Delay Time Analysis in Maintenance

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by

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Abstract

The thesis develops the application of delay time analysis to the area of mathematical modelling of planned maintenance and inspection of industrial systems. Chapter 1 gives an introduction to the history and techniques in use of maintenance modelling and surveys appropriate literature in the field. A section is devoted to papers published on delay time analysis. Chapter 2 introduces and develops mathematical models for modelling the reliability, maintenance and inspection of repairable systems. Chapter 3 gives an account of parameter estimation and model updating techniques in the light of subjective and observational data sets collected over a period of system operation. Chapter 4 addresses a bias in the probability distribution function of delay time when the data available over an operating survey is censored. Parameter estimation methods for this situation are then proposed. Chapter 5 gives an account of a simulation study of the delay time models and verifies the theory and parameter estimation techniques. Chapter 6 reports on research supported by the Science and Engineering Research Council on the application of delay time analysis to concrete structures. Finally, Chapter 7 collates the conclusion drawn on each chapter and recommends areas for further research.

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Chapter 2: Delay Time Models for Maintenance of a Repairable System

Chapter 3: Parameter Estimating and Updating for Delay Time Models

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Chapter 1

Introduction and Literature Review

1.1 Introduction

The demand for effective maintenance modelling and use of operational research (OR) to this end is evident in the recorded experiences of World War II, see Waddington (1973), OR in World War II: OR against the U-boat. Khintchine (1932) was one of first contributors to attempt to mathematically model machine maintenance. Since then, the field of planned maintenance for plant and buildings has received considerable interest and attention due to the increasing complexity of machinery and building designs, and the necessity for such systems to perform their function optimally in terms of cost, downtime and reliability. Stochastic modelling and statistical analysis have underpinned much of the OR methodology used for attempting to model and predict the consequences to and behaviour of equipment when maintenance strategies are applied. Informative accounts and reviews of mathematical techniques applied to maintenance are given by; Cox (1957), Barlow and Proschan (1965), McCall (1965), Jardine (1973), Pierskella and Voelker (1976), Christer (1984), Ascher and Feingold (1984), Barlow (1984), Thomas (1986), Valdez-Flores and Feldman (1989), Cho and Parlar (1991), Thomas et al (1991), Baker and Christer (1994).

This chapter proceeds by modelling single component items such as, for example, light bulbs, where usually it makes sense to assume that only one type of failure can be experienced. Models for reliability, cost and downtime consequences given maintenance options are then discussed along with the necessary statistical analysis and estimation techniques. Modelling procedures applied to systems of components are then addressed. In the fourth section, the concept of delay time analysis applied to maintenance and inspection (the subject of the thesis), introduced by Christer (1973), is reviewed. The research undertaken in the thesis is introduced. Finally, an overall discussion concludes the chapter.

1.2 Maintenance Modelling of Single Components

A component (or part) is a device that can fail in one failure mode when in operation and providing its service. Examples include light bulbs, valves and fuses. Failure is the state such that the component needs repair or complete replacement. The consequence of failure will induce costs due to repair or replacement. A period of downtime will be incurred until the component is restored and operating. The downtime required to replace the item will either be known prior to replacement/repair or possibly unknown. Indeed, penalty costs such as, for example, lost production, due to the component being out of service could also be incurred.

A maintenance strategy or concept is a set of directives (or policies) aimed at optimising an objective function, e.g cost or downtime, over a period of time. The set of policies considered may be constrained so that certain operating characteristics are achieved, e.g a guaranteed reliability performance. One such policy could be simply to restore failed components as they arise, that is Failure Based Maintenance (FBM). Another strategy could be to replace components after a determined period of operation or at failure, whichever comes first, and is an example of a planned preventive maintenance (PPM) policy. The period of failure free operation is the decision parameter in the model. In some cases, a component may not signal immediate failure to the operator, e.g standby devices and the deterioration to a defined failed state in concrete structures. These types of components would require an inspection or monitoring type policy.

Many components may become defective prior to failure and still remain operable, e.g a strip-light would flicker and take more time to switch on in the latter stage of its life. These types of components may benefit from an inspection policy whereby a component is inspected for the defect and consequently replaced at inspection to prevent failure. The time to inspect is the decision parameter. The defective phase would need to be included in a maintenance model. Delay time modelling has provided a tool for modelling the consequence of maintenance and inspection for components of this type. In recent years, condition monitoring techniques have been developed. A component can be continuously or periodically monitored for engineering factors, e.g stress or vibration. A decision on replacement/repair can then be made when a particular factor reaches a certain threshold.

1.2.1 Renewal Theory and Reliability

The classic methodology for modelling the maintenance of single components is the application of renewal theory and reliability. Cox (1957) and Barlow and Proschan (1965) give excellent accounts of this field of mathematics. The approach is to assume that each component has its time to failure, X say, governed by a probability law, that is X is distributed with an assumed probability density function (p.d.f), $f(x)$ say.
When a component is replaced, which may be at failure, it is then assumed that the replaced component then operates independently and statistically identical to its predecessor, in other words, a renewal point. It is also possible that a component can be repaired to an assumed 'as-new' state which is statistically equivalent to a renewal. The independence assumption, in this case, would need to be tested.

The FBM policy for a single component is an example of a renewal process, whereby the operating time between each failure is assumed to be independent and identically distributed (i.i.d) with p.d.f of time to failure, $f(x)$. The reliability, $R(x)$ say, of a component is the probability that the component will operate without failure over time interval (0, x) measured from when the component was assumed new and placed in operation. The reliability function, $R(x)$, is simply given by the probability that the failure time exceeds x, that is to say,

$$R(x) = \int_{y=x}^{\infty} f(y)\,dy = 1 - F(x) \ , \tag{1.1}$$

where $F(x)$ is the cumulative distribution function (c.d.f) of X. The mean time to failure (MTTF), $\mu = E(X)$ say, is given by,

$$\mu \;=\; \int_{x=0}^{\infty} x f(x)\,dx \;=\; \int_{x=0}^{\infty} R(x)\,dx \quad . \tag{1.2}$$

The hazard rate, $z(x)$ say, is a function such that in the small interval $(x, x + dx)$, $z(x)dx$, is the probability that the component will fail given it has not caused a failure over the operating time interval $(0, x)$, since last new, i.e $P\{X \varepsilon (x, x + dx) \mid X > x\}$, where X is the random time to failure. It follows that $z(x)$ is given by,

$$z(x) \;=\; \frac{f(x)}{R(x)} \quad . \tag{1.3}$$

The functions $f(x)$ and $R(x)$ can be written uniquely as a function of the hazard rate, $z(x)$, see Cox (1957, p.5). This can be shown by formulating the cumulative hazard function, $Z(x)$ say, that is the integral of $z(x)$, given by,

$$Z(x) \;=\; \int_{y=0}^{x} \frac{f(y)}{R(y)}\,dy \;=\; -\ln(R(x)) \quad . \tag{1.4}$$

Hence, $R(x) = \exp(-Z(x))$ and $f(x) = z(x)\exp(-Z(x))$. It can be seen that a component with a constant hazard rate, whereby the instantaneous chance of failure does not change with operating time, has an exponential distribution of time to failure.

Typical lifetime distributions commonly selected for components are exponential, Weibull, Erlang, gamma and lognormal. Depending on the selected distribution, the hazard rate, for example, could be monitonically increasing, whereby the chance of failure in the next instance of component operation, given no prior failure, increases as the component operates, i.e the component is such that it is wearing out. A decreasing hazard applies to components which become increasingly reliable as they operate without failure, e.g computer chips. It has been a common assumption that many component types have a 'bath-tub' hazard which decreases at first. This to allow for a sub-set of components to be possibly defective through manufacture, and having infant mortality. Then, the hazard is assumed approximately constant for a certain time (sometimes termed main life), and finally increasing (wear out).

The sample hazard function would be formed from the life testing of a set of components. Analysis of the failure behaviour of a component, say through condition monitoring or inspection, may reveal that a component becomes defective, giving a signal that it is about to fail or has an increasing chance of failure. In this case, the bath-tub assumption would not be appropriate as a basis for modelling this effect. An inspection would naturally increase or decrease the hazard function of failure, for component operation immediately after inspection, depending if a defect is found or not found. The defective property of the component would need to be included in a maintenance model. Delay time analysis provides a model to take this effect into account.

It is of interest in maintenance modelling to estimate the expected number of breakdowns, $B(T)$ say, over an interval $(0, T)$. For the renewal process, $B(T)$, is given by the solution to the renewal equation,

$$\begin{aligned} B(T) &= F(T) + \int_{x=0}^{T} B(T - x) f(x)\, dx \\ &= \sum_{k=1}^{\infty} F^{(k)}(T) \ , \end{aligned} \qquad (1.5)$$

where $F^{(k)}(x)$ is the k-fold convolution of $F(x)$, that is the c.d.f of time to the k'th failure. We shall also define, here, $r(T)$, as the instantaneous rate of occurrence of failures (ROCOF), given by,

$$r(T) = B'(T) = \sum_{k=1}^{\infty} f^{(k)}(T) \ , \qquad (1.6)$$

where $f^{(k)}(x)$ is the p.d.f of time to the k'th failure. The ROCOF and hazard rate has caused much confusion in the reliability field, see Ascher and Feingold (1984). The ROCOF is an absolute rate of the stochastic process of failures from the origin of the process. The hazard rate is relative, in that it is a direct property of the time between two failures. A consequence of the renewal process, is that in the limit as time increases,

$$\lim_{T \to \infty} \frac{B(T)}{T} = \lim_{T \to \infty} r(T) = \frac{1}{\mu} \ , \qquad (1.7)$$

and for large T and finite variance, σ^2 say, of time to failure, $B(T) \approx T/\mu + (\sigma^2 - \mu^2)/2\mu^2$, see Cox (1957, p.47, p.55). This implies the process becomes steady state, for example the expected number of failures in an interval, (T_1, T_2) say, selected prior to the process starting, would be approximately $(T_2 - T_1)/\mu$ for large values of T_1 and T_2. The time taken to reach this state will depend on the selected p.d.f of time to failure.

The functions introduced in this section are some of the characteristics of the behaviour of a single component system, and are important in the maintenance modelling and the estimation and testing of modelling parameters. However, it has been highlighted that properties of defects of a component would also be an important ingredient in maintenance modelling, and will prove to be so in the forthcoming chapters on delay time analysis.

1.2.2 Models for Cost

We, here, introduce two models of cost and refer to literature on other cost model structures. Jardine (1973) gives an account of many models for single-component systems. The two common types are block and age based replacement.

Block replacement can apply to a single or group of like components. The policy is to replace the component(s) at periodic points in time, T, $2T$, ..., NT, say. The decision parameter for the model is clearly T. Components which fail over the replacement periods are assumed to be replaced at failure with a statistically identical component. Let c_f be the expected cost of replacing a failed component and c_r ($< c_f$) be the expected cost of a planned replacement. It is also assumed that a failure and planned replacement is carried out with negligible time. For one component, the expected cost over each replacement cycle is $c_r + c_f B(T)$. Hence, the expected cost per unit time, $c(T)$ say, over each cycle is given by,

$$c(T) = \frac{c_r + c_f B(T)}{T} . \tag{1.8}$$

For a group of, M say, like components the expected cost would simply be multiplied by M. Also, a block replacement downtime, for this case, d_r say, may need to be considered in the denominator of $c(T)$. By differentiating $c(T)$, the optimum solution, if it exists, can be found by solving the equation,

$$c_f(r(T)T - B(T)) - c_r = 0 \quad . \tag{1.9}$$

Using the limiting results of the renewal process in the previous section, failure-based maintenance has an expected cost per unit time, c_f/μ, and the block replacement strategy would have a solution to equation (1.9) with a lower expected cost per unit time if $c_r/c_f < (\sigma^2 - \mu^2)/2\mu^2$, assuming the absence of technological improvement or condition monitoring. Dagpunar (1994) tidies up the necessary and sufficient conditions for optimality based on the mean residual life property of the failure p.d.f, $f(x)$. A disadvantage of the model is that failures may occur just before a planned replacement. Hence, the policy would apply to mainly inexpensive items, such as light-bulbs.

