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Article

Time Parametrized Motion Planning

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Abstract: Time can be treated as a free parameter to isotropically stretch the tangent space. A trajectory, which matches the boundary conditions on its configuration, is adjusted so that velocity conditions are met. The modified trajectory is found by substitution, without the computational cost of re-integrating the velocity function. This concept is extended to stretch the tangent space anisotropically. This method of time parametrization especially applies to Geometric Control, where the Pontryagin Maximum Principle minimizes some cost function and matches the boundary configuration constraints but not the velocity constraints. The optimal trajectory is modified by the parametrization so that the cost function is minimized if the stretching is stopped at any time. This is a theoretical contribution, using a wheeled robot example to illustrate the modification of an optimal velocity under multiple parametrizations.

Keywords: geometric control; motion planning; optimization; nonholonomic; time parametrization; wheeled robot

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1. Introduction

The use of autonomous vehicles is becoming increasingly important as technological advances enable them to replace humans in dangerous environments. Computer path planning and control are making automation possible, but computationally efficient methods are required. In this paper, we propose a method that extends previous work on motion planning using Geometric Control theory to match conditions on velocities via multiple time parametrizations.

A number of tools from geometric control theory have been used to tackle motion planning problems for nonholonomic systems on Lie groups. One example is where the motion planning problem is formulated in the context of an optimal control problem in order to derive a motion that fulfils the prescribed boundary conditions on the configuration space while minimizing a quadratic cost function. The resulting geometry of the configuration space determines the geometric path (or optimal trajectory). In particular, for a large class of optimal control problem on Lie groups, the systems are integrable and analytical solutions can be found. However, these formulations do not include conditions on the velocities and, therefore, the derived motions are often impractical. This paper exploits the analytic solutions and uses parametrization to modify the velocity and hence the trajectory.

Treating time as a free parameter enables the rate of change to be modified so that the tangent space is stretched or shrunk isotropically, changing the magnitude but not the direction of the velocity. The original path is tracked, but at a modified speed. For the class of systems where the associated optimal trajectory is available in closed form, the modified trajectory can be determined by reparameterizing time, thus avoiding the need to reintegrate the trajectory.

Extending this methodology, the character of the motion can be modified by anisotropically stretching the tangent space. For example, to eliminate a turning movement, the tangent space is shrunk in the direction(s) corresponding to the turn, whilst simultaneously stretching in the directions representing the linear motion. The velocity is modified so that, at any time, the cost function would be minimized if the stretching were stopped. The modified velocity is integrated numerically to identify the new trajectory.

The paper is organized by first presenting the idea of using time as a free parameter so that the tangent space is stretched isotropically. The modified trajectory is determined by substituting the parametrized time function $t = F(\tau)$ into the original trajectory. The concept is then considered as it applies to Lie groups and non-flat spaces where the optimal motion is not in a fixed direction (c.f. tumbling satellite). The novel concept of stretching the tangent space anisotropically is presented. This modifies various characteristics of the velocity, and not just its magnitude. For example, shrinking the tangent space in one direction reduces the corresponding motion (e.g., a turn), and simultaneously stretching in another direction increases that component (e.g., the linear speed). The optimal velocity minimizing the cost function depends on the interaction between velocity components and this is changed by the anisotropic stretching. With the removal of the thrusts which stretch the tangent space, the motion satisfies the optimal control problem, minimizing the quadratic cost function.

Section 3 illustrates time parametrization with the example of a wheeled robot. In a natural motion (i.e., with no controls), the linear and angular velocities interact to create a curving trajectory in the plane. Using time parametrization, the robot is stopped and then restarted without a turning motion. This is compared with anisotropic parametrization, where the robot does not have to come to rest. The anisotropic stretching slows the angular motion, causing a slowing of the linear motion. The linear velocity is increased by stretching in that direction, countering the braking effect. The velocity and acceleration are continuously modified.

2. Time Parametrization

2.1. Background and General Parametrization

Much of the previous work on modifying the velocity function (for example, Spindler [1] and Chaturvedi et al. [2]) considers only starting from rest and stopping at the end, and only for a constant speed v. This is modified by multiplying by a function $f(\tau)$, where τ is the parametrized time and $f(0) = f(T) = 0$ when T is the time at the end of the trajectory. The constant velocities were linear in Smith [3] and eigenaxis angular velocities in Spindler [1]. Flasskamp and Ober-Blobaum [4] and Frazzoli et al. [5] worked on joining trim primitives (where the velocity is fixed) into total trajectories, but only by stopping the motion and starting again. In comparison, Maclean et al. [6] used a feedback controller to achieve the optimal velocity and then reset the target velocity to zero towards the end of the trajectory.

