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Research Article

Rigid Body Trajectories in Different 6D Spaces

Carol Linton,[1] William Holderbaum,[1] and James Biggs[2]

[1] *School of Systems Engineering, University of Reading, Reading RG6 6AY, UK*
[2] *Department of Mechanical Engineering, University of Strathclyde, Glasgow G1 1XQ, UK*

Correspondence should be addressed to Carol Linton, carol1linton@btopenworld.com

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The objective of this paper is to show that the group $SE(3)$ with an imposed Lie-Poisson structure can be used to determine the trajectory in a spatial frame of a rigid body in Euclidean space. Identical results for the trajectory are obtained in spherical and hyperbolic space by scaling the linear displacements appropriately since the influence of the moments of inertia on the trajectories tends to zero as the scaling factor increases. The semidirect product of the linear and rotational motions gives the trajectory from a body frame perspective. It is shown that this cannot be used to determine the trajectory in the spatial frame. The body frame trajectory is thus independent of the velocity coupling. In addition, it is shown that the analysis can be greatly simplified by aligning the axes of the spatial frame with the axis of symmetry which is unchanging for a natural system with no forces and rotation about an axis of symmetry.

1. Introduction

The motion of a rigid body in Euclidean space E^3 consists of 3 translations and 3 rotations about the centre of mass. The ways in which these are combined determine the calculated trajectory of the body. This paper considers several methods of determining the corresponding velocities and the resultant trajectories and investigates the consequences of each method.

Previous work on Lie groups has used the special Euclidean group $SE(3)$ with no imposed structure so that it is the semidirect product of the translations in $\mathbb{R}^3$ and rotations in the group $SO(3)$. The mapping between the groups used by Holm [1] and Marsden and Ratiu [2] is expressed in two formats in this paper: as the mapping itself and as the integration using $SE(3)$. This method gives the trajectory in the body frame, which cannot be used to determine the trajectory in the spatial frame needed for many applications. The body frame trajectory is the independent of the velocity coupling. There is no natural way of weighting the rotations and translations to measure the distance (in 6 dimensions) along a trajectory using the semidirect product.

There is no bi-invariant Riemannian metric on $SE(3)$. There are natural metrics on $SO(4)$ and $SO(1,3)$—the trace form, which can be inherited by $SE(3)$ with the appropriate scaling. Etzel and McCarthy [3] used a metric on $SO(4)$ as a model for a metric on SE(3). Larochelle et al. [4] projected $SE(3)$ onto $SO(4)$ to obtain a metric. In this paper, the linear displacements are scaled so that they are small compared with a unit hypersphere. This enables $SE(3)$ to be projected vertically onto $SO(4)$ and $SO(1,3)$. This projection is extended by imposing the Lie-Poisson structure on $SE(3)$ as mentioned in [5].

The 6D Lie groups $SO(4)$, $SO(1,3)$, and $SE(3)$ with imposed Lie-Poisson structure are compared and shown to result in related trajectories, which approximate to the same values for small linear displacements. The trajectories are in a fixed frame which is the requirement for planning and controlling the motions.

Four methods of combining the translations and rotations are investigated:

(i) A semidirect product—where the rotation changes the body frame, but not the linear velocity itself: that is, there is no coupling of the angular and translational velocities.

(ii) The special Euclidean group $SE(3)$ with an imposed Lie-Poisson structure, where the rotation induces a change in the linear velocity to conserve angular momentum.

(iii) The rotation group $SO(4)$ which can approximate the previous results if the linear displacements are scaled down.

(iv) The Lorentz group $SO(1,3)$: the latter is included as a continuation from the other groups, and not as a practical alternative.

Finding the trajectory of a rigid body has 2 steps:

(1) determining how the velocity changes over time as the velocity components couple together and

(2) integrating the velocity function down to the base manifold to give the trajectory.

Both these depend on the group, for example, is the space that the group represents flat, convex or concave? The formulas for the general case are derived, but the examples are based on the simplest case with no external forces and rotation about an axis of symmetry.

Any combinations of rotations can be represented by rotation about a single axis of rotation. If this axis is also an axis of symmetry, the rotational axis does not change, and there is no precession. By choosing the axes of the fixed frame appropriately, the initial motion of anybody can be defined as an initial rotation about one axis, with some linear motion along and some motion perpendicular to that axis. Details are provided in the Appendix. In a natural system with no forces, the angular momentum remains constant, and the resulting trajectories determined using the various methods can be compared. The linear displacement of the centre of mass is used. The initial velocities used in the examples are

$$v_0 = \begin{bmatrix} v_x \\ v_y \cos(f) \\ v_y \sin(f) \end{bmatrix}, \qquad \phi_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{bmatrix}, \tag{1.1}$$

where f is a phasing constant. Using the rotation identified in the Appendix, this can be generalized to any initial conditions in a system with no forces so long as the rotation is about an axis of symmetry.

Section 2 introduces the basic ideas of geometric control theory and applies them to the rotation group $SO(3)$. The linear motion is incorporated using the semidirect product. Section 3 discusses the Lie-Poisson structure which is implicit for the rotation group $SO(4)$ and the Lorentz group $SO(1,3)$ and can be imposed for $SE(3)$. With that structure, the differential equations of motion can be derived from the Hamiltonian. The trajectories are found in Section 4. The results for each group are compared. A comparison of the results from $SO(4)$ and the structured $SE(3)$ in Section 5 show that they are interchangeable for small displacements. The strength of the coupling between the angular and translational velocities tends to the same value as the accuracy of the scaling is improved. The final Section 6 compares the different methods of combining linear and rotational motions, and when the methods fail.

2. Geometric Control Theory and Notation

This section provides the basic ideas of geometric control theory. A fuller explanation is available in many texts such as [6]. The rotations and translations are considered separately. This is used as an opportunity to introduce the necessary notation for combining them in the following sections.