In age-based replacement, each component is replaced at failure or when attaining age T, whichever comes first. Hence, each component's age needs to be monitored for the application of this policy. We will assume that each component has failure and preventive replacement expected costs, c_f, c_r respectively. The replacement cycle will end at failure or T. Hence, the expected cost per cycle is $c_fF(T) + c_rR(T)$. The cycle length is clearly random, and its expectation, $m(T)$ say, will be given by,

$$m(T) = \int_{x=0}^{T} xf(x)\,dx + TR(T) = \int_{x=0}^{T} R(x)\,dx \quad . \tag{1.10}$$

The expected cost per unit time, $c(T)$, over a finite time horizon may be complex involving the use of renewal functions. Using renewal theory, the long term expected cost per unit time is given by the ratio of the expected cost to the expected cycle length, see Cox (1957, p.118). Hence, $c(T)$, for one component is given by,

$$c(T) = \frac{c_fF(T) + c_rR(T)}{m(T)} \quad . \tag{1.11}$$

Again, a group, size M, of components would have expected cost per unit time, $Mc(T)$. An optimal solution, if it exists, can be found graphically or by differentiating, simplifying and solving the equation,

$$(c_{\mathrm{f}} - c_{\mathrm{r}})\left(z(T)\int_{x=0}^{T} R(x)\,dx - F(T)\right) - c_{\mathrm{r}} = 0 \ , \qquad (1.12)$$

where $z(x)$ is defined in function (1.3). When considering the L.H.S at $T = \infty$, it can be seen that a uniquie solution will exist, if $\mu z(\infty) > c_f/(c_f - c_r)$ and $z(x)$ is strictly increasing , since the L.H.S is negative at $T = 0$ and is monitonically increasing. Dagpunar (1994) discusses further the conditions for the existence and uniqueness of an optimum. It is must be noted that the optimal age-based or block replacement solution may not be the overall optimum maintenance strategy for the component, in that other strategies through use of inspections or condition monitoring may provide lower cost per unit time. This will be shown in Section 1.4.2

The following papers give characteristive examples of cost models. Beichelt (1981) considers a model where detection of failure can only be made by inspection. An increasing cost between failure and inspection is considered and an optimal irregular spaced inspection strategy is proposed. This is a similar situation to the modelling of concrete structures in Chapter 6. Christer and Keddie (1985) present a replacement model applied to filling valves on a canning line. Kaio and Osaki (1989) compare inspection policies for a component that can only be detected as failed by inspections. Jack (1991) considers the effect of imperfect repairs over finite time horizons. Makis and Jardine (1992) consider also a replacement and repair cost model, which takes into account the possibility of imperfect repair. Dagpunar (1994) extends the age-based replacement cost model by introducing non-zero downtimes for failure and planned replacement. Necessary and sufficient conditions are formulated and discussed. An example is given in the paper, based on a case study, in Christer and Keddie (1985). A recent application study has been published, see Vanneste and Wassenhove (1995).

1.2.3 Models for Downtime

Modelling downtime for maintenance strategies can become complex due to incorporating the finite time for renewing a component within a stochastic model.

Barlow and Proschan (1965) and Barlow and Hunter (1960b) model the failure-repair process as an alternating renewal process. We shall consider, again, the block and age-based replacement policies as examples.

In block replacement, assume the expected replacement time of failure, d_f say, is small compared to, T, the planned replacement time, so that the process of failures is approximately a renewal process over interval $(0, T)$. If d_r is the block replacement time then the expected downtime per unit time, $d(T)$ say, is given by,

$$d(T) = \frac{d_f B(T) + d_r}{T + d_r} . \tag{1.13}$$

By differentiating $d(T)$, the optimum solution, if it exists, can be found by solving the equation,

$$d_f(r(T)(T + d_r) - B(T)) - d_r = 0 . \tag{1.14}$$

Using the limiting results of the renewal process in Section 1.2.1, equation (1.14) will have a solution if $d_r(d_f - \mu)/d_f < (\sigma^2 - \mu^2)/2\mu^2$.

For age-based replacement, the restriction on d_f being small need not be imposed, and the expected downtime per unit time over a long term horizon is simply given by cost function (1.11) with $c_f = d_f$, $c_r = d_r$ and $m(T)$, the expected cycle length, given by,

$$m(T) = \int_{x=0}^{T} (x + d_f)f(x)dx + (T + d_r)R(T) . \tag{1.15}$$

It can be seen that the expected downtime per unit time for failure-based maintenance is $d_f/(\mu + d_f)$. Dagpunar (1994) gives necessary and sufficient conditions on optimality for this policy.

1.2.4 Estimation of Modelling Parameters

This section gives an account of techniques used in estimating and testing the modelling parameters of the distribution of time to failure, $f(x)$, given uncensored and censored observations.

Assume N independent components of the same type are placed in operation and each one is run to failure, with x_i say, being the observed time to failure of the i'th component, $1 \leq i \leq N$. The components need not necessarily be placed in operation at the same time. It is also assumed, for now, that the set $\{x_i\}$ do not form a series of events generated by repairs to a single component. The following sample functions can easily be calculated; probability histogram, c.d.f, reliability function, hazard and cumulative hazard. This aids in deciding the form of the p.d.f, $f(x)$, of x. Statistics such as the sample mean and variance can be calculated and confidence limits can be placed on the theoretical values of these parameters.

We now consider estimation techniques. Two formal methods of estimation are in common use, that is maximum likelihood and method of moments. Assume the p.d.f family selected has form $f(x;\, \underline{\lambda})$ where $\underline{\lambda}$ is the set of parameters to be estimated. The likelihood function of the data set $\{x_i\}$ is given by,

$$L(\underline{\lambda}) \;=\; \prod_{i=1}^{N} f(x_i;\underline{\lambda}) \quad , \tag{1.16}$$

which represents a probabilistic measure of observing the given observations. The estimated parameters are chosen at the point such that $L(\underline{\lambda})$ (or alternatively, $\mathrm{Ln}(L(\underline{\lambda}))$, the log-likelihood) is a maximum.

For the method of moments estimation, the first M sample moments about $x = 0$ are calculated, where M is the number of parameters to be estimated. Then, M simultaneous equations are set up by equating each sample moment to the corresponding theoretical moment, see Chatfield (1970, p.121). It follows that the following equations would need to be solved,

$$\int_{x=0}^{\infty} x^{j} f(x;\underline{\lambda})\,dx \;=\; \frac{1}{N}\sum_{i=1}^{N} x_i^{j} \qquad for \quad 1 \leq j \leq M \quad . \tag{1.17}$$

These equations can then be solved, if a solution exists, to obtain a point estimate of $\underline{\lambda}$.

In testing the fit of the model, the sample probability histogram can be compared to the estimated histogram and the χ^2 test can be undertaken. Alternatively, the sample and estimated c.d:f can be compared and the Kolmogorov-Smirnoff (KS) test can be carried out.

We next discuss the situation of censored observations in the context of block and age-based replacement policies. Assume, T is the current replacement practice for either policy. Over an operating survey, two sets of data on failures would arise, that is a set of size, A say, completely observed times to failure and a set of size, B say, of censored times (due to replacement at T), where it is only known the time of failure exceeds a specific value. We shall denote these sets, $\{x_j\}$, $1 \leq j \leq A$, and $\{y_k\}$, $1 \leq k \leq B$. The maximum likelihood estimation can also be applied, and is given by,

$$L(\underline{\lambda}) = \prod_{j=1}^{A} f(x_j;\underline{\lambda}) \prod_{k=1}^{B} R(y_k;\underline{\lambda}) \quad . \tag{1.18}$$

For age-based replacement, all the censored values, y_k, will be equal to T, and a test-of-fit can be carried out using the conditional c.d.f of x over the interval $(0, T)$, that is $F(x)/F(T)$, and comparing it to the sample c.d.f of failure times. Additionally, the sample proportion of planned replacements can be compared to the estimated reliability, $P\{x > T\} = R(T)$, and a binomial statistical test carried out. For the block-replacement case (or progressively censored samples, in general, where the set $\{y_k\}$ have random values), a graphical test-of-fit can be undertaken using the cumulative hazard function, see Nelson (1984), and a statistical test-of-fit undertaken by using the Kaplan-Meier estimate of the reliability function, see Kalbfleisch and Prentice (1980).

The procedures outlined above, will be applied to delay time analysis, in the appropriate modified forms, for simulated data in Chapter 5, and on the analysis of inspection records of concrete components in Chapter 6.

When considering repairs to a single component, and $\{x_i\}$ is the set of inter-arrival times of failures, then a trend could be seen by plotting cumulative failures versus operating time. An increasing gradient would show deterioration whilst a decreasing gradient would show component improvement. In these cases, the component may not always be repaired to 'as-new', and the times $\{x_i\}$ may not be independent and identically distributed, see Ascher and Feingold (1984). A renewal process is then not appropriate and the estimation techniques above would not apply. The next section on systems presents one model which can cope with this effect.

1.3 Maintenance Modelling of Systems

The aim of this section is to expand the modelling of a single component to modelling the maintenance of a multi-component system or group of systems. Effectively, a system is a collection of components. Ascher and Feingold (1984) give an extensive account on models and estimation techniques for the maintenance of repairable systems.

1.3.1 Stochastic Processes

The mathematical approach to modelling system reliability has been through the application of stochastic processes. We shall consider, here, a system whereby components are assumed to be in series and independent. A breakdown is then caused by the failure of any one component.

For an n-component series system, let $R_i(x)$ say, be the reliability function of the i'th component. The reliability function, $R(x)$, of the system from new is then given by,

$$R(x) \quad = \quad \prod_{i=1}^{n} R_i(x) \quad . \tag{1.19}$$

The p.d.f of time to first breakdown, $f_1(x)$ say, is then given by, $f_1(x) = -R'(x)$. For the case when a system needs complete replacement after breakdown, or must be repaired to an assumed statistically 'as-new' condition, then the system can be treated, here, as a single component with time to each failure, $f(x) = f_1(x)$.

We shall now consider the case when a breakdown is rectified by only replacing or repairing the failed component. The time to next breakdown would not necessarily be distributed with p.d.f $f_1(x)$. The process of breakdown arrivals in the absence of any PPM will be a superimposed renewal process (SRP), see Khintchine (1960). The SRP for a general situation would be complex, especially for large n and many non-identical components. We shall consider, here, approximating the SRP by a non-homogeneous Poisson process (NHPP). Barlow and Hunter (1960a) first introduced the NHPP for systems which are minimally repaired at breakdown. Ascher and Feingold (1984) recommend the use of the NHPP especially for complex systems when n is large. Under certain conditions, the SRP has been shown to asymptotically tend to the NHPP when n is large, limiting to a HPP as time increases, Khintchine (1960). The breakdown arrival process of a delay time model for a repairable system, Christer and Waller (1984a), has both these properties, and is discussed in detail in Chapter 2. The NHPP model may be also applied to the case of a single component when a repair does not return the component to as-new due to ageing, for example. The NHPP requires the ROCOF, $r(t)$, to be estimated. The number of breakdowns in the interval $(0, T)$ is Poisson distributed with mean value $B(T)$ given by,

$$B(T) = \int_{t=0}^{T} r(t)\,dt \quad . \tag{1.20}$$

The number of breakdowns in any interval, (T_1, T_2) say, is also Poisson distributed. The reliability function, from new, $R(x)$, is then given by,

$$R(x) = \exp(-B(x)) \quad . \tag{1.21}$$

Hence, the time to first failure has p.d.f, $f_1(x) = r(x)R(x)$.

It can be seen that the hazard rate, for first breakdown, $h_1(x)$ say, equals $r(x)$ for the NHPP, and this property causes further confusion in the reliability field, see Ascher and Feingold (1984). It can be shown that given a breakdown at time x_1 then the hazard function for the second breakdown, $h_2(x \mid x_1)$ say, equals $r(x + x_1)$, where x is measured from time x_1. Hence, breakdown repairs do not reduce or increase the chance of subsequent breakdowns. Due to this effect, the model has been termed 'minimal-repair' or 'bad-as-old'. Also, it must be noted that the p.d.f of inter-arrival time to each breakdown is dependent on the last breakdown time.

As for single components, the components of a system could become defective prior to one component causing a breakdown. Therefore, an inspection could reveal the defective components within a system and a maintenance decision to repair the components could be applied. Hence, the number of breakdowns would be reduced through this process. Delay time modelling for systems of this type takes into account the defective phase that components of a system would enter over their life.

Other models for systems have included branching processes and Markov processes. For branching processes, component dependency is modelled in that it is assumed that a component that fails may then cause the failure of other components. For Markov processes, a system is perceived to enter a set of defective states prior to failure, with each time within each state having the restriction of being exponentially distributed. A recent development has been in the introduction of a reduction factor, in that after breakdown repair, the hazard function for the time to next breakdown is set to a level between new and the time of the breakdown, see Kijima et al (1988).

We next consider cost and downtime models assuming a system is to be modelled by an NHPP.