More recently, Xargay et al. [7] adjusted the speed along a prescribed path. They consider a fleet of vehicles. Feedback is used to adjust the speed of each vehicle along its prescribed path so that they avoid each other and achieve joint objectives. Sharma [8] considers a hybrid motion planning approach where both classical and learning-based approaches are utilized in investigating optimal trajectories about some boundary condition. The classical method looks to the velocity space against constraints and optimal conditions in real time. The method described here in this paper could supplement this hybrid approach, but this is beyond the scope of this paper. Aguiar et al. [9] showed that the speed along a geometric path could be adjusted and that, for some class of systems, performance limitations are avoided. Aguiar et al. [9] showed that, for a geometric path, the original controls keep the system on the path and additional controls manage the speed adjustment is managed by additional controls. Time parametrization can be applied to variable velocities in multi-dimensional space. The method can then be extended by applying different parametrization functions F_i in each dimension. The velocity is represented by a vector in the tangent space; even for multiple bodies. Typically, for a single body, it is made up of linear and angular velocity components. The velocity vector can be written in component form $\sum_i v_i(t)e_i$, where $i \in \{1, 2, \ldots, n\}$ and $\{e_i\}$ is a set of base vectors for the n dimensional tangent space. The general form of the modification is

$$\mathbf{v}(t) = \sum_i v_i(t)e_i \implies \sum_i v_i^*(\tau)\frac{d}{d\tau}F_i(\tau)e_i \equiv \mathbf{v}^*(\tau) \tag{1}$$

Any trajectory can be represented by a vector $x(t)$, typically made up of position and attitude components which change over time. In general, $x(t)$ is any point on the manifold of possible configurations.

The effect of isotropic time parametrization is summarized in the following theorem.

Theorem 1. *Any velocity vector $v(t)$ can be modified isotropically by the application of a parametrization function $F(\tau)$, where τ is the parametrized time, the original time $t = F(\tau)$, and the stretching factor $f(\tau) = \frac{d}{d\tau}F(\tau)$. The modified velocity vector $v^*(\tau)$ becomes*

$$\mathbf{v}^*(\tau) \equiv \mathbf{v}(F(\tau))f(\tau) \tag{2}$$

The trajectory represented by the vector $x(t)$ of positions and orientations is modified by the parametrization to a new vector

$$\mathbf{x}^*(\tau) = \mathbf{x}(F(\tau)) \tag{3}$$

Proof of Theorem 1. This result follows from differentiating the position vector $\mathbf{x}^*(\tau)$.

$$\begin{aligned} \frac{d}{d\tau}\mathbf{x}^*(\tau) &\equiv \frac{d}{d\tau}\mathbf{x}(F(\tau)) \\ &= \frac{d}{dF}\mathbf{x}(F(\tau))\frac{d}{d\tau}F(\tau) \\ &= \frac{d}{dt}\mathbf{x}(t)\frac{d}{d\tau}F(\tau), \text{ since } t = F(\tau) \\ &= \mathbf{v}(t)\frac{d}{d\tau}F(\tau), \text{ since } \frac{d}{dt}\mathbf{x}(t) = \mathbf{v}(t) \end{aligned}$$

□

Corollary 1. *If the parametrization function has the property that $T = F(T)$, then the reparameterized motion achieves the same position and attitude as the original motion. That is, $\mathbf{x}(T) = x^*(T)$ by substitution using Theorem 1.*

2.2. Parametrization Function

Any non-decreasing differentiable function on the time interval $\tau \in [0, T]$ which starts at zero can be used as the parametrization function $F(\tau)$.

$$F(0) = 0, \frac{dF}{d\tau} = f(\tau) \geq 0 \tag{4}$$

To achieve the original destination $x(T)$ in the same time T, there is the extra condition that $F(T) = T$ (refer to Corollary 1). Any constraints on the velocity may limit the point that can be reached in any prescribed time T. With any time reparameterization which has $F(T) = T$, the maximum values of the velocity components will increase (since the average velocity is unchanged).

Three possible examples are shown in Figure 1, where the stretching factor $f = \frac{dF}{d\tau}$ is plotted.

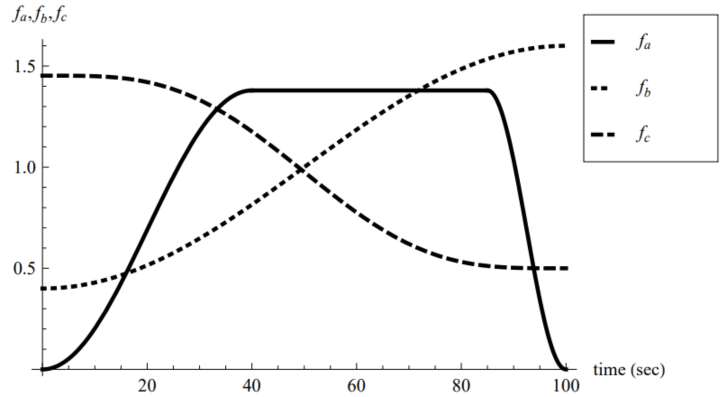

Figure 1. Possible time reparametrization functions.

1. f_a (the solid line) parametrizes the velocity to start and finish at rest, maintaining a constant multiple, denoted a, of the original velocity during the middle of the time interval, between T_1 and T_2. The acceleration is smooth, starting at zero, increasing to a maximum and then reducing to zero. The braking phase reverses this, in a shorter time span. The total time is unchanged so $\int_0^T f_a(\tau)d\tau = T$.

$$f_a(\tau) = \begin{cases} a - a\cos\left(\frac{\pi\tau}{T_1}\right) & 0 \leq \tau \leq T_1 \\ a & T_1 \leq \tau \leq T_2 \\ a + a\cos\left(\frac{\pi(\tau - T_2)}{(T - T_2)}\right) & T_2 \leq \tau \leq T \end{cases}$$

a is adjusted so that $\int_0^T f_a(\tau)d\tau = T$. The average velocity is unchanged. Since the motion starts and finishes at rest, the maximum speed will increase as a result of the parametrization.