2.1. Lie Theory

A trajectory is represented by a matrix $g(t) \in G$, where G is the matrix Lie group which reflects the structure or shape of the space on which the trajectory lies. In Lie theory, the trajectory is pulled back to the origin by the action of $g^{-1}(t)$. The tangent field at the origin $\mathrm{X} \in \mathfrak{g}$ (where $\mathfrak{g}$ is the Lie algebra) determines the trajectory through the expression

$$g^{-1}(t)\frac{dg}{dt}(t) = \mathrm{X}. \tag{2.1}$$

One finds that

$$g(t) = \exp(\mathrm{X}t) \quad \text{if X is time independent.} \tag{2.2}$$

If $X = X(t)$, time dependent, an analytic solution is difficult to find in most cases, so the forward Euler method is used in this paper to demonstrate the general form of the solutions as follows:

$$g_{n+1} = g_n \exp(\mathrm{X}_n s), \tag{2.3}$$

where s is the step length, and n is the step number so that $X_n = X(t_n)$ is the tangent field at time $t_n = ns$. The trajectory started at $g(0) = I$.

Fact. The forward Euler method uses

$$\frac{g_{n+1} - g_n}{s} = \frac{dg_n}{dt} = g_n \mathrm{X}_n \tag{2.4}$$

so that

$$g_{n+1} = g_n(1 + sX_n) \simeq g_n \exp(sX_n). \tag{2.5}$$

Alternatively, assume that X_n is constant between times t_n and t_{n+1} so that the incremental motion is given by $\delta g_n = \exp(X_n s)$. This increment is applied to the previous configuration using $g_{n+1} = g_n \delta g_n$ which gives (2.3).

2.2. *Rotations*

Rotation in 3 dimensions is used to demonstrate the ideas of the previous section. A body rotating in space about its fixed centre of mass can have angular velocity $\{w_i\}$ for $i \in \{1, 2, 3\}$ about three orthogonal axes $\{e_i\}$. The resultant velocity X can be written in both coordinate and matrix forms as

$$X = \sum_{i=1}^{3} w_i e_i = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}. \tag{2.6}$$

From this notation, the base matrices $\{e_i\}$ of the Lie algebra $\mathfrak{so}(3)$ can be extracted.

Any rotation in Euclidean space can be represented by a rotation about a single axis, so, by choosing the axes of the body frame appropriately, all initial rotation can be represented about the e_1 axis. The Appendix gives the required rotation of the body frame to achieve this. The initial rotational velocity can then be written as

$$\phi_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{bmatrix}. \tag{2.7}$$

In a natural motion of a body rotating about an axis of symmetry, there are no forces and no coupling with other motions. (The discussion in Section 3 can be applied to $SO(3)$ to confirm this, with $o_2 = o_3$). The angular velocity is unchanging. The axis of rotation does not change (no precession). The attitude of the body $\Phi(t) \in SO(3)$ is the solution of

$$\Phi^{-1}(t) \frac{d\Phi(t)}{dt} = \phi_0. \tag{2.8}$$

The resulting attitude at time t is

$$\Phi(t) = \exp(\phi_0 t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(wt) & -\sin(wt) \\ 0 & \sin(wt) & \cos(wt) \end{bmatrix}. \tag{2.9}$$

2.3. *Semidirect Product* $\mathbb{R}^3 \ltimes SO(3)$

To represent a rigid body rotating about its centre of mass and moving through Euclidean space, the rotational and translational motions need to be combined. The semidirect product is used to represent the motion from the body frame prospective. The rotational motion has already been determined. The linear velocity can be written in matrix and coordinate representations of $\mathbb{R}^3$ as

$$v = [v_1, v_2, v_3]^T = \sum_{i=1}^{3} v_i e_i, \tag{2.10}$$

where here $\{e_i\}$ is the orthogonal basis in $\mathbb{R}^3$.

The semidirect product enables the rotation to influence the linear motion. The two elements act on each other through an action $\circ$ defined by

$$(x_1, \Phi_1) \circ (x_2, \Phi_2) = (x_1 + \Phi_1 x_2, \Phi_1 \cdot \Phi_2) \tag{2.11}$$

as used by Marsden [2, page 22] and Holm [1, page 110]. This rotates the body frame. It can also be written in matrix form as

$$\begin{bmatrix} 1 & 0 \\ x_1 & \Phi_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ x_2 & \Phi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x_1 + \Phi_1 x_2 & \Phi_1 \Phi_2 \end{bmatrix} \tag{2.12}$$

which is the format of the Lie group $SE(3)$ with no structure imposed. The attitude $\Phi(t)$ is found using (2.1). The displacement x at time t is found by integration as

$$x = \int_{s=0}^{t} \Phi(s) v(s) ds, \tag{2.13}$$

where $\Phi(s)$ is the rotation achieved at time s (since integration is the method of adding incremental changes in $\mathbb{R}^3$). Alternatively the 2 velocity functions can be combined into one 4×4 matrix to give the equation

$$g^{-1}(t) \frac{dg}{dt} = \begin{bmatrix} 0 & 0 \\ v(t) & \phi(t) \end{bmatrix} \tag{2.14}$$

which has the solution $SE(3) \ni g(t) = \begin{bmatrix} 1 & 0 \\ x(t) & \phi(t) \end{bmatrix}$. The same result is obtained from the integration in (2.13).

In the simple example defined in expression (1.1), the initial translation velocity is $v_0 = [v_x, v_y \cos(f), v_y \sin(f)]^T$. In a natural motion of a body moving in Euclidean space, there are no forces and no coupling with other motions. The translational velocity is unchanging. The combined velocity function is

$$v \ltimes \phi = \begin{bmatrix} v_x \\ v_y \cos(f) \\ v_y \sin(f) \end{bmatrix} \ltimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -w \\ 0 & w & 0 \end{bmatrix}, \tag{2.15}$$

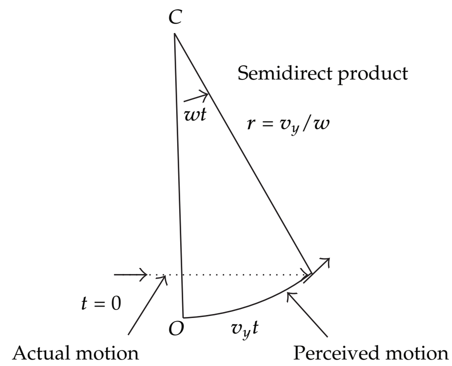

Figure 1: Actual motion in determined using semidirect product.