1.3.2 Models for Cost

In this section, cost models are discussed for a complex system, assuming a NHPP model for the process of breakdown arrivals. Barlow and Hunter (1960a) consider a form of block replacement model whereby at time T the system is completely replaced, and breakdowns are minimally repaired over the replacement period. Assuming instantaneous breakdown repairs, the expected cost per unit time model has identical form in function (1.8) but with $B(T)$ being the expected number of breakdowns assuming an NHPP model. The model could also be applied to a system whereby at time T the system is overhauled, repairing defective components, so that it is assumed to return to a statistically 'as-new' system. However, the cost of overhaul would be dependent on the number of defects within the system at time T, and thus needs to be modelled. Chapter 2 presents a delay time model which takes this effect into account for a complex repairable system and is based on the paper, Christer and Waller (1984a).

Other maintenance options exist, for a complex system. For example, one could replace after N breakdowns where N is another decision option in the model, or a form of age-based model whereby replacement occurs after the system operates without failure for time T. Evidently, the model chosen would be dependent of the failure characteristics of the system, cost levels of repair and feasibility of applying the maintenance strategy.

Practically all age-based and block replacement strategies assume the assymptotic cost function as an objective function to optimize. Some attention has been given to modelling an equivalent finite time horizon cost, for example, Christer (1978, 1987b), Christer and Jack (1991) and Jack (1991).

Christer and Scarf (1994) consider a replacement model for medical equipment. The system is assumed to breakdown in accordance with a NHPP. Two decision variables are considered, K and L. K is the number of years to replace old equipment and L is period of use for new equipment prior to replacement. A cost model is formulated taking into account discounting factors, over the time period $K + L$. There is scope to extend the model through use of inspections. Scarf and Bouamra (1995) apply a similar model to a set of inhomogeneous bus fleets and consider the effect of a penalty factor for delaying replacements. Kobbacy et al (1995) assume the corrective repair process of system is modelled by a delayed renewal process after each preventive maintenance activity. Sensitivity analysis for the change in optimum cost policy is carried out for varying modelling parameter values.

1.3.3 Modelling Downtime

As for single components downtime models can become complex. The two maintenance strategies given in Section 1.2.3 could be applied to the system given identical assumptions, except we replace the renewal assumption by the NHPP assumption. Morumora (1970) considers a policy whereby a system is minimally repaired for operating time T and replaced at first breakdown after operating time exceeds T. A use-based delay time model is addressed in Chapter 2. Dagpunar and Jack (1993) consider a policy whereby a system is minimally repaired over an elapsed time T (operating + cumulative repair) and replaced at first breakdown after T. Christer and Waller (1984b) present a case study of a canning line. An approximate model for downtime is proposed whereby breakdowns are rectified over a period T and an inspection is carried to repair defects at T so as to return the system to a statistically 'as-new' condition. The appendix to the thesis outlines a delay time model to take into account the effect of stochastic downtime.

1.3.4 Estimation of Modelling Parameters

The section details the maximum likelihood estimation technique for the NHPP assumption of breakdown arrivals. Assume a breakdown arrival process has been observed over an interval of time $(0, T)$ whereby breakdowns are minimally repaired and the system is assumed new at time 0. Let B be the number of breakdowns observed at the ordered times, $\{t_i\}$ say, where $t_i < T$, for $1 \leq i \leq$ B. Cox and Hinkley (1974) derive the likelihood as,

$$L(\underline{\lambda}) = \exp(-B(T;\underline{\lambda}))\prod_{i=1}^{B} r(t_i;\underline{\lambda}) \quad , \tag{1.22}$$

where $\underline{\lambda}$ is the set of modelling parameters to be estimated and $r(t; \underline{\lambda})$ and $B(T; \underline{\lambda})$ are the respective parameterised models for the ROCOF, for $t > 0$, and the expected number of breakdowns for time $(0, T)$.

In testing the fit of the model, the sample cumulative failures vs. operating time could be plotted against $B(x)$ for $0 \leq x \leq T$. For a numerical test, it is observed that a random breakdown time over time interval $(0, T)$ has the c.d.f, $B(x)/B(T)$, given in Ross (1983). Therefore, the empirical distribution of breakdown times over $(0, T)$ could be compared to the estimated c.d.f and the K-S test carried out. An additional test for the NHPP model could also be carried out on the number of breakdowns occurring over a set of intervals $(0, T)$. The sample distribution can then be compared to the Poisson distribution of mean $B(T)$.

1.4 Delay Time Inspection Modelling

The mathematical modelling of maintenance using the technique of delay time analysis was introduced by Professor A. H. Christer and first mentioned in the appendix to the paper, Christer (1973). Since then, a series of papers have been written successfully developing the concept and applying the model to many areas of industrial maintenance. The model arose from the observation that a component can become defective and enter a phase prior to causing a failure such that it can be detected by an inspection and be repaired. Evidently a component's life without any maintenance intervention, see Fig.1.1, can be defined as three states namely :

(1) When it is not defective (or not noticeably defective by current inspection procedures).
(2) When it is defective, and can be inspected and repaired.
(3) When it causes a failure and needs immediate repair or replacement.

The *delay time*, h say, is the interval of time spent in state 2, i.e from the instant when the component becomes defective to its necessary repair or replacement. The initiation time, u say, is the interval of time spent in state 1, to when a defect becomes first detectable by inspection.

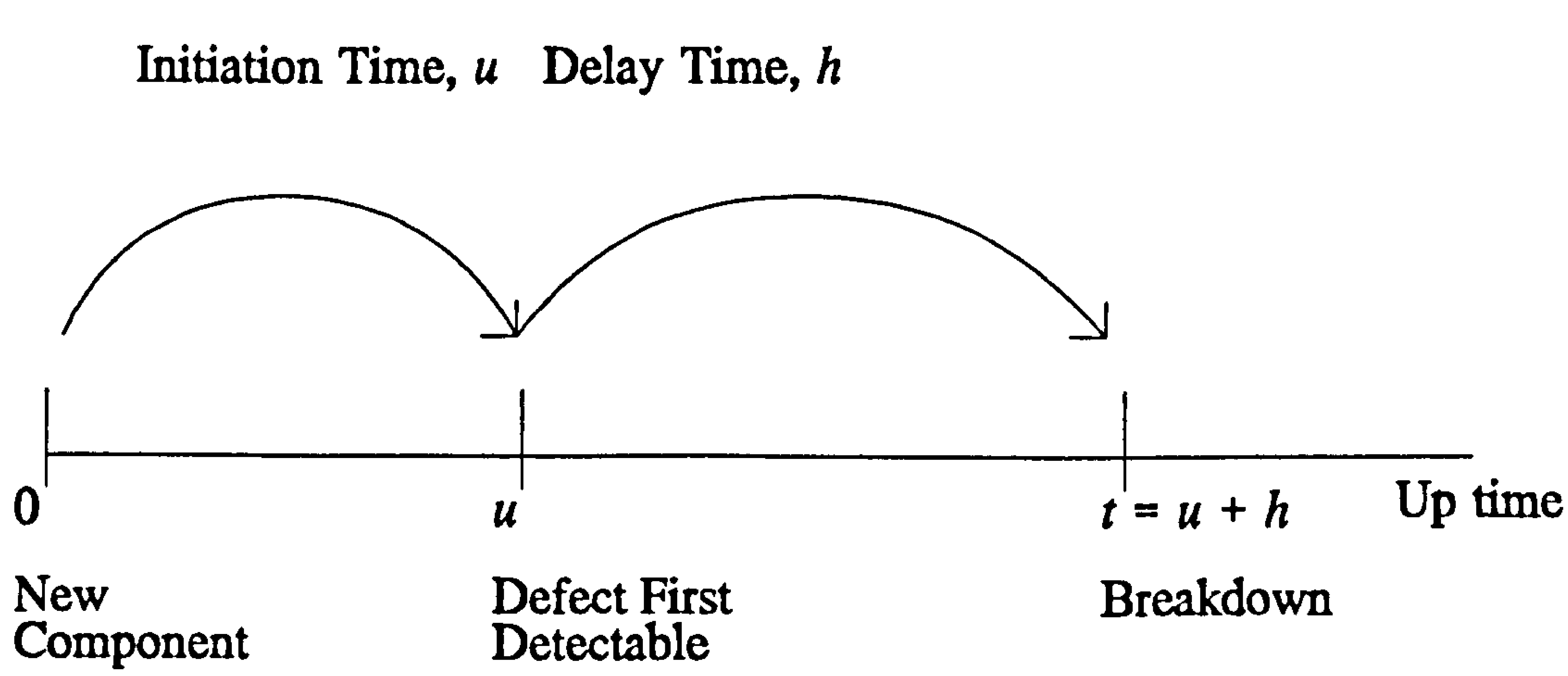

Fig. 1.1. The initiation time and delay time of a component.

The judgement that a component is defective would be made by the maintenance engineer. A component which is replaced whilst in the defective phase would reduce cost and downtime compared to failure replacement. However, a compromise must be sought based on cost/downtime levels of repair and planned inspection activity. It is evident that there is an extended scope for modelling options using the concept of delay time.

1.4.1 Review of Papers

Christer (1982) introduces the delay time model in the context of building maintenance. Here, a cost based system model, assuming perfect inspections, is formulated with expected repair costs assumed to be varying over the delay time period. A method is suggested to estimate the repair cost function by subjective estimates from engineers and inspectors.

Christer and Waller (1984a) formally introduce the delay time model for complex industrial systems. Models of downtime and cost are formulated assuming perfect inspections and HPP defect arrivals. The model is then extended to NHPP defect arrivals. The method to model imperfect inspections is then proposed. Numerical examples are provided.

Christer and Waller (1984b) present a case study of a canning line plant. A snapshot model is proposed to aid in locating the component types where planned maintenance would be an effective option. The delay time p.d.f is estimated through subjective estimates of engineers and inspectors via a questionnaire. The system model is proposed and the predicted proportion of defects which arise as failures is modelled accurately to the observed value for the current inspection practice.

Christer (1987a) models the reliability function of a single component subject to one known defect type. The component is assumed to be inspected perfectly and periodically. The model is then expanded to consider the reliability of n components in parallel. There is scope to further expand the model to the case of imperfect inspections.

Christer (1988) develops a cost based model for the maintenance of civil engineering structures. A system model is assumed with expected repair costs varying over the delay time period. Due to delay times being most likely to be in the order of years, the probability of detecting a defect is also assumed to vary over the delay time period.

Christer and Redmond (1990) discuss a bias in the parameter estimation of the delay time p.d.f when data collected over an operation period is censored. For example, delay time estimates may only be obtainable at failures. The biased (or conditional) p.d.fs of delay time are formulated for the two cases of breakdowns and inspected defects, assuming a complex system is periodically and perfectly inspected. A maximum likelihood estimation technique is then proposed to estimate the parameters of the true delay time distribution. There is scope to extend the analysis to imperfect inspections and considering the bias in the initiation time of defects. Also, the parameter estimation bias for single components and n-component systems provides further research.

Cerone (1991) presents an approximation technique for the reliability function formulated in Christer (1987) for a single component. Cubic splines are fitted to the reliability function over each inspection interval. Pelligrin (1991) presents a graphical procedure to determine the optimal inspection interval for a system delay time model allowing for imperfect inspections.

Chillcot and Christer (1991) present a case study of applying delay time analysis to the maintenance of coal mining equipment. A system model is assumed to predict downtime. Due to repair downtimes being large compared to the inspection period, the modelled process of faults arising will halt during a downtime period, and an iterative model is proposed to take this effect into account.

Baker and Wang (1992) fit a single-component delay time model for estimating delay time parameters by objective means given observations of times of failures and inspection outcomes, where maintenance was carried out irregurarly. The reliability of the components is estimated when various inspection policies are applied. This was an important paper in extending the applicability of delay time modelling.

Christer and Redmond (1992) introduce model updating techniques when the p.d.f of delay time has been subjectively derived. The objective is to model the known proportion of defects arising as breakdowns for the current inspection practice. The actual delay time is assumed to be linearly related to the subjective estimate. The uniqueness and existence of parameter estimates are then discussed. The scope to change the model through assuming imperfect inspections is also addressed. An estimation technique to estimate modelling parameters given breakdown time and defect observations is also proposed. The case study, Christer and Waller (1984b), is used to demonstrate the techniques proposed.

Christer and Wang (1992) present a case study of a model for condition monitoring of a production plant. A component's wear property is modelled and a delay time model proposed based on replacing a component when its wear reaches a certain threshold.

Baker and Christer (1994) present a review of delay time modelling. Further research topics are outlined and a method to estimate modelling parameters from both subjective and objetive data is proposed.