2. f_b (the dotted line) illustrates a parametrization which smoothly changes the velocity from $v(0)f_b(0)$ to $v(F_b(T))f_b(T)$, where a_1 and a_2 are some constant multiples.

$$f_b(\tau) = a_1 + a_2\cos\left(\frac{\pi t\tau}{T}\right) \tag{5}$$

so that $f_b(0) = a_1 + a_2$, $f_b(T) = a_1 - a_2$ and $\int_0^T f_b(\tau)d\tau = Ta_1$. If the requirement is for $f = \frac{d}{d\tau}F$ to increase, the total time must decrease, so that $F(T) > T$. For instance, an accelerating car will reach its destination before a car that stays at the same speed. This type of function can be used to increase the velocity if the modified velocity function is made continuous with the velocity before $(t < 0)$ and after $(t > T)$, the period of acceleration. An easy way of enforcing a continuous velocity at times t equal 0 and T is to make the acceleration zero at those times. The acceleration is given by

$$\frac{d}{d\tau}(vf) = f\frac{d}{d\tau}v + v\frac{d}{d\tau}f$$

The second differential of the parametrization function $\frac{d^2}{d\tau^2}F_b$ is zero at the start and end of the time interval.

3. f_c (the dashed line) illustrates a parametrization based on a polynomial function with first and second order differentials, which are zero at each end of the time interval.

$$f_c = a_1 - a_2\frac{280}{T^7}\left(\frac{t^7}{7} - \frac{t^6T}{2} + \frac{3t^5T^2}{5} - \frac{t^4T^3}{4}\right)$$

As in the previous example, $f_c(0) = a_1 + a_2, f_c(T) = a_1 - a_2$ and $\int_0^T f_c(\tau)d\tau = Ta_1$. A continuous acceleration at times t equal 0 and T is most easily achieved by a zero jerk (rate of change of acceleration) at those times. Jerk is given by

$$\frac{d^2}{d\tau^2}(vf) = f\frac{d^2}{d\tau^2}v + 2\frac{d}{d\tau}v\frac{d}{d\tau}f + v\frac{d^2}{d\tau^2}f$$

Versions of f_b are used in the examples that follow.

2.3. Isotropic Parametrization and Lie Theory

In Lie theory, the motion is often represented by elements $g(t)$ of a differentiable group. Most Lie groups used in applications are matrix groups (refer to any text book on the subject, such as Hall [10]). The group is a homogeneous space, and has a natural transitive self-action with a corresponding transitive group of diffeomorphisms, so one can pull back to the identity along the path by the action $g^{-1}(t)$. In particular, the body frame of reference and the tangent space are pulled back to the identity. The body frame (and moments of inertia) are determined at the identity. The trajectory is found using

$$g^{-1}(t)\frac{d}{dt}g(t) = X(t) \tag{6}$$

where $X(t)$ belongs to the tangent space at the identity, known as the Lie algebra of the group. $X(t)$ is represented by a matrix. Marsden and Ratiu [11] used this expression on page 437. It reflects the matrix multiplication operation, which is used in evaluating the expression.

The time parametrization can be applied to Lie theory.

Theorem 2. *The isotropic parametrization $t = F(\tau)$ modifies the motion $g(t)$, where $g^{-1}(t)\frac{d}{dt}g(t) = X(t)$, to*

$$g(F(\tau)) = g^*(\tau) \tag{7}$$

with $(g^*)^{-1}\frac{d}{d\tau}g^*(\tau) = X^*(\tau)$ and $X^*(\tau) = X(F(\tau))f(\tau)$ and $f(\tau) = \frac{d}{d\tau} = F(\tau)$

The modifies trajectory can be found by substituting the parametrization function.

Proof. The proof follows the same steps as for Theorem 1. □

2.4. Anisotropic Parametrization

In this section, the tangent space will be stretched anisotropically. The (linear or angular) speed in each direction is modified independently. For instance, a sideways motion or a turn can be introduced to the trajectory of a hovercraft by stretching the corresponding dimension.

In Geometric Control theory, there are many situations when the optimal velocity can be expressed analytically in terms of several parameters and time. These parameters are dependent on integration constants, such as momentum and energy, and are themselves constants. With anisotropic parametrization, these parameters change as the velocity is modified. In general, the velocity components $\{v_i\}$ are modified

$$v_i(t) \implies v_i^*\left(\{F_j\}, \left\{\frac{d}{d\tau}F_j\right\}\right)$$

where any of the parametrization functions $F_j(\tau)$, and their derivatives can influence the modified velocity component v_i^*. This is illustrated for the wheeled robot example in Section 3.5.

The resulting acceleration can be written as

$$\frac{d}{d\tau}v_i^* = \sum_j \frac{\partial}{\partial F_j}v_i^* \frac{d}{d\tau}F_j + \sum_j \frac{\partial}{\partial F_j}v_i^* \frac{d^2}{d\tau^2}F_j \tag{8}$$

The first term is the acceleration along the geometric (optimal) path. It has been modified by the factor $\frac{d}{d\tau}F_j$, which is the stretching factor. The second term is the acceleration, which is applied to stretch the tangent space and modify the velocity.

3. Examples

The concepts in Section 2 are illustrated using the example of a wheeled robot. Realigning a satellite is another example where the optimal velocity is time dependent. The spin stabilized rotation of a satellite is generally more efficient than the standard eigenaxis rotation; see Maclean et al. [6].