and the configuration at time t is

$$\begin{bmatrix} v_x t \\ \dfrac{v_y}{w}\left(\sin(wt+f)-\sin(f)\right) \\ \dfrac{v_y}{w}\left(\cos(f)-\cos(wt+f)\right) \end{bmatrix} \ltimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(wt) & -\sin(wt) \\ 0 & \sin(wt) & \cos(wt) \end{bmatrix}. \tag{2.16}$$

Thus, the semidirect product produces a trajectory as perceived in the body frame. Although the translational velocity is fixed at v_0, and the perceived trajectory is curved. In the body frame, the velocity in the $y - z$ plane is v_y so, at time t, the origin O is seen at an angle wt behind the body having traveled a distance $v_y t$. The perceived radius of travel r is given by

$$rwt = v_y t \tag{2.17}$$

so the perceived radius is v_y/w as seen in Figure 1 and (2.16).

In a spatial frame, a rigid body rotating about its centre of mass and moving through Euclidean space does not move in a straight line but moves by spirally about an axis. If the rotation is about a fixed axis with constant translational speed, then the trajectory is a screw motion with constant pitch as proved by Chasles in his Screw Theory of motion. The angular momentum induces a change in the translational velocity.

In the semidirect product, the two velocity matrices were combined as $\begin{bmatrix} 0 & 0 \\ v & \phi \end{bmatrix}$ to calculate the trajectory, where v is the 3×1 column matrix of linear velocities, and ϕ is the 3×3 angular velocity matrix given above. In Section 4, the velocity matrix is taken as

$$\begin{bmatrix} 0 & -\varepsilon v^T \\ v & \phi \end{bmatrix} \tag{2.18}$$

with $\epsilon \in \{1, 0, -1\}$ for $\mathfrak{so}(4), \mathfrak{se}(3), \mathfrak{so}(1,3)$, respectively. Other variations in the order of columns and rows are possible but produce equivalent results. Alternative values of ε create nonclosed groups and are not considered here.

In the next section, a structure is imposed on these groups which creates a relationship between the rotational and linear elements. With the Lie-Poisson structure, the velocities interact, with rotations inducing changes in the linear motion. The motions are described in the spatial frame, rather than the body frame.

3. Structure of the Lie Algebras

In this section, the structure of the Lie groups is identified in terms of structure constants. This structure (or shape of the space) determines how motion in one direction influences motion in another, and how the motions add together to arrive at a configuration in space.

(i) The rotation group $SO(4)$ has an obvious structure in that it represents rotations. The space has positive curvature which is related to the fixed Casimir $C = \sum_{i=1}^{6} p_i^2$ where the $\{p_i\}$ are the momentum components. The group has a bilinear map and is a Poisson manifold.

(ii) $SO(1,3)$ has some similar characteristics. It is a space with negative curvature in some directions since one of the fixed Casimirs is $C = \sum_{i=1}^{3} p_i^2 - \sum_{i=4}^{6} p_i^2$.

(iii) For the Lie group, $SE(3)$, a Lie-Poisson structure is imposed. The equivalent Casimir is $C = \sum_{i=1}^{3} p_i^2$. There is thus no automatic relative weighting between the rotational and linear elements.

More comparative data is provided by Jurdjevic [7]. In all cases, there is another fixed Casimir $C = \sum_{i=1}^{3} p_i p_{i+3}$ which pairs the momentum types but adds no information about the relative weighting of the two types.

The base matrices for the 6-dimensional Lie algebras being considered can be seen from the equation

$$\sum_{i=1}^{3} v_i e_i + w_i e_{i+3} = \begin{bmatrix} 0 & -\varepsilon v_1 & -\varepsilon v_2 & -\varepsilon v_3 \\ v_1 & 0 & -w_3 & w_2 \\ v_2 & w_3 & 0 & -w_1 \\ v_3 & -w_2 & w_1 & 0 \end{bmatrix} \tag{3.1}$$

with $\epsilon \in \{1, 0, -1\}$ for $\mathfrak{so}(4), \mathfrak{se}(3), \mathfrak{so}(1,3)$, respectively. The orthogonal basis for the dual Lie algebra $\mathfrak{g}^*$ is $\{e^i\}$ given by $e^i = \mathbb{I} e_i$, where $\mathbb{I} = I_{4\times 4}$, the unit matrix, is a bilinear form.

From these base matrices, the structure of the group is quantified in terms of the structure constants in Section 3.1. For a group with a Lie Poisson structure, these same constants provide an interaction between the functions on the algebra in Section 3.2. If those functions are the Hamiltonian and the momentum components, the equations of motion can be expressed in terms of the structure constants in Section 3.3. Finally in Section 3.4, the velocity matrix is found for the three Lie algebras.

3.1. Structure Constants

In many situations, the addition of two motions depends on the order in which they occur-rotate then move, or move then rotate. This is reflected in the nonassociative matrix multiplication: $AB \neq BA$ in most cases. Structure constants c_{ij}^k are used to describe this nonassociative action in a Lie algebra $\mathfrak{g}$. They are defined using the Lie bracket by (see [6, page 56])

$$c_{ij}^k e_k = [e_i, e_j] = e_i e_j - e_j e_i. \tag{3.2}$$

The value of the structure constants is easily shown by matrix multiplication to be, by using $\epsilon \in \{1, 0, -1\}$ for $\mathfrak{so}(4)$, $\mathfrak{se}(3)$, and $\mathfrak{so}(1,3)$,

$$\begin{gathered} c_{15}^3 = c_{26}^1 = c_{34}^2 = c_{42}^3 = c_{45}^6 = c_{53}^1 = c_{56}^4 = c_{61}^2 = c_{64}^5 = 1 \\ c_{16}^2 = c_{24}^3 = c_{35}^1 = c_{43}^2 = c_{46}^5 = c_{51}^3 = c_{54}^6 = c_{62}^1 = c_{65}^4 = -1 \\ c_{12}^6 = c_{23}^4 = c_{31}^5 = \epsilon \\ c_{13}^5 = c_{21}^6 = c_{32}^4 = -\epsilon. \end{gathered} \tag{3.3}$$

They are the same for the dual algebra, $\mathfrak{g}^*$.