Christer et al (1995) present a case study for the maintenance modelling of a copper production plant. A system model is assumed and the objectively based estimation of the delay time modelling parameters undertaken using the observed times of breakdowns and the number of defects detected at inspections. The mean downtime per breakdown is assumed to increase prior to a planned inspection due to likely occurrence of more severe breakdowns as the system operates. A downtime model is formulated and a weekly planned maintenance activity is recommended.

1.4.2 Single-Component Models

We consider in this section, a delay time model for a single component. Papers and reports on the single-component model are given by; Christer (1987), Baker (1991), Baker and Wang (1991), Cerone (1991), Baker (1992), Baker and Wang (1992), Christer and Wang (1992) and Baker and Christer (1994). The component is assumed to enter a defective state prior to failure such that if detected by inspection, then repair or replacement options exist to prevent failure. When a component is in the defective state, it is assumed that it is still able to provide its necessary service.

Assume the initiation time, u, has the p.d.f and c.d.f, $g(u)$ and $G(u)$ say, respectively. Likewise, the delay time, h, has p.d.f and c.d.f, $f(h)$ and $F(h)$ say, respectively, independent of u. The c.d.f of time to failure, $P(x)$ say, would then be given by the convolution of u and h such that $u + h \le x$. Therefore, $P(x)$, is given by,

$$P(x) = \int_{u=0}^{x} g(u)F(x-u)\,du \quad , \tag{1.23}$$

and the reliability, $R(x) = 1 - P(x)$.

Consider a maintenance strategy, whereby a component is replaced or repaired at failure and *only* when detected in the defective phase at an inspection. Cox (1957) presents a similar strategy based on wear level of a component at inspection We shall consider, here, perfect inspections, and the special case that $g(u)$ is exponentially distributed. Therefore, an inspection would in effect be a renewal point, in that if the component is not defective at an inspection (and consequently not replaced), then the time u to becoming defective after inspection has the same exponential p.d.f, $g(u)$, due to the memoryless property. Assume the expected cost of failure replacement, planned defect replacement and inspection have costs c_f, c_r and c_I respectively. The objective is find the optimum time to inspect T, after each failure-free period $(0, T)$ of component operation. The expected cost over each cycle, $C(T)$, is given by,

$$\begin{aligned} C(T) &= c_f P(T) + (c_r + c_I)\int_{u=0}^{T} g(u)(1-F(T-u))\,du + c_I(1-G(T)) \\ &= (c_f - c_r - c_I)P(T) + c_r G(T) + c_I \quad , \end{aligned} \tag{1.24}$$

after simplification. Assuming instantaneous inspection and replacement times, the expected cycle length, $M(T)$, is given by,

$$M(T) = \int_{x=0}^{T} xP'(x)\,dx + TR(T) = \int_{x=0}^{T} R(x)\,dx \quad . \tag{1.25}$$

Hence, the long-term expected cost per unit time, $c(T)$ say, is given by,
A comparison will be made, here, with the age-based policy of Section 1.2.2, whereby a component is replaced after time T, regardless of whether it is in the defective state

$$c(T) = \frac{C(T)}{M(T)} . \tag{1.26}$$

or not, that is not inspecting. It will be shown by numerical example that the optimum inspection policy can have a lower expected cost per unit time than the age-based policy. We will assume the following values of the modelling parameters, where the delay time distribution is also taken to be exponential:

$c_r = 10$,

$c_f = 80$,

$c_I = 5$,

$G(u) = 1 - \exp(-0.2u)$ and

$F(h) = 1 - \exp(-0.3h)$.

It will be assumed that the cost of a planned replacement of a component at inspection, c_r, is equal to the cost of an age-based planned replacement. The graphs of the expected cost per unit for a range of T values are given in Fig. 1.2 for the inspection policy, function (1.26), and the age-based policy, function (1.11). Clearly, the optimum inspection policy results in lower expected cost per unit time. The optimum inspection time is shorter than the age-based replacement time. This is to be expected, as inspections will reduce the frequency of failures and component replacements.

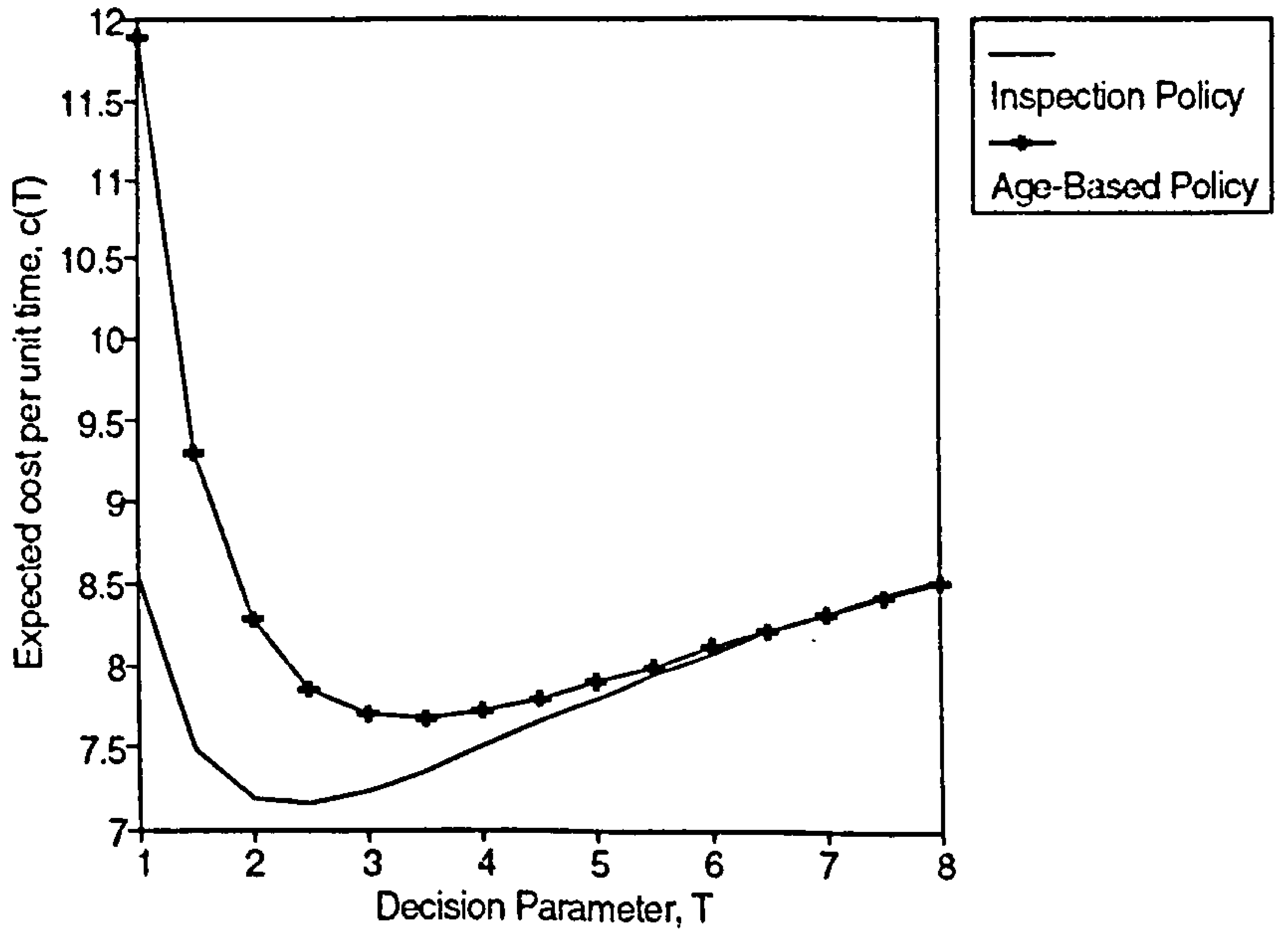

Fig. 1.2. Comparison of an inspection policy and age-based replacement.

1.4.3 System Models

We will consider here, briefly, a delay time model for a repairable system. Chapter 2 discusses the model in greater detail. Papers on the system model are given by; Christer (1982), Christer and Waller (1984a,b), Christer (1988), Christer and Redmond (1990,1992), Pellegrin (1991), Chilcott and Christer (1991) and Baker and Christer (1994). It will be assumed that a system comprises of many independent component parts, and a breakdown can be caused by any one component. Defects are assumed to arise as a stochastic process with each defect having a delay time period before causing a breakdown. When a breakdown occurs, it is assumed that only the component that caused the breakdown is repaired or replaced and other defects go undetected. In the absence of inspections, an example of the breakdown arrival process is given in Fig. 1.3. It can be seen that a perfect inspection prior to the first breakdown, would detect defects within the system, and if the faults are corrected, then future breakdowns would be prevented.

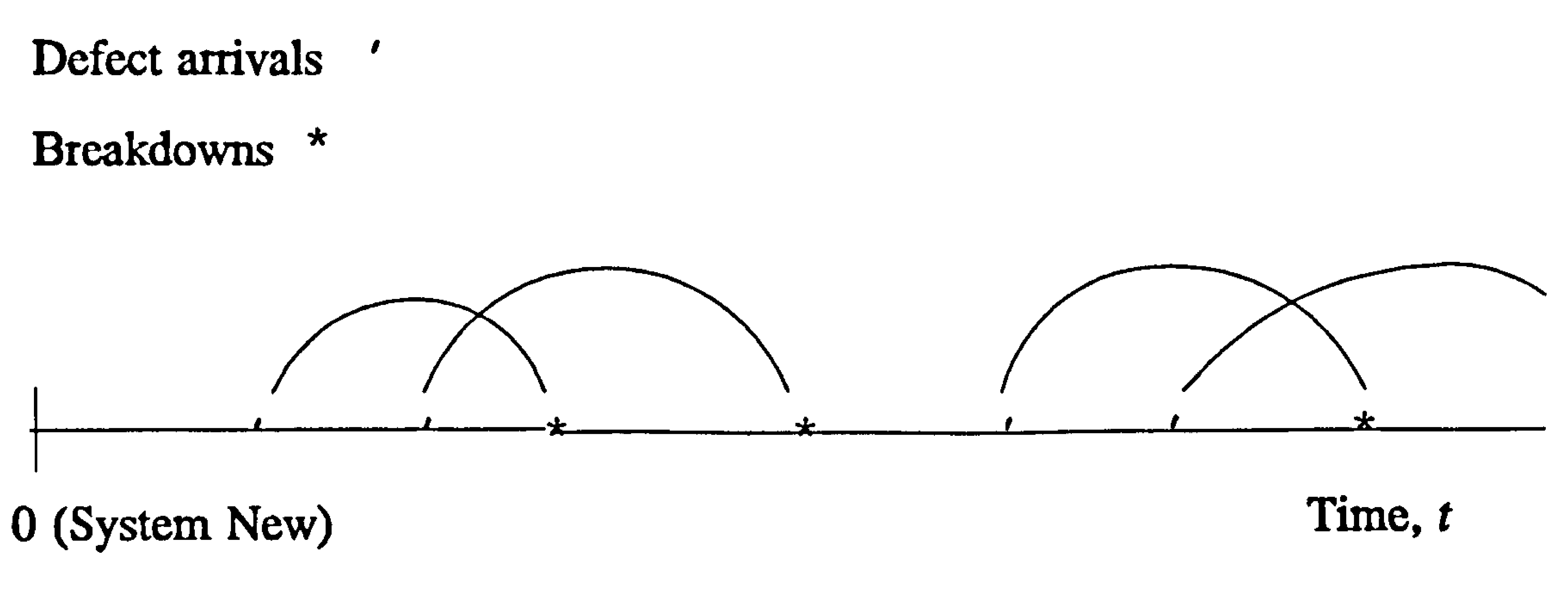

Fig. 1.3. An example of the breakdown arrival process for a system.

The following presents the set of initial assumptions concerning modelling parameters for a technical system under study:

(a) At time 0, the system is in a new or 'as new' state, that is defect free.

(b) Defects arise within the system, independently, as a homogenous Poisson process (HPP) with rate parameter k, over time.

(c) Delay times, h, are independent of arrival time u and are described by the probability density function (p.d.f), $f(h)$ say, with finite mean μ.

(d) Those defects which cause breakdowns over time are repaired with negligible time at failure.

(e) A breakdown has a repair cost, which is independent of the defect arrival time, the delay time and the repair time. The repair cost is assumed to have a mean c_b.

(f) Inspections are perfect.

(g) There is a constant time T between successive inspections.

(h) The expected cost of an inspection is c_I.