3.1. Wheeled Robot Introduction

To illustrate time parametrization of a time dependent velocity, a wheeled robot is considered (as used by Maclean and Biggs [12]). The position in the plane of the centre of the rear axle is given by $\{x, y\}$ and its orientation by Φ (see Figure 2). The linear velocity and momentum components in the body frame are denoted by $\{v_1, v_2\}$ and $\{p_1, p_2\}$, with w and p_3 for the angular velocity and momentum. The rear wheels cannot slip sideways so $v_2(t) = 0$. They can be powered to change the forward motion. The front wheels can be steered to change the turning motion. With no steering or forward power, there are two invariants: the linear momentum $P = \sqrt{p_1^2 + p_2^2}$ and the optimal Hamiltonian $H = \frac{1}{2}(p_1 v_1 + p_3 w)$. Using the standard equations for the Lie algebra $\mathfrak{se}(2)$, Maclean and Biggs [12] found the optimal velocity function $X(t)$ and the associated trajectory $g(t)$.

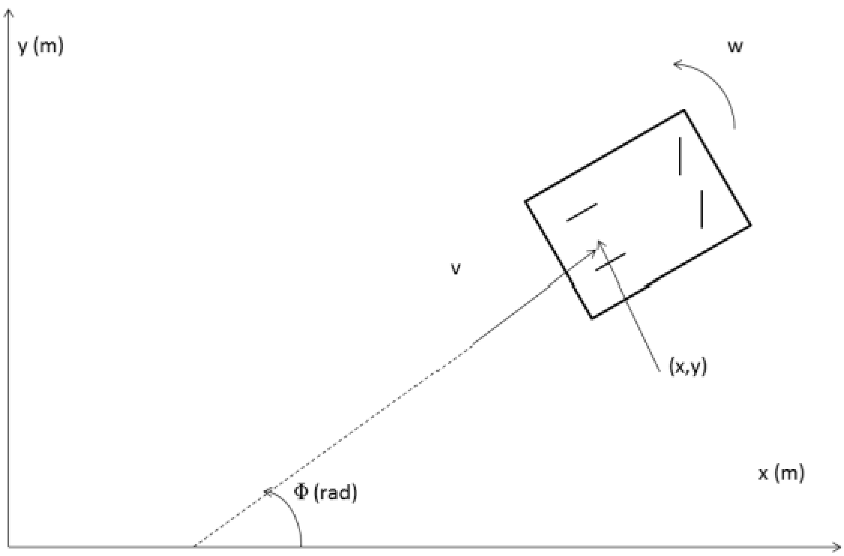

Figure 2. Coordinate system for the wheeled robot.

The optimal velocity components can be written in vector form (denoted by the hat symbol) as

$$\hat{X}(t) \equiv \begin{bmatrix} v_1(t) \\ v_2(t) \\ w(t) \end{bmatrix} = \begin{bmatrix} \frac{P}{m_1}\mathrm{sn}(ut, m) \\ 0 \\ u\mathrm{dn}(ut, m) \end{bmatrix} \tag{9}$$

with the corresponding trajectory expressed in component form as

$$\hat{g}(t) \equiv \begin{bmatrix} x(t) \\ y(t) \\ \Phi(t) \end{bmatrix} = \begin{bmatrix} \frac{P}{m_1 u m}(1 - \mathrm{dn}(ut, m)) \\ \frac{P}{m_1 m}(t - \frac{1}{u}E(\mathrm{am}(ut, m), m)) \\ \mathrm{am}(ut, m) \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} \tag{10}$$

where $\{x_0, y_0\}$ are the values required to start the trajectory at the identity $\{0, 0\}$. The constants $\{m_1, m_3\}$ are defined as $m_1 \equiv p_1/v_1$ and $m_3 \equiv p_3/w$. The Jacobi elliptic functions are written as $\mathrm{sn}(ut, m) \equiv \sin(\mathrm{am}(ut, m))$, $\mathrm{cn}\ (ut, m) \equiv \cos(am(ut, m))$ and $\mathrm{dn}\ (ut, m)$ with the Jacobi parameters determined as (see also Maclean and Biggs [12]):

$$m = \frac{P^2}{2Hm_1} \text{ and } u = \sqrt{\frac{2Hm_1 - P^2}{m_1 m_3}} \tag{11}$$

Standard results from any handbook of mathematical formulas give $\mathrm{am}(ut, m) = \int_0^t u\mathrm{dn}(ut, m)dt$ as the amplitude of the rotation, and $E(\mathrm{am}(ut, m), m) \equiv \int_0^t u\mathrm{dn}^2(ut, m)\ \mathrm{dt}$ as the elliptic integral of the second kind.

The natural trajectory of the robot given in Equation (10) with no external forces applied is illustrated in Figure 3. The robot starts pointing forward $(\Phi(0) = 0)$ at the identity $\{0, 0\}$ with no forward motion $(v_1 = 0)$, but with some initial angular motion $(w \neq 0)$, which causes it to turn anticlockwise. There is no sideways slip at the rear wheels, so the turning motion is partially converted into a forward motion. The forward motion reaches a maximum at the point A, when the angular speed is at its minimum. The rate of turn then increases again until, at point B, $\Phi(t) = \pi$ and the forward motion is zero again $(v_1 = \sin(\Phi(t)) = 0)$. The robot then moves backwards, but continues to rotate anticlockwise for the second loop.

