>
- Scutari, G., Barbarossa, S., Palomar, D.P.: Potential games: a framework for vector power control problems with coupled constraints. In: *Proc. IEEE International Conference on Acoustics Speech and Signal Processing Proceedings (Vol. 4)*, p. IV (2006)
- Hespanha, J.P.: *Noncooperative Game Theory: An Introduction for Engineers and Computer Scientists*. Princeton University Press (2017)
- Mai, R., Nguyen, D.H., Le-Ngoc, T.: Linear precoding game for mimo mac with dynamic access point selection. *IEEE Wireless Communications Letters* 4(2), 153–156 (2015). <https://doi.org/10.1109/lwc.2014.2387833>
- Zhong, W., et al.: Relay selection and discrete power control for cognitive relay networks via potential game. *IEEE Trans. Signal Process.* 62(20), 5411–5424 (2014). <https://doi.org/10.1109/tsp.2014.2347261>
- La, Q.D., et al.: Potential games. In: *Potential Game Theory: Applications in Radio Resource Allocation*, pp. 23–69 (2016)
- Yamamoto, K.: A comprehensive survey of potential game approaches to wireless networks. *IEICE Trans. Commun.* 98(9), 1804–1823 (2015). <https://doi.org/10.1587/transcom.e98.b.1804>
- Shah, V., Mandayam, N.B., Goodman, D.J.: Power control for wireless data based on utility and pricing. In: Ninth IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (Cat. No. 98TH8361), vol. 3, pp. 1427–1432. *IEEE* (1998)
- Hong, E.J., Yun, S.Y., Cho, D.-H.: Decentralized power control scheme in femtocell networks: a game theoretic approach. In: 2009 IEEE 20th International Symposium on Personal, Indoor and Mobile Radio Communications, pp. 415–419. *IEEE* (2009)
- Sung, C.W., Fu, Y.: A game-theoretic analysis of uplink power control for a non-orthogonal multiple access system with two interfering cells. In: 2016 IEEE 83rd Vehicular Technology Conference (VTC Spring), pp. 1–5. *IEEE* (2016)
- Chaiel, H.K., Alabdel Abass, A.A.: Game theoretical model for information transmission in structure-free wireless sensor networks. *IET Commun.* 14(17), 3080–3086 (2020). <https://doi.org/10.1049/iet-com.2019.1216>
- Abass, A.A.A., Mandayam, N.B., Gajic, Z.: Evolutionary random access game with objective and subjective players. Under Review
- Zhou, J., et al.: A game theory control scheme in medium access for wireless body area network. (2014)
- Chen, G., et al.: Reinforcement learning-based sensor access control for wbans. *IEEE Access* 7, 8483–8494 (2018). <https://doi.org/10.1109/access.2018.2889879>
- Tsiropoulou, E.E., Katsinis, G.K., Papavassiliou, S.: Utility-based power control via convex pricing for the uplink in cdma wireless networks. In: 2010 European Wireless Conference (EW), pp. 200–206. *IEEE* (2010)
- Tsiropoulou, E.E., Katsinis, G.K., Papavassiliou, S.: Distributed uplink power control in multiservice wireless networks via a game theoretic approach with convex pricing. *IEEE Trans. Parallel Distr. Syst.* 23(1), 61–68 (2011). <https://doi.org/10.1109/tpds.2011.98>
- Balevi, E., Gitlin, R.D.: Stochastic geometry analysis of ieee 802.15. 6 uwbn performance with game theoretical power management. In: 2018 IEEE 19th Wireless and Microwave Technology Conference (WAMICON), pp. 1–5. *IEEE* (2018)
- Wang, X., Cai, L.: Interference analysis of co-existing wireless body area networks. In: 2011 IEEE Global Telecommunications Conference-GLOBECOM 2011, pp. 1–5. *IEEE* (2011)
- Jameel, F., et al.: High snr analysis of inter-body interference in body area networks. In: 2017 International Conference on Communication, Computing and Digital Systems (C-CODE), pp. 117–121. *IEEE* (2017)
- Zhang, Z., et al.: Power control and localization of wireless body area networks using semidefinite programming. In: 2015 2nd International

Symposium on Future Information and Communication Technologies for Ubiquitous HealthCare (Ubi-HealthTech), pp. 1–5. IEEE (2015)