(j) An inspection takes time, d_I, to undertake, and all defective components that may be found are repaired/replaced within this time.

(i) The expected cost of a repair to a defective component at an inspection is c_d.

It will be shown in Chapter 2, that the process of breakdown arrivals of the system described above, under failure-based maintenance (FBM), is a non-homogeneous Poisson process (NHPP). A characteristic function of the inspection policy is the proportion of defects which would arise as breakdowns, $b(T)$ say. This has been derived by Christer and Waller (1984a), and is given by,

$$b(T) = \frac{1}{T}\int_{h=0}^{T} (T - h)f(h)dh \ . \tag{1.27}$$

The model for expected cost per unit time, $c(T)$ say, is then given by,

$$c(T) = \frac{kTb(T)c_b + kT(1-b(T))c_d + c_I}{T + d_I} \ , \tag{1.28}$$

over each cycle, $(0, T + d_I)$.

To model downtime, assume d_b is the expected downtime of a breakdown repair, and that d_b is small compared to T. Hence, the approximate expected downtime per unit time, $d(T)$ say, over each cycle $(0, T + d_I)$, is given by,

$$d(T) = \frac{d_b kTb(T) + d_I}{T + d_I} \quad . \tag{1.29}$$

The delay time model can further be applied to imperfect inspections, non-homogeneous rates of defect arrivals and to other inspection-type policies. These model developments and others will be discussed in Chapter 2.

1.4.4 Estimation of Modelling Parameters

Central to the application of delay time analysis, is the accurate estimation and testing of modelling parameters concerning the selected initiation time and delay time model of components and systems. There have been essentially two approaches to the problem, that is the subjective method and the objective method.

The subjective method was introduced in the context of building maintenance, Christer and Waller (1984b). Questionnaires were compiled and presented to engineers and inspectors. The questions were aimed at obtaining subjective estimates of the initiation time and delay time of specific parts. At a breakdown, the engineer was asked how long ago the defect would have arisen. Thus, yielding a point estimate of both the initiation and delay time. At an inspection, an estimate is also required on how much longer a defective component could be delayed before it would cause a breakdown, in order to obtain an estimate of its delay time. Standard estimation and test-of-fit procedures can then be carried out to estimate the distribution (for single-components) or rate process (for systems) of the initiation time, u, and the delay time p.d.f, $f(h)$. However, a model formulated from only subjective data, would not necessarily model observational characteristics of the known inspection practice, such as the observed proportion of defects arising as breakdowns and the sample properties of the observed breakdown times. Hence, a form of revision or updating is necessary after subjective estimation. Chapter 3 presents methods for a system model to update and test estimated modelling parameters. Chapter 4 presents the case when subjective data is further censored, by considering the situations when a subjective data set can only be obtained at breakdowns or, perhaps, only at inspections. A maximum likelihood technique is proposed to remove this observational bias.

Objective methods were proposed by Christer and Redmond (1992) for a system model and applied by Christer and Wang (1992) for a single-component model. The method utilizes the observational information that can be obtained when operating a component or system under an inspection practice. The types of data that can be obtained for a system which is periodically inspected, say, are; the number of breakdowns over each inspection interval, the number of defect repairs at each inspection, the times of inspections and the times of breakdowns. The sample distributions of the observed number of breakdowns, defect repairs and breakdown times can then easily be formulated in terms of the inspection period, T, the defect arrival rate, $g(u)$, and the delay time p.d.f, $f(h)$. A model for u and h can be proposed and modelling parameters estimated via the maximum likelihood process. The estimated distribution of the number of breakdowns, defect repairs and breakdown times can then be compared to the corresponding sample distributions and the appropriate statistical tests-of-fit carried out. If statistical tests fail, then the proposed models for u and h, for example Weibull distributions, would need to be revised. Subjective measures of u and h can help in this case to decide on appropriate models. The objective method for a system will be demonstrated with simulated data in Chapter 5 and with inspections records of the deterioration of concrete structures in Chapter 6.

Evidently, a fusion of both methods would be beneficial due to an increased sample of data information, especially if there is only a small sample of objective data. Baker and Christer (1994) discuss methods of achieving this.

1.4.5 Introduction to Chapters

In Chapter 2, the type of complex repairable system under study and the delay time concept used to model maintenance will be defined. A non-homogeneous (or time-dependent) Poisson process model is used to describe the arrival process of breakdowns. The downtime and cost consequence due to a purely failure based maintenance policy are then modelled and discussed. The effects of inspecting the system over time are considered. Models for downtime and cost are derived for a periodic based inspection policy. Initially, inspection will be assumed perfect and this requirement is then relaxed to include the case of imperfect inspections. Extensions to these models will be developed and conditions involving modelling parameters are derived for the decision to optimize cost or downtime by an inspection based policy. Numerical examples will be provided. An alternative inspection policy based on inspecting a system after a specific period of use (or operation) will also be discussed. The chapter is based on the paper, Christer and Waller (1984a).

In Chapter 3, procedures are constructed for estimating the parameters necessary to formulate the models derived in Chapter 2. These will be constructed and based upon the experience gained and the data collected in operating repairable systems over time. Two types of data will be focused on, namely subjective and objective. Subjective data can arise from engineers' estimation of the delay time of specific defects at breakdowns and inspections. Thus, data of this type is expected to be in error. However, the collection of this data has been shown to be possible and prior delay time distributions have been estimated in specific cases, see Christer and Waller (1984b), Chilcott and Christer (1992), Christer and Desa (1992). The objective data for estimating the delay time distribution is based upon observations of times of breakdowns and defect detections. This data will aid the estimation of delay time parameters and the testing of the fit of the subsequent maintenance model. A maintenance model formulated with a substantial subjective input to delay time parameter estimates could not guarantee to automatically model the "status quo" characteristics of the system. That is subjective data may not imply that which is currently observable. Management interest may be in cost, downtime or proportion of defects which arise as failures under a current inspection practice. Eitherway, updating procedures are given to force the subjectively based model to agree with "status quo" observation. The situation is demonstrated in Fig. 1.4, where $b(T)$ is the prior model for the probability a defect arises as a breakdown and b_0^* is the observed estimate of this probability for the inspection practice, T_0. This updating procedure could be considered as a "model tuning process". We will find in Chapter 3 that there is not necessarily a unique option for updating. However, a selection criteria is given based on other information, which may be available, over the system data collection period. The chapter is substantially based on the paper, Christer and Redmond (1992).

In Chapter 4, the case of having a censored data sample with which to estimate a delay time distribution is discussed. As previously outlined, one situation which could arise is that delay times and initiation times of defects may only be readily estimated from either breakdown events or when defects are detected at inspections. Another situation could be a non-balanced mixture of these two extremes in that we may not be able to obtain an estimate of delay time and initiation time for each defect which has arisen over a survey period. In the case of censored data, a bias in the estimated distribution of delay times or initiation time would exist. This will be established by deriving the respective conditional p.d.f of delay time and initiation time associated with defects which arise as breakdowns, and those which are detected at inspection. The p.d.fs will be derived for both perfect and imperfect inspection policies. An example of the unconditional p.d.f and conditional p.d.fs of delay time is given in Fig. 1.5, assuming a delay time distribution $f(h)$, defect arrival rate k and inspection period $T = 10$. A maximum likelihood estimation technique and appropriate tests of model fit are then recommended to cope with the observational bias introduced. Much of the work of this chapter is based on the paper, Christer and Redmond (1990).

In Chapter 5, a simulation study is undertaken to further investigate and verify the delay time models and proposed method of analysis. Simulation programs, written in Pascal, have been used to simulate the delay time process given sets of input parameters and assumptions. Methods of simulation are shown for the case of perfect inspections and instantaneous repair of breakdowns. Then, these assumptions are relaxed to imperfect inspection and non-instantaneous breakdown repairs. The output of simulation experiments are analyzed and compared to the appropriate theoretical values of the models of the earlier chapters. An investigation is undertaken into the accuracy and effectiveness of the parameter estimation procedures given in Chapters 3 and 4. Correction of bias is carried out on censored simulation data. The effects of not correcting for bias but using an updating method, as a further option, is also explored. Fig. 1.6 demonstrates the situation when correction of bias has not been undertaken compared with the theoretical model and an observed value for the proportion of defects arising as breakdowns. An iteration method is developed which alternates between updating the scale parameter of a Weibull delay time distribution and only estimating the shape parameter using maximum likelihood. The estimation of delay time distribution parameters based upon only observational data (failure times and number of defects detected at inspections) is also demonstrated. Results are shown for simulated data sets and conclusions drawn.

In Chapter 6, an account is given on research supported by the Science and Engineering Research Council (grant: GR/F 61196) over a three year period. The project was in collaboration with members of the Civil Engineering Department at Queen Mary and Westfield College, London University. The main objectives were to collect and publish data on the observed rates of deterioration of particular defect types in a large number of concrete bridges and to develop predictive mathematical models that relate inspection frequency to maintenance costs. The delay time model is expanded to an extra phase in order to model the deterioration process of cracking and spalling in concrete, see Fig. 1.7. Costs of repairs are then expected to increase over the cracking and spalling phases. Due to the inspection records of components, only intervals containing the times to the states are available. A model for the distributions of times within each phase is proposed and estimation, via maximum likelihood, is undertaken, given the censored observations. Appropriate tests of model fit are carried out using the Kaplan-Meier estimate of the reliability function. Single component and multi-component cost models are then formulated for a variety of inspection and maintenance options. The motivation for the project was in part associated with the prototype modelling paper for inspection practices of major concrete structures, Christer (1988). A paper is to be published on this research project in the European Journal of Operational Research in 1996.

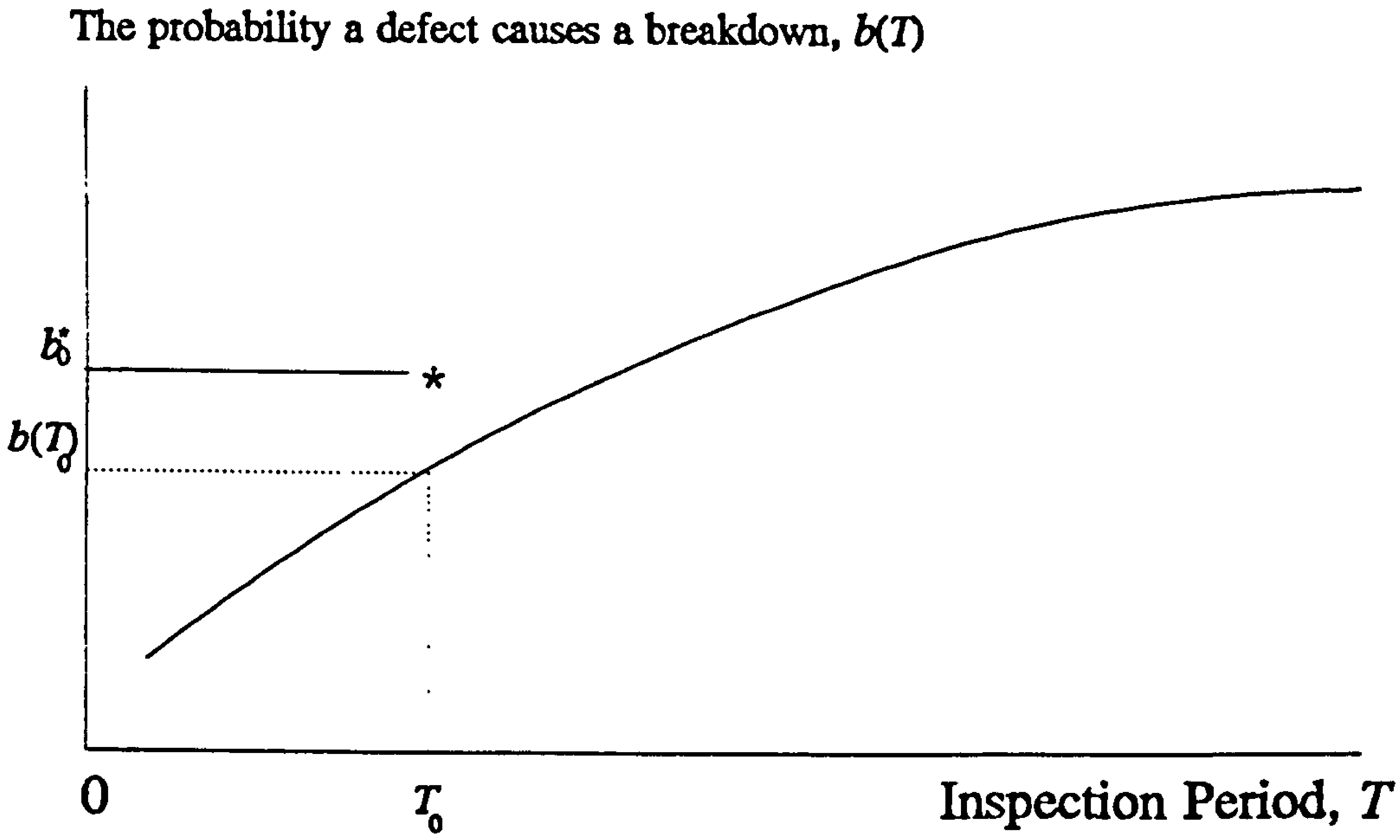

Fig. 1.4. The prior model, $b(T)$, compared to the known current practice.