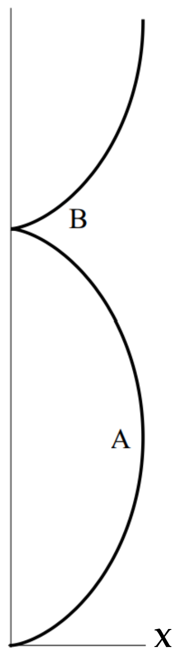

Figure 3. Optimal trajectory of the wheeled robot.

This trajectory satisfies the necessary conditions for optimality, but only sufficiently for a small time. The trajectory used in the following section is before the reversal at point B. At that point, the direction of the force preventing the sideways slip becomes important.

3.2. Start And Finish at Rest

The motion of a wheeled robot can be reparametrized to start and finish at rest at time $\tau = T$ by using any parametrization function $t = F(\tau)$, which has $\frac{d}{d\tau}F = f$ and $f(0) = f(T) = 0$. Using Theorem 2, the velocity and trajectory of the wheeled robot become, after substitution into Equations (9) and (10),

$$\hat{X}(F(\tau))f(\tau) = \begin{bmatrix} v_1(F(\tau))f(\tau) \\ v_2(F(\tau))f(\tau) \\ w(F(\tau))f(\tau) \end{bmatrix} = \begin{bmatrix} \frac{P}{m_1}\mathrm{sn}(uF(\tau),m)f(\tau) \\ 0 \\ u\mathrm{dn}(uF(\tau),m)f(\tau) \end{bmatrix} \tag{12}$$

$$\hat{g}(t)(F(\tau)) \equiv \begin{bmatrix} x(F(\tau)) \\ y(F(\tau)) \\ \Phi(F(\tau)) \end{bmatrix} = \begin{bmatrix} \frac{P}{m_1 um}(1-\mathrm{dn}(uF(\tau),m)) \\ \frac{P}{m_1 m}(F(\tau) - \frac{1}{u}E(\mathrm{am}(uF(\tau),m),m)) \\ \mathrm{am}(uF(\tau),m) \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix} \tag{13}$$

Using the parametrization function $f(\tau) = 1 - \cos(\frac{2\pi}{T}\tau)$, the original trajectory is tracked, with modified speed, as shown in Figure 3. The parametrization has changed the time at which each point is achieved. On the left of Figure 4, the x and y displacements from the origin are illustrated. The dotted line is the original y displacement before parametrization. The dashed line is the original x displacement which reaches a maximum as shown in Figure 3 before turning back. The solid lines are the displacements after parametrization. The displacements are smaller at the start until the speed increases. The same final position is achieved (and also the same half point due to the parametrization function chosen). On the right of Figure 4, the linear velocity is shown, with and without the parametrization.

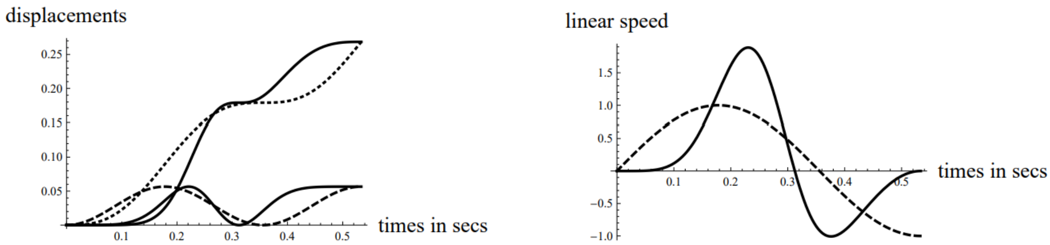

Figure 4. Wheeled robot starting and finishing at rest.

Since the tangent space is stretched isotropically by the parametrization function, the momentum P is modified to $Pf(\tau)$, and the Hamiltonian H to $H(f(\tau))^2$. The Jacobi parameter m, which effects the shape of the function, is unchanged (Equation (11)).

3.3. Eliminating The Turning Motion

A second example of isotropic stretching is to eliminate the turning motion by stopping the robot and restarting with no angular motion.

The stop is achieved by using a parametrization function, which has $\frac{dF}{dt} = 0$ at time T_2, as shown in Figure 5. The restart involves only linear motion. The unmodified velocity v is constant, and is multiplied by a stretching factor to give $v^* = vf(\tau)$.

The angular and linear velocities (w and v) are both discontinuous, but, because the parametrization factor $\frac{d}{dt}F$ is zero at time T_2, the modified velocities (w^* and v^*) and accelerations are continuous.

