Urban Maths: Car Park Mayhem*

Several years ago, my employer introduced an exit barrier on its main car park. In order to leave, drivers wind down their windows and insert tokens or swipe passes. One cold wintry day, I burst out laughing on the way to retrieve my car, as some imaginative local youths stood by the barrier with a pyramid of snowballs. I was still laughing as I went through the ritual of winding down my window and receiving an icy wallop in the mush. Perhaps I would have received more facial wallops if I were to have suggested that a cannonball-like, face-centred cubic lattice would be more efficient for packing snowballs than would their body-centred cubic lattice. On the other hand, I might have escaped scot free if I were to have suggested that these happy teenagers charge some token protection money. But that's all by the by. About the same time, car park bays were re-painted in the usual rectangular pattern shown in Figure 1.

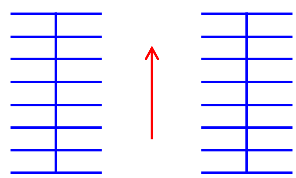


Figure 1: rectangular pattern.

However, this traditional conformity set me thinking. First of all, the car park included some dead ends whereby the grid in Figure 1 continued to the boundary fence. How mad: cars need to reverse against the flow in order to escape the bedlam. One solution is to leave a free lane for traffic to move adjacent and parallel to the fence, rather like aisles along the ends of pews in churches. Another would be to move this lane slightly away from the fence so that a single row of parking bays could be placed along the fence. These possibilities are illustrated in Figures 2(a) and 2(b) respectively, and the latter clearly packs more spaces into the bounded region. That is not all though. There was no flow direction specified and vehicles can travel in both directions. The problem with this is that the separation between columns of parking bays is wider than it would be for one-way flow, so this was a missed opportunity to fit more bays into this busy car park.

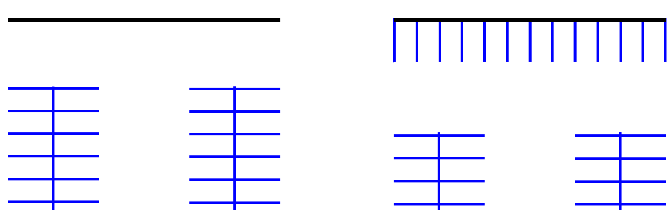


Figure 2: (a) boundary with lane; (b) boundary with bays.

Now the challenge becomes more interesting. A nearby hospital has parking spaces that form a herring-bone pattern as illustrated in Figure 3(a). This pattern is attractive and appears in such things as woollen

*or Mayhematics as Professor Tony Mann's letter in June's issue might suggest scarves, tweed jackets, parquet flooring and masonry. It can offer denser packing of parking spaces than the rectangular pattern for small car parks. However, it is generally impractical for larger car parks because traffic cannot flow in opposite directions along adjacent aisles unless vehicles nose into some bays and reverse into others, which is a recipe for disaster. A good solution for avoiding this problem is to consider a diagonal pattern as illustrated in Figure 3(b). The question of interest is whether a diagonal layout is more efficient than a unidirectional rectangular layout.

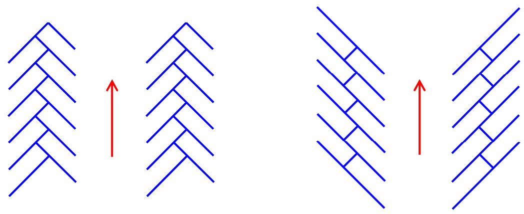


Figure 3: (a) herringbone pattern; (b) diagonal pattern.

The advantages of a diagonal layout over a rectangular layout are that the column width is less, because of the angled bays, and the gap between columns is less, because of the reduced turning circle arc. Conversely, the column length is substantially greater, so which arrangement is better? To answer this question, we formulate the problem using three parameters (bay length l, bay width w, turning circle radius r) and one variable (bay angle θ), as displayed in Figure 4. In time-honoured fashion, we assume an infinite car park in two dimensions.

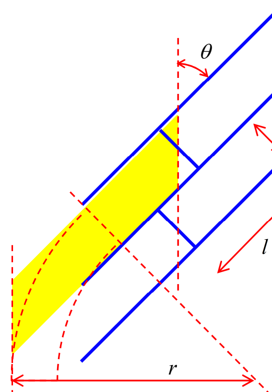


Figure 4: bay angle optimisation.

The total space required for each bay roughly corresponds to the area enclosed by a parallelogram, as illustrated in yellow within Figure 4. We specify that the car must be travelling parallel to the painted line segments on entry to its bay, which allows it to cut the corner of the neighbouring bay. This is reasonable, as the bay width is already generous and the turning circle radius is considerably larger than this. If either condition were violated, we could instead require the car to be travelling parallel on reaching the start of its neighbouring bay.