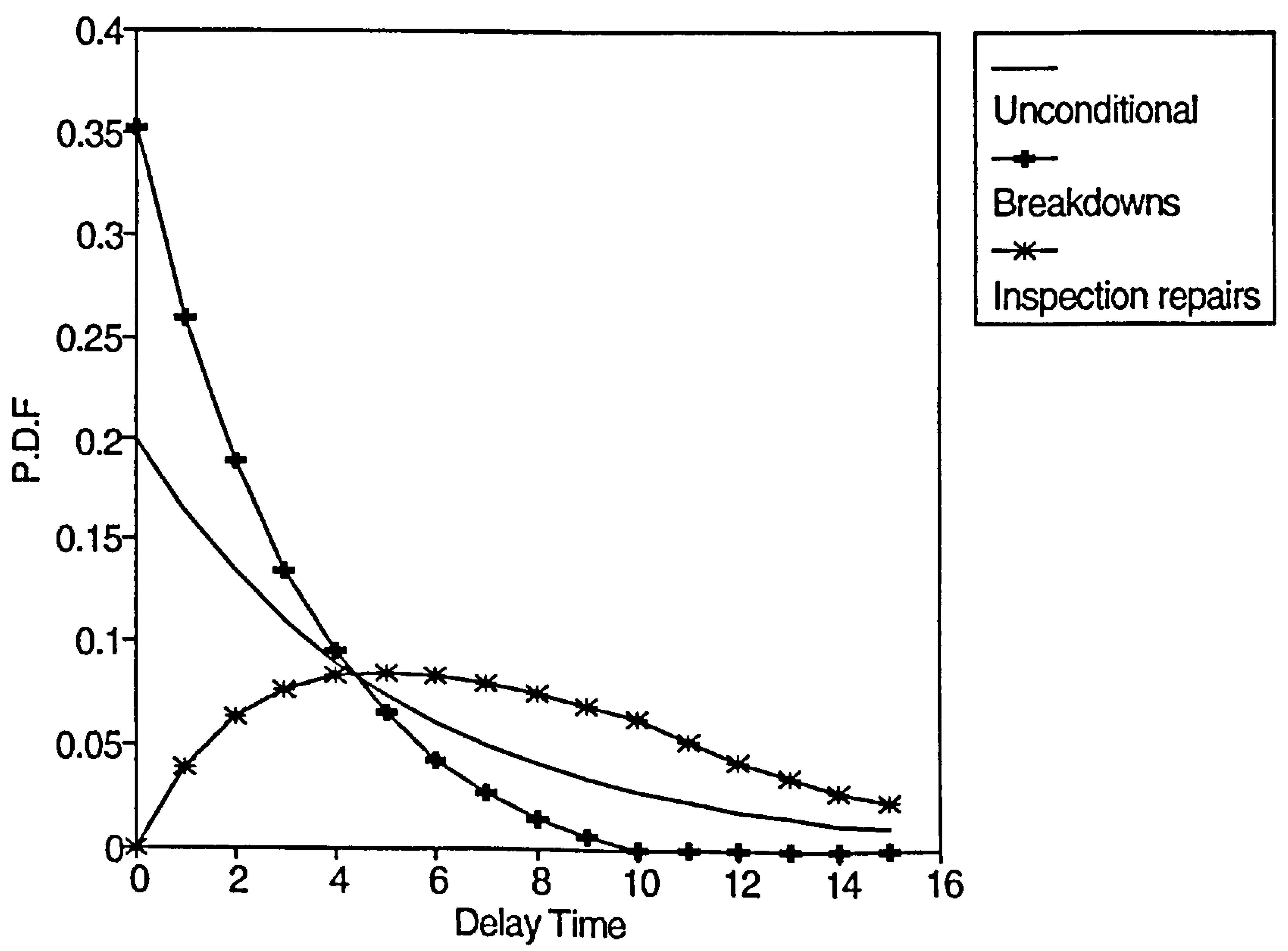

Fig. 1.5. The unconditional and conditional p.d.fs of delay time.

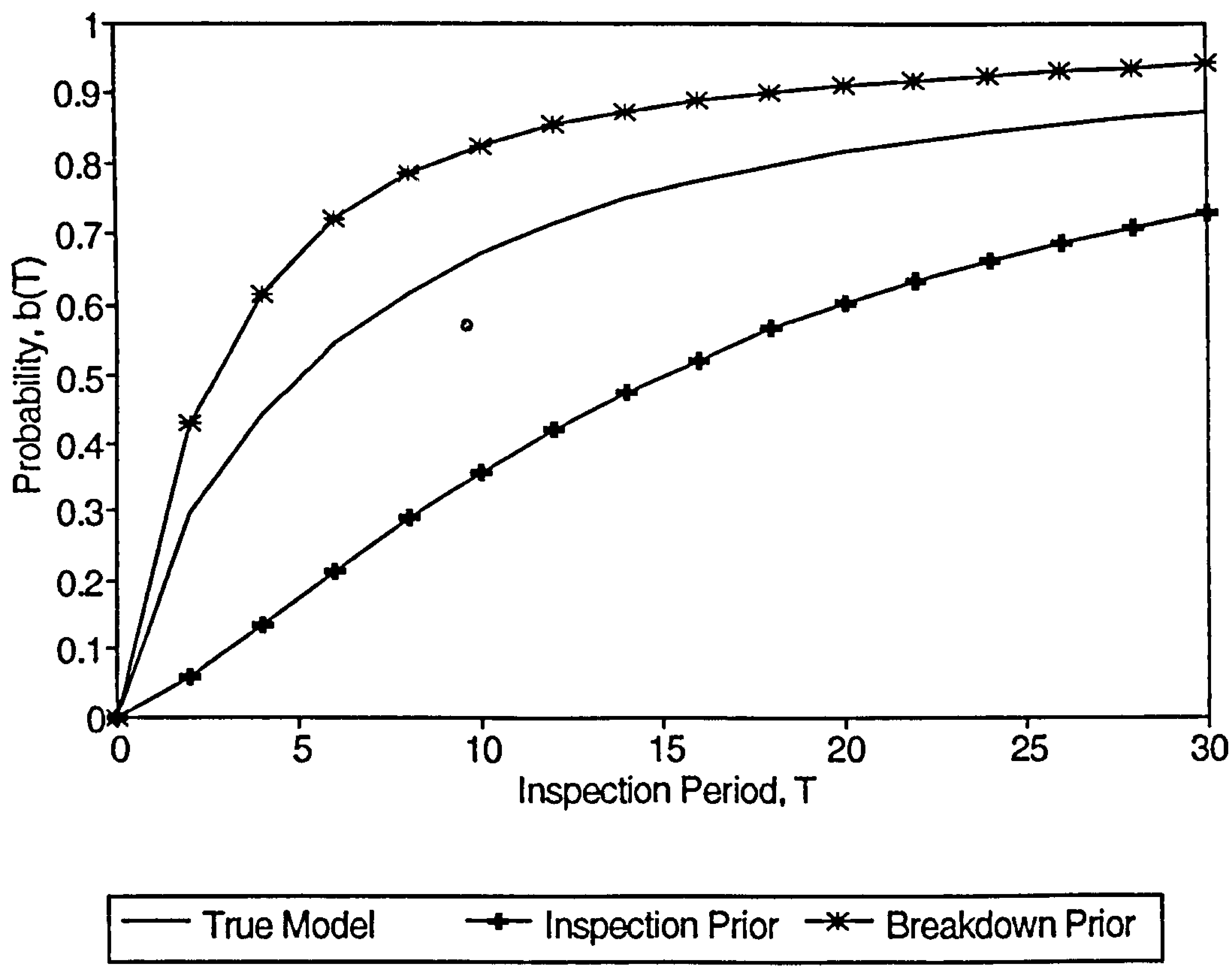

Fig. 1.6. The effects of not correcting for bias of conditional delay time sets.

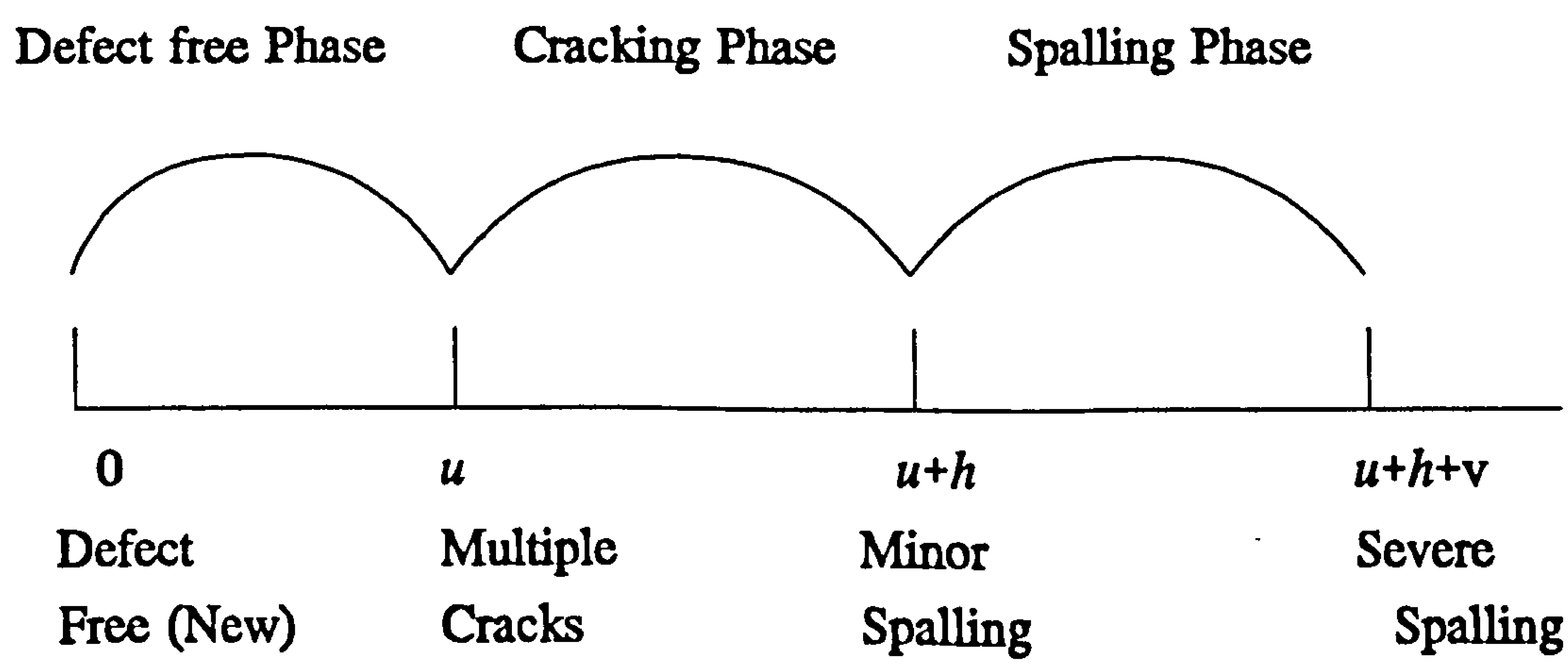

Fig. 1.7. The deterioration phases of a concrete component.

1.5 Discussion

This chapter has presented an overview of past and current developments in maintenance modelling. It is evident that delay time models and other new models are now increasingly being applied and tested through case studies. However, there is evidence of a deficiency of models for maintenance that takes into account the physical process, be it chemical, electrical or mechanical, that leads to a component failure. Geraerds (1972) regards the selection of statistical models for component and system failure behaviour as a subsection of a complex maintenance model of an organisation, that also takes into account such factors as maintenance planning and control, designs of systems, inventory problems and the feedback of results. Dekker et al (1995) consider also the planning of the maintenance activities for a group of components with different estimated optimal policies. It is shown that combining the maintenance activities, by delaying or bringing forward planned maintenance for some components with increased cost penalty, can reduce overall maintenance costs. This is due to the setup cost being shared. Hence, the necessary fusion between mathematical models and organizational planning and constraints are evolving in the maintenance field.