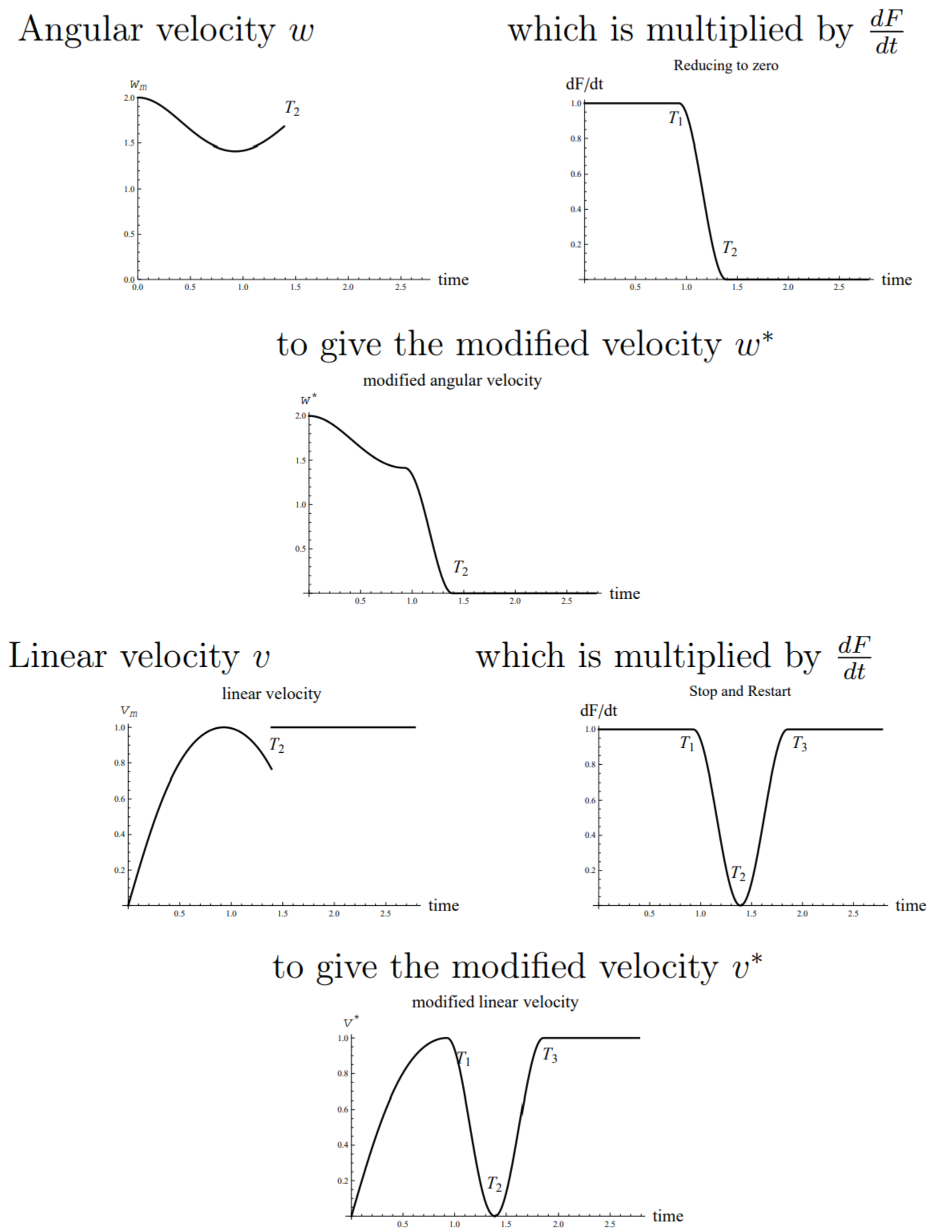

Figure 5. Velocity components of a Stop Start Trajectory.

3.4. Summary of Isotropic Stretching

This parametrization method provides a simple means of modifying the velocity to match boundary conditions. The parametrization function can be constructed to meet various constraints, such as the maximum acceleration forces and the smoothness of applying those forces. These considerations influence the time taken to accelerate. The advantage of the method is that the modified trajectory is easy to evaluate by substitution.

3.5. Anisotropic Stretching

In this section, anisotropic stretching is used to eliminate the turning motion of the wheeled robot without coming to a stop by shrinking the direction which corresponds to angular motion, and stretching the direction of the linear motion.

Wheeled Robot Example

The wheeled robot described earlier was moving with linear and angular momentum. In this section, the angular momentum is eliminated by time T_2, as shown in Figure 6. Simultaneously, the linear motion is continued and then increased after the turning has stopped.

