Dynamics and chaos in extensible pendulum systems

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The extensible pendulum is a generalization of the classic mass-on-a-spring problem to regimes where vibrations are allowed to occur in either a vertical plane or a three-dimensional region of space. Even when the spring is linear in nature—where restoring forces are proportional to extension—the equations of motion are highly nonlinear since Hooke's law is operating in more than one spatial dimension. Mathematically, it is Pythagoras's theorem which gives rise to this type of *geometric nonlinearity* and simple experimental arrangements can provide intriguing insights into incredibly rich dynamics.

Despite being purely deterministic, the 'pendulum' (side-to-side) and 'spring' (up-down) modes of an extensible pendulum are coupled together and can give rise to oscillations that appear to be wildly unpredictable—that is, *chaotic*. This paper adds both dissipation (from low-speed air resistance) and periodic forcing (from prescribed movement of the suspension point) into the mix. The derivation of the governing equations will be detailed, based on Lagrangian analysis and the incorporation of a velocity-dependent potential to capture energy losses. External forcing adds complexity by increasing the dimensionality of the system's phase space. A set of new results from recent computations will be presented, including phenomena such as limit-cycle attractors, period-doublings, and fractals.

Keywords: Extensible pendulum, nonlinear dynamics, Lagrangian mechanics, limit-cycle attractors, period doublings, chaotic systems.