38. Roy, M., et al.: Finding optimal transmission strategy for intra-wban communications. *Electron. Lett.* 56(23), 1283–1286 (2020). <https://doi.org/10.1049/el.2020.2169>
39. Zhou, Z., et al.: Age of information aware scheduling for dynamic wireless body area networks. *IEEE Sensor. J.* 23(16), 17832–17841 (2023). <https://doi.org/10.1109/jsen.2023.3290612>
40. Basar, T., Olsder, G.J.: *Dynamic Noncooperative Game Theory*, vol. 200. SIAM (1995)
41. Brown, G.W.: Iterative solution of games by fictitious play. *Act. Anal. Prod Allocation* 13(1), 374 (1951)
42. Boyd, S.P., Vandenberghe, L.: *Convex Optimization*. Cambridge university press (2004)

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## APPENDIX A

### Proof of claim 1

*Proof* The entries of the potential function (matrix)  $\Phi$  are derived based on the conditions given in Equation (5) [20]. The matrix  $\Phi$  is  $3 \times 3$  and when combining this with the conditions given in Equation (5), there are  $n \times \frac{m(m-1)}{2} + m \times \frac{n(n-1)}{2}$  equations to be solved according to [20]. In this case, 18 linear simultaneous equations with 9 unknowns. The potential game exists if these equations have a solution. We show only the first four equations so that the reader knows how to proceed with the derivation.

$$a_{11} - a_{21} = \phi_{11} - \phi_{21}, \quad (A1)$$

$$a_{11} - a_{31} = \phi_{11} - \phi_{31}, \quad (A2)$$

$$a_{21} - a_{31} = \phi_{21} - \phi_{31}, \quad (A3)$$

$$a_{12} - a_{22} = \phi_{12} - \phi_{22}, \quad (A4)$$

We choose  $\phi_{22} = \phi_{33} = 0 \rightarrow \phi_{12} = a_{12} - a_{22}$ . Using  $\phi_{12}$  with  $a_{11} - a_{21}$  we get  $\phi_{11}$ . We proceed in the same manner with the of the rest calculations to get all the entries of Equation (7). This completes the proof. ■

## APPENDIX B

### Proof of claim 2

*Proof* According to [20], the game is constructed according to the conditions in Equation (5) which guarantee the existence of NE. To prove the uniqueness of the NE if  $\Phi$  is a positive definite matrix, we recognise that Equation (8) is a quadratic programming problem with constraints that can be written as follows:

$$\begin{aligned} \min_{z_i \in \mathbf{Z}} \quad & -\mathbf{z}^T \Phi \mathbf{z}, \\ \text{subject to} \quad & \mathbf{z}^T \mathbf{1} = 1, \\ & \mathbf{z} < \mathbf{1}, \quad -\mathbf{z} < \mathbf{0}, \end{aligned} \quad (B1)$$

where  $\mathbf{1}$  is a column vector of all ones. Equation (B1) is a quadratic optimisation problem that is convex (and hence has a unique solution) if  $\Phi$  is a positive definite matrix [42]. ■

## APPENDIX C

### Proof of claim 3

*Proof* Proving the pure NE in the first three parts stems from applying the NE definition given in Equation (9). As a result, we show how to prove the first strategy  $s_1^*$  and the rest follow the same approach. According to Equation (9) (or Equation 10),  $s_1^*$  is a pure NE if it gives the highest payoff among all other strategies. This leads to comparing the entries of the first row of the payoff matrix Equation (7) with the entries of the second and third rows which gives the conditions in Claim 3. Proving the mixed NE in the last part of the claim uses the concept that players are indifferent when using mixed strategies, meaning that they have the same payoff for all strategies. In other words, this can be written as follows:

$$\sum_{i=1}^3 z_i^* \phi_{ji} = \sum_{i=1}^3 z_i^* \phi_{ki}, \quad \forall j \neq k \in \{1, 2, 3\}$$

and by solving these equations with taking into consideration that  $\sum_{i=1}^3 z_i = 1$ . In fact, we solve for

$$\sum_{i=1}^3 z_i \phi_{2i} = \sum_{i=1}^3 z_i \phi_{3i}$$

and

$$\sum_{i=1}^3 z_i \phi_{1i} = \sum_{i=1}^3 z_i \phi_{3i},$$

taking into consideration that  $\phi_{22} = \phi_{33} = 0$ . ■