3.2. *Poisson Bracket*

In order to develop the coordinate equations for these Lie groups, it is necessary to introduce the Poisson bracket and show that it describes the structure in the same way as the Lie bracket.

The canonical form of the Poisson bracket is (see [8, page 20] onwards)

$$\{F, E\} = \frac{\partial F}{\partial q}\frac{\partial E}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial E}{\partial q} \tag{3.4}$$

with $F(p,q)$ and $E(p,q)$ being functions on the cotangent space with canonical coordinates (q,p), representing position and momentum. To express this in other coordinates, write $q = q(z_i)$ and $p = p(z_j)$ and get, using partial differentiation,

$$\{F, E\} = \frac{\partial F}{\partial z_i}\{z_i, z_j\}\frac{\partial E}{\partial z_j}. \tag{3.5}$$

If these functions are pulled back to the origin, there is no positional dependence and so

$$\{F, E\} = \frac{\partial F}{\partial p_i}\{p_i, p_j\}\frac{\partial E}{\partial p_j}. \tag{3.6}$$

For a Lie algebra with a Poisson structure, it can be shown that $\{p_i, p_j\} = -c_{ij}^k p_k$ (see [8, page 50]), and the Poisson bracket describes the structure in a similar way to the Lie Bracket. Hence the Poisson relationship between any functions E and F on $\mathfrak{g}^*$ becomes

$$\{E, F\} = -c_{ij}^k p_k \frac{\partial E}{\partial p_i}\frac{\partial F}{\partial p_j}. \tag{3.7}$$

3.3. Hamiltonian Flow

For a Poisson manifold, Bloch [9, page 121] defines X_H, the Hamiltonian vector field of H, as the unique vector field such that

$$\frac{df}{dt} = X_H(f) = \{f, H\} \quad \forall f. \tag{3.8}$$

From (3.7) above, the Hamiltonian vector field is

$$X_H = -c_{ij}^k p_k \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} \tag{3.9}$$

and, for all functions f,

$$\frac{df}{dt} = -c_{ij}^k p_k \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i} f. \tag{3.10}$$

With a Poisson structure naturally or imposed, the coordinate equation for Lie algebras is found by setting $f = p_i$ for $i \in \{1, 2, 3, 4, 5, 6\}$

$$\dot{p}_i = -c_{ij}^k p_k \frac{\partial H}{\partial p_j}, \tag{3.11}$$

where $\{p_i\}$ are the components of momentum. The relationship with the velocities is $p_i = m_i v_i$ for $i \in \{1, 2, 3\}$ and $p_{i+3} = o_i w_i$, where the linear velocities are $\{v_1, v_2, v_3\}$, the angular velocities are $\{w_1, w_2, w_3\}$, the moments of inertia are $\{o_1, o_2, o_3\}$, and the added masses are $\{m_1, m_2, m_3\}$ allowing for a different mass in each direction. The Hamiltonian for a natural motion, with no forces, is the kinetic energy so

$$H = \frac{1}{2} \sum_{i=1}^{3} \left(m_i v_i^2 + o_i w_i^2 \right). \tag{3.12}$$

3.4. Velocity Functions for the Groups

The velocity functions change with time in each direction as determined in this section and are shown in Table 1. The velocities for $E^3 \ltimes SO(3)$ arise from the separate subgroups and were found earlier. For the Lie groups, the differential equations for the momentum components are found using (3.11), the Hamiltonian given above, and the structure constants found by matrix multiplication of the base matrices. First the Hamiltonian is written in terms of momentum components

$$H = \frac{1}{2} \sum_{i=1}^{3} \left(\frac{p_i^2}{m_i} + \frac{k_i^2}{o_i} \right), \tag{3.13}$$

Table 1: Velocity functions for the various groups.

	$\mathbb{R}^3 \ltimes SO(3)$	$SO(4)$	$SE(3)$	$SO(1,3)$
$v_1(t)$	v_x	v_x	v_x	v_x
$v_2(t)$	$v_y \cos(f)$	$v_y \cos(w_4 t + f)$	$v_y \cos(w_5 t + f)$	$v_y \cos(w_6 t + f)$
$v_3(t)$	$v_y \sin(f)$	$v_y \sin(w_4 t + f)$	$v_y \sin(w_5 t + f)$	$v_y \sin(w_6 t + f)$
$w_1(t)$	w	w	w	w
$w_2(t)$	0	0	0	0
$w_3(t)$	0	0	0	0
		$w_4 = (1 - o_1/m)w$	$w_5 = w$	$w_6 = (1 + o_1/m)w$

where $\{k_i\} = \{o_i w_i\} = \{p_{i+3}\}$ are the angular momentum components. The required differential equations for the momentum are, from (3.11) and the structure constants,

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} \dfrac{-p_3 k_2}{o_2} + \dfrac{\epsilon p_3 k_2}{m_3} + \dfrac{p_2 k_3}{o_3} - \dfrac{\epsilon p_2 k_3}{m_2} \\ \dfrac{p_3 k_1}{o_1} - \dfrac{\epsilon p_3 k_1}{m_3} - \dfrac{p_1 k_3}{m_1} + \dfrac{\epsilon p_1 k_3}{o_1} \\ \dfrac{\epsilon p_2 k_1}{m_2} - \dfrac{p_2 k_1}{o_1} + \dfrac{p_1 k_2}{o_2} - \dfrac{\epsilon p_1 k_2}{m_1} \\ \dfrac{k_2 k_3 (o_2 - o_3)}{o_2 o_3} + \dfrac{p_2 p_3 (m_2 - m_3)}{m_2 m_3} \\ \dfrac{k_3 k_1 (o_3 - o_1)}{o_1 o_3} + \dfrac{p_1 p_3 (m_3 - m_1)}{m_1 m_3} \\ \dfrac{k_1 k_1 (o_1 - o_2)}{o_2 o_3} + \dfrac{p_2 p_1 (m_1 - m_2)}{m_2 m_1} \end{pmatrix}. \tag{3.14}$$