Using simple trigonometry, the vertical distance (parallelogram base) occupied by each bay is $w/\sin\theta$. Similarly, the horizontal distance (parallelogram height) occupied by each bay is found by summing three components to give $r(1-\cos\theta)+l\sin\theta+(w\cos\theta)/2$. The total area required for each bay is then the product of base times height, which is

$$\begin{aligned} A &= \frac{w}{\sin\theta}\left\{r(1-\cos\theta)+l\sin\theta+\frac{w\cos\theta}{2}\right\} \\ &= w\left\{l+r\csc\theta+\left(\frac{w}{2}-r\right)\cot\theta\right\}. \qquad (1) \end{aligned}$$

In order to minimise this area, we differentiate Equation (1) with respect to θ to give

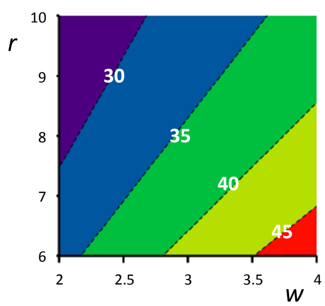
$$\begin{aligned} \frac{dA}{d\theta} &= -wr\csc\theta\cot\theta - w\left(\frac{w}{2}-r\right)\csc^2\theta \\ &= -w\csc\theta\left\{r\cot\theta + \left(\frac{w}{2}-r\right)\csc\theta\right\}. \end{aligned} \tag{2}$$

Setting the derivative in Equation (2) to zero for $\theta \in [\arctan(w/2l), \pi/2]$ gives

$$\hat{\theta} = \arccos\left(1 - \frac{w}{2r}\right). \tag{3}$$

This implies that the optimal angle is independent of the bay length, though the bay width and turning circle are important. Although we should show that the second derivative is positive to deduce that this turning point is a minimum, we rely upon a graph for the purposes of this article.

Now let's be specific. A Rolls-Royce Phantom (and why not?) is 5.84 metres long and 1.99 metres wide, with a turning circle of about 14 metres. For illustration, we round these parameters up to $l = 6m$, $w = 3m$ and $r = 8m$ to allow for pedestrian access, collision avoidance, door opening and axle track. Although the owner of such a fine vehicle is more likely to order home delivery than jostle for a supermarket parking space, this convenient template ensures that our bays accommodate almost all cars. Substituting these values of w and r into Equation (3) gives $\hat{\theta} \approx 0.62$ radians, which corresponds to about 36°. Interestingly, Figure 5(a) displays a contour graph of $\hat{\theta}$ (degrees) against w and r, from which it is clear that the optimal angle is fairly robust against measurement errors for the bay width and turning circle radius.



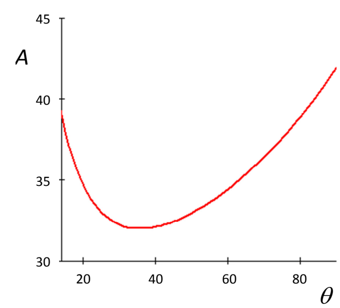


Figure 5: (a) contour graph of $\hat{\theta}$ against w and r; (b) graph of A as a function of θ.

From Equation (1), this angle requires an area of about $32m^2$ per bay. For comparison, the equivalent area for a rectangular pattern, corresponding to $\theta = \pi/2$ radians, is $42m^2$. Hence, the optimal diagonal pattern compares favourably and represents an efficiency saving of about 24%. Figure 5(b) displays a graph of area as a function of angle (degrees), to illustrate this relationship in more detail. Ignoring our assumption of infinite space, this analysis suggests that a car park accommodating 500 vehicles in a rectangular grid would accommodate 619 vehicles in an optimal diagonal grid. The corresponding area for a 45° diagonal pattern as shown in Figure 3(b) is about $32\frac{1}{2}m^2$, which is similar to the area for the optimal diagonal pattern and would still accommodate 613 cars (23% efficiency saving). However, a grid using a 45° angle might be considerably easier to paint accurately than a grid using a 36° angle.

Perhaps there is some analogy between this optimal parking bay problem and the recent use of beautiful diagrids in architecture, which help to reduce the total mass of steel beams required. In any case, this theory could also be adapted to address more important applications in finite space, such as the design of terminal gates at airports and the allocation of yachting berths in marinas. Now to tackle the equivalent three-dimensional problem ready for when future commuters travel in Chitty-Chitty-Bang-Bang style flying cars . . .

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