Over the past ten years, delay time modelling has undergone considerable development and is increasingly being accepted as an important concept for the real world modelling of maintenance of components and systems. There have been models which have touched on the concept, for example, Cox (1957, p.121) introduced a wear model such that a component can be defect free or enter a defective state prior to failure. This is equivalent to having a finite probability of zero delay time. An inspection model is presented to take into account this effect. Cozzoloni (1968) formulates a model whereby a system is assumed to have an unknown number of defects after a planned maintenance activity with each defect having a delay to cause failure. It is shown that the process of breakdowns will be a non-homogeneous Poisson process, as with the delay time system model. Butler (1979) classifies a component as functioning, functioning but defective, and failed. An inspection is also assumed to possibly increase the chance of failure due to the chance of observing a component in the defective state. However, a Markov model is formulated, thereby restricting the distribution of u and h to exponential. Lewis (1972) suggests an accumulation model whereby defective components which are detectable and do not cause failures, i.e having infinite delay time, are repaired at a failure. The expected repair time is then correlated with operating time. Jansen and Van der Duyn Schouten (1995) present a model for maintenance optimization of parallel production units. In the discussion of extensions to this model, it is assumed that the lifetime distribution of a component may be modelled by a convolution of two non-identical exponential distributions. The unit is described as 'good' in the first phase and 'doubtful' in the second phase. The condition can be tested and a decision can then be made on whether to overhaul. Hence, the model allows an extra decision option through the use of inspections. In addition, the maintenance cost could be further optimized when it is decided to take only the doubtful (or defective) units out of production for overhaul.

Statistical methods, testing and policy formulations are evidently being developed and formalised for the delay time model with the growing experience through applications. It is important that statistical tests are carried out in confirming all postulated assumptions, e.g the renewal assumption of a perfect inspection. A method to identify the optimal and feasible policy type (e.g periodic or age-based inspections) for a component or system also needs to be addressed. The following chapters of the thesis develop the theory of the delay time model, formalise statistical methods for revising, estimating and testing modelling parameters and apply the model to the deterioration of concrete structures.

Chapter 2

Delay Time Models for Maintenance of a Repairable System

2.1 Introduction

In this chapter, the type of repairable system under study and the delay time concept used to model maintenance will be defined. A non-homogeneous (or time-dependent) Poisson process model is used to describe the arrival process of breakdowns. The downtime and cost consequence due to a purely failure based maintenance policy are then modelled and discussed.

The effects of inspecting the system over time are considered. Models for downtime and cost are derived for a periodic based inspection policy. Initially, inspection will be assumed perfect and this requirement is then relaxed to include the case of imperfect inspections. Extensions to these models will be developed. Conditions involving the modelling parameters are derived for the existence and uniqueness of an optimal inspection based policy. Numerical examples will be provided. An alternative inspection policy based on inspecting a system after a specific period of use (or operation) will also be discussed.

Finally, conclusions are drawn and suggestions are given for further areas of research.

2.2 Delay Time Concept for a Repairable System

The repairable system under study can be a simple or complex electrical or mechanical plant where the objective is to model the cost or downtime consequences for various maintenance policies available to the engineer. The following assumptions are assumed to characterise the system being modelled:

(a) The system is modelled as a two state system where, over its service life, it can be either operating acceptably or down for necessary repair or planned maintenance.

(b) The system is comprised of many component parts or sections which are prone to become defective independently of each other when the system is operating.

(c) Defects which may have arisen in the system, deteriorate over an operating time. The deterioration may be due to operating conditions, such as vibration or environmental effects. However, the system can remain functioning in an acceptable manner until breakdown.

(d) The breakdown will be assumed to have been caused by one of the defects which has deteriorated sufficiently to affect the operating performance of the system as a whole (essentially a series type configuration of independent component parts.) Failure is assumed evident to the user and corrective maintenance is essential. Hence, the system would then cease operating due to the failed component. Once the failed component has been replaced or restored to a 'statistically new' condition, the system is assumed to be able to return to the operating state. However, other defective components can still be present if only corrective repair of the failed component is carried out.

The concept used to model the system described is *delay time analysis*, conceived by Christer (1973) and introduced into the context of building maintenance by Christer (1982) and then to industrial maintenance by Christer and Waller (1984b). A defective component is assumed to become defective at a point in time, u say, when the system is operating such that it is detectable by current inspection procedures. This epoch will be called the *initiation time* for the defect and a similar process applies to all other defects. The arrival process of defects in the system will be a superposition of u values across all the components. Due to a large number of component parts in the system, the superposition of defect arrivals in the system, u, will be assumed to be approximated by a Poisson process in the homogeneous (HPP) or non-homogeneous (NHPP) form, unaffected by breakdown repairs over time.

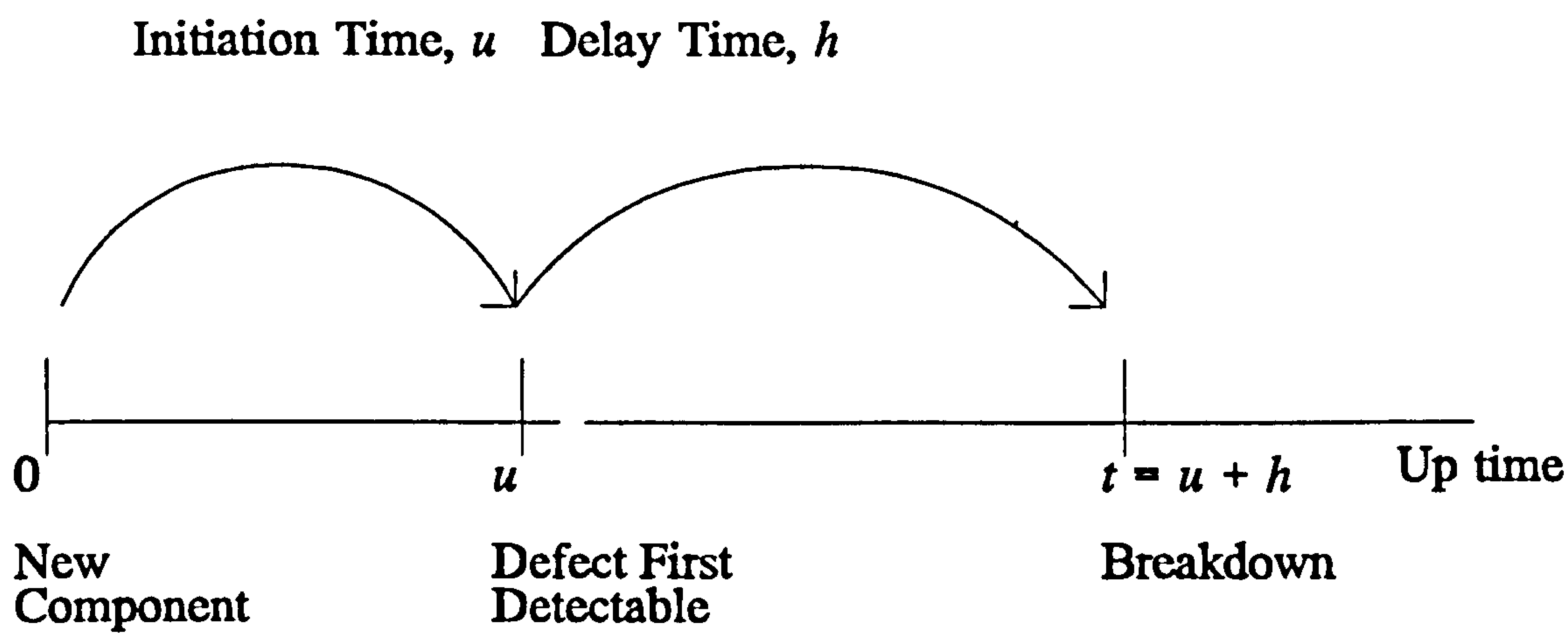

Fig. 2.1. The initiation time and delay time of a defect.

The *delay time*, h, of a defective component is the interval of operating time from the initiation time, u, to the point at which the defective component will cause a breakdown (or failure), at time $t = u + h$, see Fig. 2.1. Over the delay time period $(u, u + h)$, it is possible that the defective component in question can be repaired or replaced at a planned inspection thus preventing breakdown and consequences associated with it. The delay time of a component will be assumed random and independent of the initiation time. In general, components which are non-identical or not subjected to similar operating conditions would have delay times which are not identically distributed. However, it will be assumed that each random defect arrival, at time u, in the superposition process of defect arrivals across all components, will have a delay time, h, acceptably modelled as being identically distributed and independent of u.

The above assumptions are characteristic of those which have been successfully used in modelling inspection policies for building and industrial maintenance, see Christer and Waller (1982, 1984b), Chilcott and Christer (1991), Christer and Wang (1992), Baker and Wang (1992, 1993), Christer and Wang (1995), Christer et al (1995).

2.3 Type of Maintenance Activities

We, here, introduce the various types of maintenance activities that can be carried out on the system. Gits (1986) defines a set of maintenance activity options to be considered in the design stage of a maintenance concept. Three main types will be investigated :

a) *Breakdown repair*, a procedure to repair only the defective component which caused failure.

b) *Inspection*, an observation process or intervention where defects could be located and consequently repaired thus preventing breakdowns (if repairs are perfect). Inspection is then classed as a preventive maintenance activity.

c) *Overhaul* or *replacement*, a maintenance process where the whole system is assumed to return to a statistically 'as new' condition.

Inspections or overhauls are planned and can be carried out at planned points in time (e.g periodically), or could be initiated at breakdowns (1st or 2nd etc.) The type of policy for a particular system in question would depend on the failure characteristics of the system, such as defect rate and delay time distribution, for example, as well as the objectives of maintenance. We shall see, for example, that as may be expected, the quality of inspections and the cost and downtime levels of inspections and breakdown repairs contribute to determining the appropriate maintenance policy.

2.4 Failure Based Maintenance (FBM)

The models formulated in this section describe the process of occurrence of breakdowns and so provide a conceptual framework for when inspection models are considered. Our task is to derive models of repair, downtime and cost over time when performing corrective repair to components which cause breakdowns. It is assumed that breakdowns are dealt with by only repairing the defective component, that is FBM. The situation is demonstrated in Fig. 2.2, where it is assumed, at first, that breakdowns are instantaneously repaired. The following presents the set of initial assumptions concerning modelling parameters for the first technical system under study, and are based upon the previous section:

(a) At time 0, the system is in a new or 'as new' state, that is defect free.

(b) Defects arise within the system, independently, as a homogenous Poisson process (HPP) with rate parameter k, over time.

(c) Delay times, h, are independent of arrival time u and are described by the probability density function (p.d.f), $f(h)$ say, with finite mean μ.

(d) Those defects which cause breakdowns over time are repaired with negligible time at failure.

(e) A breakdown has a repair cost, which is independent of the defect arrival time, the delay time and the repair time. The repair cost is assumed t o have a mean c_b.

It is also convenient, here, to define $F(h)$ as the cumulative distribution function (c.d.f) of delay time h.

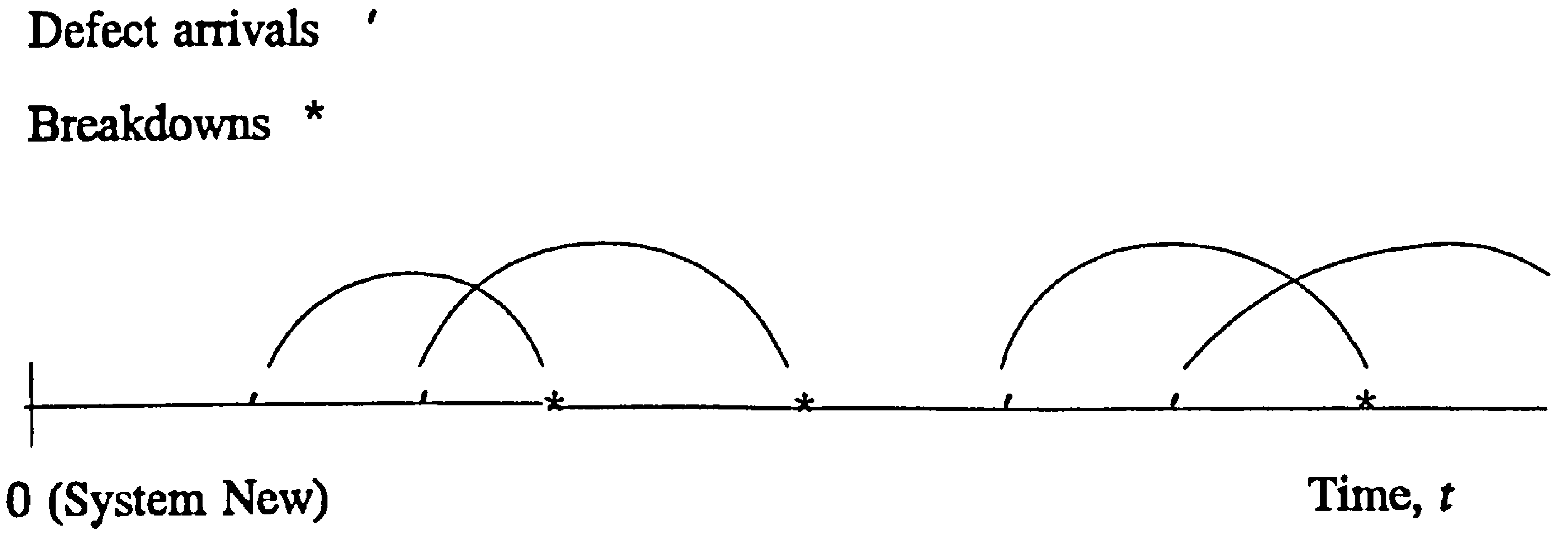

Fig. 2.2. An example of the breakdown arrival process.