In order to demonstrate the properties of the various groups, the following simplifying assumptions are made: $m_i = m$ for all i. The rotation is assumed to be about the axis of symmetry so $o_3 = o_2$. The momentum equations become

$$\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} (m - \epsilon o_2)\left(\dfrac{k_3 p_2 - k_2 p_3}{(m o_2)}\right) \\ \dfrac{k_1 p_3}{o_1} - \dfrac{k_3 p_1}{o_2} + \dfrac{\epsilon(k_3 p_1 - k_1 p_3)}{m} \\ \dfrac{k_2 p_1}{o_2} - \dfrac{k_1 p_2}{o_1} + \dfrac{\epsilon(k_1 p_2 - k_2 p_1)}{m} \\ 0 \\ \dfrac{k_1 k_3 (o_2 - o_1)}{o_1 o_2} \\ \dfrac{-k_1 k_2 (o_2 - o_1)}{o_1 o_2} \end{pmatrix}. \tag{3.15}$$

Continuing with the simple example from expression (1.1), the initial momentum components are

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} mv_x \\ mv_y \cos(f) \\ mv_y \sin(f) \\ o_1 w \\ 0 \\ 0 \end{pmatrix}. \tag{3.16}$$

The angular velocities are easily found to be constants

$$\begin{pmatrix} w_1(t) \\ w_2(t) \\ w_3(t) \end{pmatrix} = \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix}. \tag{3.17}$$

The linear velocity components can be determined from

$$\frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \left(1 - \epsilon \dfrac{o_1}{m}\right) w v_3 \\ -\left(1 - \epsilon \dfrac{o_1}{m}\right) w v_2 \end{pmatrix}. \tag{3.18}$$

This has solution

$$\begin{pmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \cos\left(\left(1 - \epsilon \dfrac{o_1}{m}\right) wt + f\right) \\ v_y \sin\left(\left(1 - \epsilon \dfrac{o_1}{m}\right) wt + f\right) \end{pmatrix}. \tag{3.19}$$

These velocities are shown in Table 1.

In summary, the initial velocities remain unchanged in the semidirect product. There is no interaction between the components. In all three Lie groups with the Lie-Poisson structure, the rotation induces a change in the linear velocities perpendicular to the axis of rotation. In $SE(3)$ (with imposed structure), the frequency of the change is equal to the initial rotation w. In $SO(4)$ and $SO(1,3)$, the frequency of the change is influenced by the ratio of the moment of inertia and the mass of the body.

4. Trajectories Determined for the Lie Groups

Having found the velocity functions, they are integrated in this section to determine the trajectory. The resulting trajectories are analyzed in the next section.

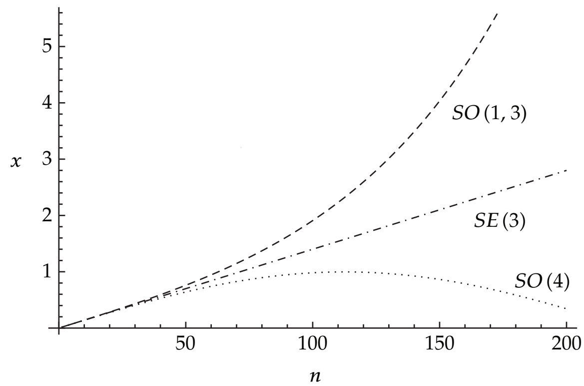

Figure 2: Displacement along the axes of rotation.

From Table 1 using the base matrices shown in (3.1), the velocity matrix X is given by

$$X(t) = v_x e_1 + v_y \cos(w_\epsilon t + f) e_2 + v_y \sin(w_\epsilon t + f) e_3 + w e_4 \tag{4.1}$$

with $w_\epsilon = (1 - \epsilon(o_1/m))w$ and $\epsilon \in \{1, 0, -1\}$ for $\mathfrak{so}(4), \mathfrak{se}(3), \mathfrak{so}(1,3)$, respectively. The trajectory is found by (2.1). Because the velocity matrix is time dependent, the trajectory is calculated by the forward Euler method of numerical iteration using the simple equation (2.3)

$$g_{n+1} = g_n \exp(X_n s), \tag{4.2}$$

where s is the step length, and n is the step number so that $X_n = X(t_n)$ and $t_n = ns$.

The figures show the displacements x, y and the rotation θ for the three Lie groups, with initial velocities $\{v_x, v_y, w\} = \{0.014, 0.01, 0.09\}$, $f = 0$, $o_1/m = 0.6$ and step length $s = 0.1$.

The displacement along the axis of rotation as shown in Figure 2 is a straight line for $SE(3)$ (the dot-dash line). It has a sinusoidal function in $SO(4)$ (the dotted line) with a maximum of one, the unit sphere. In $SO(1,3)$, it is a sinh curve.

The rotation of the body is fixed at the original angular velocity, as shown in Figure 3, the plot of $\sin(\theta)$ where θ is the orientation at time t.

The trajectories in all 3 spaces approximate a screw motion about the original axis of rotation. The main difference in the trajectories is determined by the ratio of the moment of inertia and the mass: o_1/m. With $o_1/m = 0.6$, the difference in angular frequency and amplitude can be seen in Figure 4. The angular frequency is $w_y \simeq w_\epsilon = w(1 - \epsilon o_1/m)$, and the amplitude is $y_m \simeq v_y/w_y$.

For all four combinations of linear and rotations, the linear and rotational displacement functions shown in Table 2 indicate the form of the solutions. There are additional small magnitude terms which add lower-frequency variations and curve the trajectory away from the main axis of the screw. The largest difference is in $SO(4)$ and is shown in Figure 5.