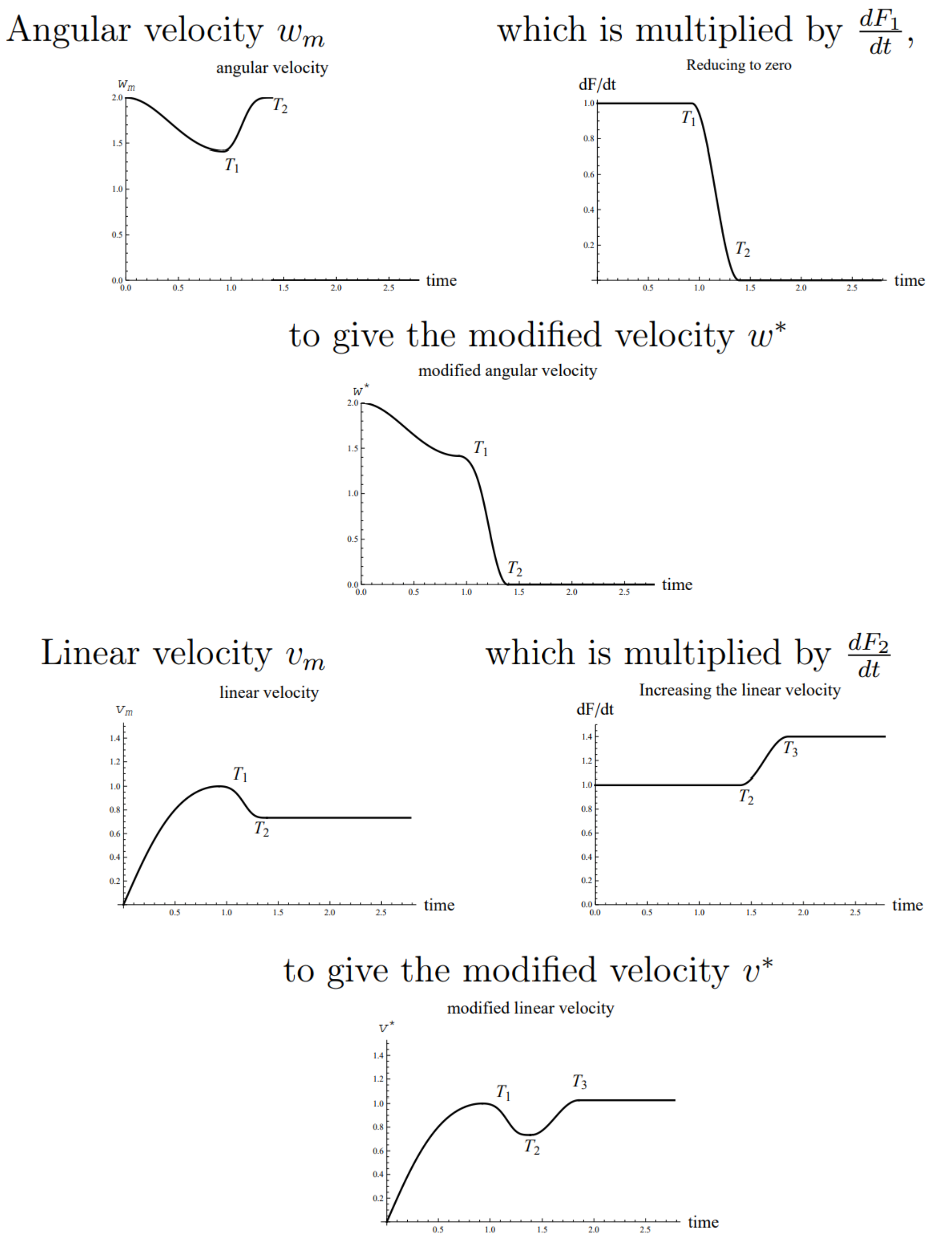

Figure 6. Velocity components without the stop and restart.

There are two parametrization functions: $F_1(\tau)$, which stops the angular motion, and $F_2(\tau)$, which maintains the linear motion and then increases it back to its original maximum. This is required because reducing the turning motion of the wheeled robot effectively brakes the forward motion. A forward thrust is required until the required forward speed is reached.

The 'constants' P and H used in determining the optimal velocity components are modified by the parametrization functions to

$$P_F = f_2 P, \text{and}, H_F = \frac{1}{2}(f_2^2 p_1 v_1 + f_1^2 p_3 w) \tag{14}$$

The Jacobi parameters (originally m and u) are also modified (by substitution into Equation (6)) to

$$m_F = \frac{P_F^2}{2H_f m_1} = \frac{f_2^2 P^2}{f_2^2 P^2 + f_1^2 u^2 m_3 m_1} = \left(\frac{f_1^2}{f_2^2}\left(\frac{1}{m} - 1\right) + 1\right)^{-1} \tag{15}$$

and $u_F = f_1 u$ with $m_F = m$ when $f_1/f_2 = 1$ (the isotropic stretching of the previous section). It follows that $u_t \implies uF_1(\tau)$. When the angular velocity is eliminated, $f_1 = 0$ and $m_F = 1$. Once the angular motion stops, the linear velocity remains constant. The parametrization function for the linear component increases the velocity, as seen in the last diagram of Figure 6.

The resultant velocity function is, after substituting into Equation (12),

$$\hat{X} = \begin{bmatrix} f_2(\tau)v^*(\{F_1(\tau), F_2(\tau)\}) \\ 0 \\ f_1(\tau)w^*(\{F_1(\tau), F_2(\tau)\}) \end{bmatrix} \equiv \begin{bmatrix} f_2(\tau)\frac{P}{m_1}\mathrm{sn}(uF_1(\tau), m_F) \\ 0 \\ f_1(\tau)u\mathrm{dn}(uF_1(\tau), m_F) \end{bmatrix} \tag{16}$$

The displacement is given by

$$\Phi(\tau) = \int_0^\tau f_1(\tau)u\mathrm{dn}(uF_1(\tau), m_F)d\tau \tag{17}$$

$$x(\tau) = \int_0^\tau f_2(\tau)P\mathrm{sn}(uF_1(\tau), m_F)\cos(\Phi(\tau))d\tau \tag{18}$$

$$y(\tau) = \int_0^\tau f_2(\tau)P\mathrm{sn}(uF_1(\tau), m_F)\sin(\Phi(\tau))d\tau \tag{19}$$

Figure 7 compares the stop start trajectory from the previous section with the one above, which stops the turning motion without coming to rest. The initial stage (shown by the dashed line in Figure 7b) is the natural motion of the wheeled robot with no forces applied, and is the same in both cases. The turning motion is then stopped at time T_2 (the solid line). Since the linear motion does not stop in Figure 7b, the robot travels further before the start of the linear only motion. The dotted line (to time T_3) shows the increase in linear motion, before the trajectory continues at constant speed (the dot dashed line). This dotted part of the trajectory is found using parametrization of a linear only motion in the fixed direction.