To conclude this section, the trajectory functions shown in Table 2 are summarized. The motion determined in the semidirect product $\mathbb{R}^3 \ltimes SO(3)$ is that experienced in the body

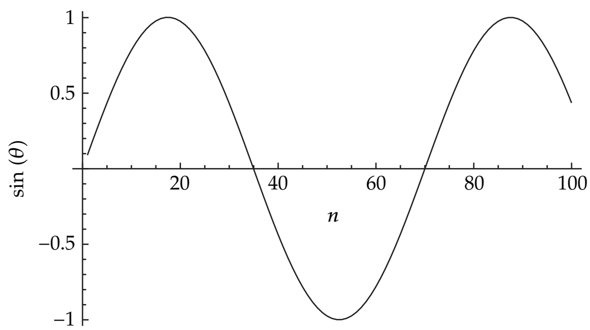

Figure 3: Constant rotation about original axis of rotation.

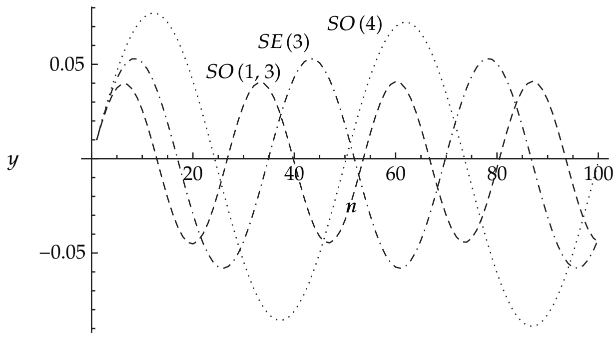

Figure 4: Induced screw motion about the axis of rotation.

frame. In $SE(3)$, with the imposed Poisson structure, the motion is in the spatial frame—see Figure 6. The actual and perceived distances traveled are both $v_y t$. The perceived angle of rotation is the actual rotation of the trajectory plus the rotation of the body. The actual radius of travel (v_y/w_5) is half the perceived radius since $w_5 = 2w$. In order to conserve angular momentum about the origin, the body performs a screw motion, and the linear velocity changes in the spatial frame. In $SO(4)$ and $SO(1,3)$, the radius of the screw motion is influenced by the ratio of the moment of inertia and the mass of the body.

The perceived trajectory (calculated using the semidirect product) is independent of the actual trajectory (whether calculated using the Lie-Poisson structure of $SE(3)$ or working with the uncoupled velocities). Further details are given in the Appendix.

Over longer-time frames, the structure of the group causes the trajectories to diverge from this simple model.

5. Comparing the Different Trajectories

In this section, the trajectories identified previously are compared in more detail. The configuration in $SO(4)$ is shown to approximate the configuration in $SE(3)$ for small linear displacements. Thus, $SE(3)$ can be projected onto $SO(4)$ (and vice versa). The induced Lie-Poisson structure has coupling of the velocities for $SE(3)$. The variation in the frequency of the spiral motion across the groups is analyzed and found to be eliminated for small displacements.

Table 2: Approximate displacement functions for small displacements.

	$\mathbb{R}^3 \ltimes SO(3)$	$SO(4)$	$SE(3)$	$SO(1,3)$
$x(t)$	$v_x t$	$\sin(v_x t)$	$v_x t$	$\sinh(v_x t)$
$y(t)$	$v_y(\sin(wt+f)-\sin(f))/w$	$v_y(\sin(w_4 t+f)-\sin(f))/w_4$	$v_y(\sin(w_5 t+f)-\sin(f))/w_5$	$v_y(\sin(w_6 t+f)-\sin(f))/w_6$
$z(t)$	$v_y(\cos(f)-\cos(wt+f))/w$	$v_y(\cos(f)-\cos(w_4 t+f))/w_4$	$v_y(\cos(f)-\cos(w_5 t+f))/w_5$	$v_y(\cos(f)-\cos(w_6 t+f))/w_6$
$\theta_1(t)$	$w\mathrm{t}$	wt	wt	wt
$\theta_2(t)$	0	0	0	0
$\theta_3(t)$	0	0	0	0
		$w_4 = (2 - o_1/m)w$	$w_5 = 2w$	$w_6 = (2 + o_1/m)w$

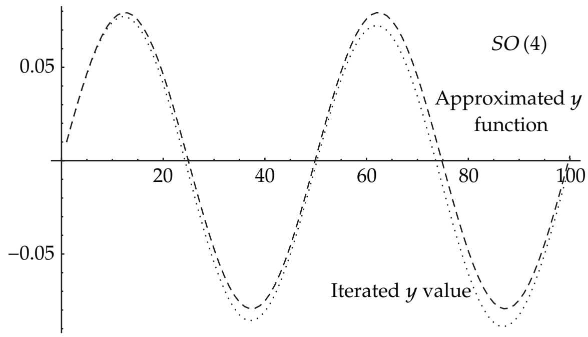

Figure 5: Comparison of iterated and calculated values of y displacement for $SO(4)$.

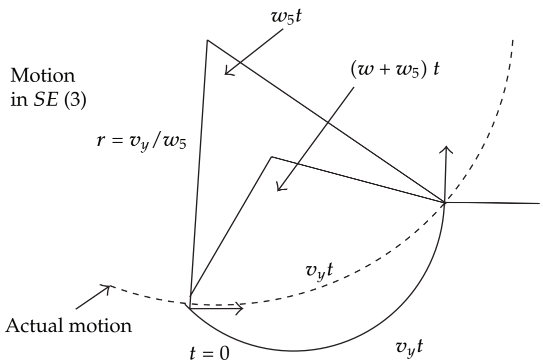

Figure 6: Actual motion determined using $SE(3)$.

5.1. Projecting $SO(4)$ *Vertically onto* $SE(3)$

Intuitively, the 2 linear displacements and one rotation of a planar body can be represented on the surface of a large sphere. Equivalently, the environment can be scaled down to a unit sphere, so that the linear displacement is small compared to 1, using some linear unit (such as kilometer or light-year). Similarly 3D spatial displacements can be approximated on the surface of a unit hypersphere. Three axes are the rotation axes of the body frame, and there is no constraints on the size of those rotations $(\theta_1, \theta_2, \theta_3)$. The small angular displacements (x, y, z) from the other axes are the spacial displacements, after scaling down by a factor R.