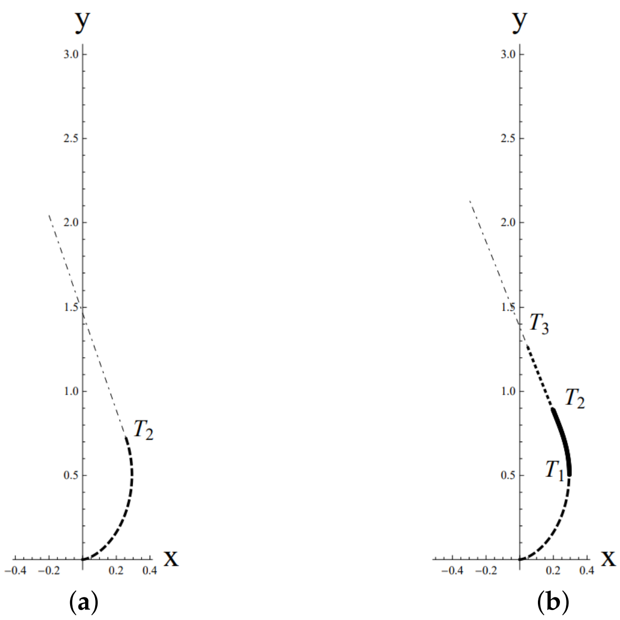

Figure 7. (**a**) Stop start trajectory; (**b**) trajectory avoiding the stop.

4. Conclusions

Time parametrization changes the velocity by isotropically stretching the tangent space. In this paper, this concept is extended to anisotropic stretching of the tangent space. For a basic example, this can introduce or eliminate turns and sideways motion.

With isotropic stretching, the original path is tracked with modified speed, enabling velocity boundary conditions to be achieved. The modified trajectory can be found by substitution, without the need to reintegrate the modified velocity function.

With anisotropic stretching of the tangent space, the velocity is modified to change other features of the motion, and not just its magnitude. For example, a turn can introduced by stretching the tangent space in the corresponding direction. This stretching changes the relationship between components of the optimal velocity function as expressed by various parameters. The parameters effecting the velocity become time-dependent, and the resultant velocity function has to be integrated numerically to determine the trajectory.

This parametrization method provides a simple means of modifying the velocity to pass through a series of velocity states that satisfies the original optimal control problem, which minimized the quadratic cost function. This means that, when the forces causing the change induced by the parametrization functions are removed, the robot continues to move along a trajectory that minimizes the cost function.

This paper generalizes the previous one direction velocity case to optimal velocity functions which are time dependent. It also extends from always starting and finishing at rest to starting and finishing at any multiple of the original velocity. These velocity functions can be modified anisotropically by applying multiple parametrization functions, avoiding the need to stop and restart when changing direction.

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References

1. Spindler, K. New Attitude Control algorithms with applications to Space telescopes. In Proceedings of the 12th International Symposium on 'Space Flight Dynamics', Darmstadt, Germany, 2–6 June 1997; pp. 227–232.
2. Chaturvedi, N.; Sanyal, A.K.; McClamroch, N.H. Rigid-Body Attitude Control. *IEEE Control Syst.* **2011**, *31*, 30–51.
3. Smith, R.N. Geometric Control Theory And Its Application To Underwater Vehicles. Ph.D. Thesis, University of Hawaii at Manoa, Honolulu, HI, USA, 2008.
4. Flasskamp, K.; Ober-Blobaum, S. Energy efficient control for mechanical systems based On inherent dynamical structures. In Proceedings of the 2012 American Control Conference, Montreal, QC, Canada, 27–29 June 2012; pp. 2609–2614.
5. Frazzoli, E.; Dahleh, M.A.; Feron, E. Maneuver-based motion planning for nonlinear systems with symmetries. *IEEE Trans. Robot.* **2005**, *21*, 1077–1091. [CrossRef]
6. Maclean, C.; Pagnozzi, D.; Biggs, J.D. Computationally light attitude controls for resource limited nano-spacecraft. In Proceedings of the 62nd International Astronautical Congress, Cape Town, South Africa, 3–7 October 2011; pp. 1–9.
7. Xargay, E.; Kaminer, I.; Pascoal, A.M.; Hovakimyan, N.; Dobrokhodov, V.; Cichella, V.; Aguiar, A.P.; Ghabcheloo, R. Time-Critical Cooperative Path Following of Multiple Unmanned Aerial Vehicles over Time-Varying Networks. *J. Guid. Control Dyn.* **2013**, *36*, 499–516. [CrossRef]
8. Sharma, V.D.; Lee, J.; Andrews, M.; Hadžić, I. Hybrid Classical/RL Local Planner for Ground Robot Navigation. *arXiv* **2024**, arXiv:2410.03066.
9. Aguiar, A.P.; Hespanha, J.P.; Kokotović, P.V. Performance limitations in reference tracking and path following for nonlinear systems. *Automatica* **2008**, *44*, 598–610. [CrossRef]
10. Hall, B.C. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction (Graduate Texts in Mathematics)*; Springer: Berlin/Heidelberg, Germany, 2004.

11. Marsden, J.E.; Ratiu, T.S. *Introduction to Mechanics and Symmetry*, 2nd ed.; Springer: Berlin/Heidelberg, Germany, 1999.
12. Maclean, C.; Biggs, J.D. Path planning for simple wheeled robots: Sub-Riemannian and elastic curves on SE (2). *Robotica* **2013**, *31*, 1285–1297. [CrossRef]