Any element $g \in SO(4)$ can be written as

$$g_a = \exp \Omega_a = \exp \begin{bmatrix} 0 & -x^T \\ x & \theta \end{bmatrix} = \begin{bmatrix} a & -Y \\ X_a & \Phi_a \end{bmatrix}, \tag{5.1}$$

where $x = [x, y, z]^T$ and $\theta = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$ are the two subelements (and $a = 1 - \|x\|$).

The corresponding group element in $SE(3)$ is

$$g_b = \exp \Omega_b = \exp \begin{bmatrix} 0 & 0 \\ x & \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ X_b & \Phi_b \end{bmatrix}. \tag{5.2}$$

The difference between the two groups is found from the exponential series expansion to be

$$\begin{aligned} g_a - g_b &= (\Omega_a - \Omega_b) + \frac{1}{2}\left(\Omega_a^2 - \Omega_b^2\right) + \cdots \\ &= \begin{bmatrix} 0 & -x^T \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} xx & x^T\Phi \\ 0 & x^T x \end{bmatrix} + \cdots . \end{aligned} \tag{5.3}$$

Ignoring the values in the first row, the difference is of order 2 in x as follows:

$$SO(4) \ni \begin{bmatrix} a & -Y \\ X_a & \Phi_a \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 \\ X_b & \Phi_b \end{bmatrix} \in SE(3) \tag{5.4}$$

with $\Phi_a = \Phi_b = \exp\theta$, $X_a = X_b$, and $a = 1 - \|x\|$ to second order in x. The projection $\pi_3 : SO(4) \to SE(3)$ is

$$\pi_3\left(\begin{bmatrix} a & -Y \\ X_a & \Phi_a \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ X_a & \Phi_a \end{bmatrix}. \tag{5.5}$$

A similar projection can be done from $SO(1,3)$ onto $SE(3)$.

5.2. Scaling the Linear Displacements

Since the projection from $SO(4)$ to $SE(3)$ is accurate to second order in x, a scaling factor $S \in \mathbb{R}$ (as suggested in [4]) can be chosen so that the plotted displacement x_p of the rigid body in $\mathbb{R}^3$ is $x_p = Sx$, and x is the calculated displacement which is kept within the required condition $|x| < \sqrt{\delta}$ where the required accuracy is δ (a proportion).

This scaling has a consequence on the inertia terms. The moment of inertia is a function of the mass m and the square of the size of the body r. So scaling the linear dimensions to be small compared with the unit hypersphere reduces the moment of inertia to order of $\|S^2\|$.

The angular frequency of the linear velocity terms tends to the angular velocity, as can be seen from Table 1

$$w_\epsilon = \left(1 - \epsilon\frac{o_1}{m}\right)w \longrightarrow w \tag{5.6}$$

for all three groups as the accuracy of the projection increases. For small displacements, the three groups produce the same results as follows:

$$g^{-1}(t)\frac{dg}{dt} = v_x e_1 + v_y\cos(wt + f)e_2 + v_y\sin(wt + f)e_3 + we_4 \tag{5.7}$$

has solution

$$\begin{bmatrix} v_x t \\ \dfrac{v_y(\sin(2wt+f)-\sin(f))}{w} \\ \dfrac{v_y(\cos(f)-\cos(2wt+f))}{w} \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(wt) & -\sin(wt) \\ 0 & \sin(wt) & \cos(wt) \end{bmatrix} \tag{5.8}$$

for $\{v_x t, v_y t\} \ll 1$. This gives the displacement in the spatial frame.

The scaling factor S also has the effect of changing the relative weighting of the rotational and linear displacements. The linear displacements are shrunk and have less weighting in any comparison, such as distance (in 6 dimensions) traveled along the trajectory. This was demonstrated (5.6) where the angular frequency tends to the same value irrespective of the linear size of the body. There is an innate difference between rotations and translations which cannot be overcome by some arbitrary metric. There is no bi-invariant Riemannian metric on $SE(3)$. The requirements of each system will determine what (if any) a metric can be defined for that situation.

6. Discussion

Four methods of combining linear motion in 3 dimensions with a rotation have been considered. A single rotation is used since any 3 rotations about orthonormal axes can be replaced by a single rotation about one rotated axis, and if that axis is also an axis of symmetry, precession is avoided. The rotation of the axes necessary to achieve this is given on the Appendix. The linear motion results in linear displacements, and these can be scaled by a factor S so that they are small compared to a unit hypersphere. Small linear displacements on a sphere are generated by small angular displacements.

The four methods considered are as follows.

(1) Semidirect product, where the motion is described in the body frame: there is no coupling of the angular and translational velocities. This is often referred to as $SE(3)$ because the integration of the velocities can be done using that group.

(2) $SE(3)$ with induced Lie-Poisson structure: the angular momentum induces a change in the linear velocity. The resulting trajectory is in a fixed frame and is a spiral motion about an axis parallel to the original axis of rotation. The frequency of the screw motion is twice the original angular rotation. Whatever the frequency of the screw motion, the perceived motion in the body frame is always that produced by the semidirect product. Higher frequencies reduce the radius of the screw maintaining the angular momentum. A frequency of w_ϵ generates a radius of v_y/w_ϵ and maintains a velocity of v_y around the screw—see Table 2.

(3) $SO(4)$ where the linear motions are interpreted as small angular displacements from three of the six axes: the frequency of the spiral motion is reduced as the motion of inertia about the original axis of rotation increases. When considering small linear dimensions, this reduction in frequency is eliminated—see (5.6). The trajectories for small linear displacements are the same as those produced using $SE(3)$ with induced Lie-Poisson structure. The difference arises from the term

$x(t) = \sin(v_x t)$ compared with $x(t) = v_x t$, which feeds back into the displacements in other directions. As the rate of change of $x(t)$ decreases, the momentum is transferred to the other directions. The rate of change of $y(t)$ increases faster. This is not shown in the approximate functions in Table 2.

(4) $SO(1,3)$: the same considerations apply to this group with the following differences. The frequency of the spiral motion is increased as the motion of inertia about the original axis of rotation increases. As the displacement $x(t) = \sinh(v_x t)$ increases exponentially, the displacement in the other directions decreases exponentially. The displacement $y(t)$ decreases by an extraterm with the form of cosh function.

7. Conclusion

Trajectories of a rigid body in Euclidean space are determined by the linear and rotational velocities, which can interact. This paper has separated the step of finding the velocities from integrating the velocities into a trajectory. The semidirect product of the linear and rotational velocities consider the velocities separately without coupling them. The three 6D Lie groups use the Lie-Poisson structure to reflect the rotational influence on the linear velocity. The angular moments of inertia decrease the frequency of the spiral motion in the rotation group $SO(4)$, have no impact in $SE(3)$, and increase it in $SO(1,3)$. For small displacements, the impact on the linear velocity is the same for all three groups.

The velocities of the semidirect product can be integrated using the matrix format of $SE(3)$ and result in the usual identification between semidirect product and Special Euclidean group. The resulting trajectory is in the body frame, and this cannot be interpreted into the spatial frame since the perceived trajectory is independent of the velocity coupling. The three Lie groups integrate the motion in a spherical, flat, or hyperbolic space to give the trajectory in a spatial frame. For small displacements, these are the same (within any chosen degree of accuracy) and reflect the trajectory in a fixed frame. It differs from the body frame trajectory created by the semidirect product.

Any combinations of rotations can be represented by rotation about a single axis of rotation. If this axis is also an axis of symmetry, the rotational axis is fixed. In a natural system with no forces and a body symmetric about the axis of rotation, the analysis can be simplified by aligning one axis of the spatial frame with the rotation.

Due to the innate differences between rotational and translational motion, there cannot be a metric that applies to a rigid body motion in Euclidean space, without imposing some relationship between the relative motions which will depend on the system being measured.

Appendices

A. Rotation to Generalize the Motion

The simplified example of rotation about one axis, with initial linear motion along and perpendicular to that axis, can be generalized by rotating the coordinate frame. This is valid for $SO(4)$, $SE(3)$, and $SO(1,3)$, and also $SO(3)$ by dropping the first row and first column of the rotation matrix.

Assume that the generalized initial velocities are given by

$$V = v_1 e_1 + v_2 e_2 + v_3 e_3 + w_1 e_4 + w_5 e_5 + w_6 e_6. \tag{A.1}$$

By applying a rotation R to this, the initial velocities are simplified to

$$RVR^{-1} = v_x e_1 + v_y e_2 + v_z e_3 + w e_4, \tag{A.2}$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{w_1}{w} & \dfrac{w_2}{w} & \dfrac{w_3}{w} \\ 0 & 0 & \dfrac{w_3}{\sqrt{w^2 - w_1^2}} & -\dfrac{w_2}{\sqrt{w^2 - w_1^2}} \\ 0 & -\dfrac{\sqrt{w^2 - w_1^2}}{w} & \dfrac{w_1 w_2}{w\sqrt{w^2 - w_1^2}} & \dfrac{w_1 w_3}{w\sqrt{w^2 - w_1^2}} \end{pmatrix}. \tag{A.3}$$

The relationships between the general components $\{v_i, w_i\}$ and the simplified ones $\{v_x, v_y, v_z, w\}$ can be found by matrix multiplication and justified as follows.

(i) Conserving angular momentum $w^2 = \sum_{i=1}^{3} w_i^2$: if $w = w_1$, then no simplification is required, and R is the unit matrix.

(ii) From the invariant Casimir $\sum_{i=1}^{3} v_i w_i$, the x component is fixed at $v_x = 1/w \sum_{i=1}^{3} v_i w_i$.

(iii) Consider $v_y = (v_2 w_3 - v_3 w_2)/\sqrt{w_2^2 + w_3^2}$.

(iv) Conserving linear momentum $v_z^2 + v_y^2 = (\sum_{i=1}^{3} v_i^2) - v_x^2$.

It follows that $v_y^2 + v_z^2$ is a constant and so the 2 linear components can be written as

$$\begin{aligned} v_y &= \sqrt{\left(\sum_{i=1}^{3} v_i^2\right) - v_x^2}\cos(wt + f), \\ v_z &= \sqrt{\left(\sum_{i=1}^{3} v_i^2\right) - v_x^2}\sin(wt + f). \end{aligned} \tag{A.4}$$

B. Conversion from Body Frame to Spatial Frame

Selig [10] states that the body frame X' and an alternative frame viewpoints X are related through the expression

$$X = \begin{bmatrix} 1 & 0 \\ u & R \end{bmatrix} X' \begin{bmatrix} 1 & 0 \\ u & R \end{bmatrix}^{-1}, \tag{B.1}$$

where u is the displacement vector between the two spaces, and R is the relative rotation between them. The body frame trajectory (in the y-z plane) in the simple example of this paper is

$$X' = \begin{bmatrix} 1 & 0 & 0 \\ \frac{v}{w}\sin(wt) & \cos(wt) & -\sin(wt) \\ \frac{v}{w}(1-\cos(wt)) & \sin(wt) & \cos(wt) \end{bmatrix}. \tag{B.2}$$

If the displacement vector u is set to

$$u = \begin{bmatrix} \frac{v}{w}\left(\sin(mtw) - \frac{\sin((n-1)tw/2)\sin ntw}{n\sin(tw/2)}\right) \\ \frac{v}{2w}\left(1 - 2\cos(mtw) - \frac{\sin((2n-1)tw/2)\sin ntw}{n\sin(tw/2)}\right) \end{bmatrix} \tag{B.3}$$

and the rotation R is mtw for any value of m, then the trajectory in the alternative frame is given as

$$X = \begin{bmatrix} 1 & 0 & 0 \\ \frac{v}{wn}\sin(nwt) & \cos(wt) & -\sin(wt) \\ \frac{v}{wn}(1-\cos(nwt)) & \sin(wt) & \cos(wt) \end{bmatrix}. \tag{B.4}$$

A trajectory with any value of n can be found by changing the displacement vector u. The radius of the trajectory in the spatial frame cannot be determined. This confirms the geometric diagrams provided in the paper. The body frame trajectory cannot be interpreted into a spatial frame without additional information.

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