2.4.1 Number of Breakdowns arising over Time

If we define t to be a time scale in units of time since the system was in the new or 'as new' state, then breakdowns would arise as a stochastic point process along the t axis. The failure arrival pattern is a superposition of failure arrivals across all the components within the system. It is clear that over an indefinite period of time and in the absence of inspection based maintenance, each defect can be expected to cause a breakdown event.

The process of breakdown arrivals can be described by defining, B_t say, to be the cumulative number of breakdowns after operating the system for time t and define $B(t) = E(B_t)$, the expected number of breakdowns in $(0, t)$. To derive, $B(t)$, we observe that a defect arising in an interval $(u, u + du)$, for $u < t$, where du is small, will cause a breakdown in interval $(0, t)$ if its delay time $h < t - u$, see again Fig.2.1. The probability that $h < t - u$ is $F(t - u)$. Clearly, the expected number of defects to arrive in $(u, u + du)$ is kdu, due to HPP defect arrivals. Hence the expected number of breakdowns caused by defects arriving in $(u, u + du)$ is,

$$F(t - u)kdu \quad . \tag{2.1}$$

The expected number of breakdowns, $B(t)$, can then be obtained by integrating over all possible $u < t$. Hence,

$$B(t) = k\int_{u=0}^{t} F(t - u)du = k\int_{h=0}^{t} F(h)dh \quad , \tag{2.2}$$

which is given in Christer and Waller (1984a). As expected, $B(t)$ is a monotonically increasing function, with its differential being $kF(t)$ when $F(t)$ is differentiable. Considering the integrand as a product of 1 and $F(h)$, the expression for $B(t)$ can be integrated by parts and re-written,

$$B(t) = k\left(tF(t) - \int_{h=0}^{t} hf(h)dh\right) \quad , \tag{2.3}$$

which can be seen to tend to the line $k(t - \mu)$ as t tends to infinity, where μ is the mean delay time. This suggests that a property of the model is that breakdown occurrences arise asymptotically as a HPP when defects arrivals arise as a HPP independently of the delay time distribution. It may be necessary and intuitive to obtain the distribution of B_t. An analogy of the stochastic process, B_t, arises in queuing theory. The process is an example of an $M/G/\infty$ queue, see Ross (1983) or Medhi (1983, p.321). In this situation customers arrive according to a Poisson process and each one is immediately served with a commonly distributed service time. Customers are analogous to defect arrivals and the service time is analogous to the delay time. A breakdown is analogous to a completed service for a customer. Ross (1983) shows that the arrival process of completed services is a non-homogeneous Poisson process (NHPP), see also Parzen (1962, p.125). Hence, B_t is Poisson distributed with mean $B(t)$, i.e,

$$p\{B_t = n\} = \frac{B(t)^n e^{-B(t)}}{n!} \quad . \tag{2.4}$$

Hence, for example, the reliability of the system can be determined, that is $e^{-B(t)}$, the probability no failures arise in interval $(0, t)$. The NHPP requires a time-dependent rate function, say $r(t)$, such that $r(t)dt$ represents the expected number of breakdowns in interval $(t, t + dt)$. The function $r(t)$ has been called arrival intensity, see Parzen (1962,p.125), or rate of occurrence of failures (ROCOF), see Ascher and Feingold (1984,p.4). For this process, $r(t)$ is given directly by equation (2.2) as,

$$r(t) = B'(t) = kF(t) \quad , \tag{2.5}$$

which can be seen to be a multiple k of the delay time c.d.f. As $t \to \infty$, we have from equation (2.5) that the rate of arrival of breakdowns tends to the defect arrival rate, k. Hence for this system the breakdown process tends asymptotically to an HPP, identical to the process of defect arrivals, that is the system performance would tend to a limit. It follows that in the limit the inter-arrival time between breakdowns would tend to the exponential distribution of mean $1/k$, which will prove to be an important property in asymptotic considerations of cost and downtime. Properties of the inter-arrival times of breakdowns for a general NHPP, can be found in Parzen (1962, p.138).

2.4.2 Modelling Cost

The repair cost of a breakdown can include factors such as manpower, materials, lost production and environmental damage. Independence is assumed between the number of breakdowns occurring in time t and the cost of each repair, with an a^sumed mean cost per repair of c_b. Letting, c_n say, be the repair cost of the n'th failure, then the cumulative repair cost, say C_t, is given by,

$$C_t = \begin{cases} \sum_{n=1}^{B_t} c_n & for \quad B_t \geq 1 \\ 0 & for \quad B_t = 0 \end{cases} , \tag{2.6}$$

which is a cumulative Poisson process when the repair costs, c_n, are independent and identically distributed, see Cox (1957). Defining $C(t)$ as the total expected cost after operating the system for time t we have,

$$C(t) = E(C_t) = c_b B(t) , \tag{2.7}$$

which is given in Christer and Waller (1984a), though derived slightly differently there. The expected asymptotic repair cost per unit time over all time, c_∞ say, is then given by,

$$c_\infty = \lim_{t \to \infty} \left(\frac{C(t)}{t} \right) = k c_b . \tag{2.8}$$

This is an expected result, since k defects are expected to arrive in unit time, each with an expected subsequent repair cost, c_b, incurred when the defect ultimately causes a breakdown.

2.4.3 Modelling Downtime

Here we relax assumption (d) so that repair times are finite and our interest is in modelling the effects of downtime. It will be assumed that each breakdown is repaired with a repair time independent of each other, and with an identically distributed repair time with mean, d_b say. The situation is depicted in Fig.2.3. When the system is in the down state, it will be assumed that new defects do not arise within a breakdown repair time and that other defects are effectively frozen, that is deterioration (i.e the expiry of delay time) does not take place for other defects which may be present in the system at the time of failure. When d_b is small, the repair downtime can be thought of as being a repair cost, that is cost is measured in units of downtime. Hence, the expected cumulative downtime over real time t can be approximated by, $D(t)$ say, given by,

$$D(t) = d_b B(t) , \tag{2.9}$$

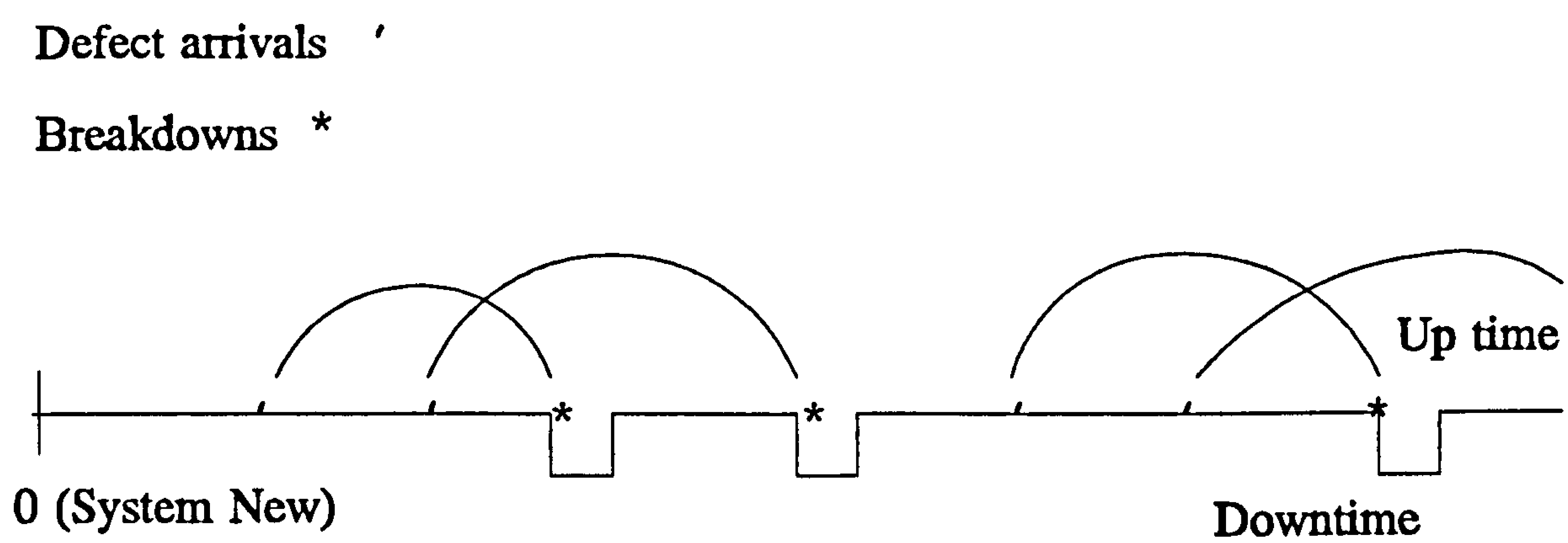

Fig. 2.3. An example of breakdown arrivals with repair times.

and the approximate downtime per unit time, d_∞, over an indefinite time period is given by,

$$d_\infty = k d_b \ . \tag{2.10}$$

It is evident that the condition, $k.d_b \leq 1$, needs to be satisfied in the absence of inspection intervention because the expected downtime per unit time must lie in the interval (0, 1).

For a system with parameters such that $k.d_b > 1$, or when d_b is not small, the appropriate theoretical stochastic model for expected cumulative downtime would need to be derived. This is outlined in the appendix. Chillcot and Christer (1991) propose an iteration method to model the expected downtime per unit time when d_b is not small. However, for the case of FBM over an infinite time horizon, the theoretical expected downtime per unit time, d_∞, and cost per unit time, c_∞, can be easily derived.

For the case where we are interested in modelling downtime, d_∞ will essentially be the long term proportion of time the system is in the down state. In Section 2.4.1, it has been stated that the time between breakdowns, when breakdowns are instantaneously repaired, tend to the exponential distribution, mean $1/k$. When a period of downtime occurs, the stochastic process of breakdown arrivals effectively stops for the duration. Hence, the operating time between breakdowns would tend to the exponential distribution in the long term. We assume here that repair times are independent and identically distributed, with mean d_b. Hence in the long term, the FBM process can be thought of as being an alternating renewal process of up and downtimes, see Cox (1957,p.80). The long term proportion of time spent in the down state, d_∞, is then given by,

$$d_\infty = \frac{E(\text{downtime})}{E(\text{uptime}) + E(\text{downtime})} = \frac{kd_b}{1 + kd_b} \quad . \tag{2.11}$$

which, as required, lies in the interval (0, 1) for any positive values of k and d_b. The result is also given in Smith (1985, p.27) for systems which have exponential time between failure and arbitrary downtime distribution. It is worth noting that d_∞ is independent of the delay time distribution or parameters from it. However, the speed at which the limit is achieved will depend upon the delay time distribution form. Also, it can be seen that the approximate model needs the condition that kd_b is small compared to unity.

In the long term, performing FBM, the up-down cycle length, that is the uptime before repair plus downtime of repair, has the expected value $1/k + d_b$. An expected cost, c_b, would be incurred over this interval. Therefore, the theoretical asymptotic cost per unit time, c_∞, can be derived by considering the process as a renewal reward process. The value c_∞ is then the ratio of expected cost for one repair, c_b, to the expected cycle length $1/k + d_b$, giving,

$$c_\infty = \frac{kc_b}{1 + kd_b} \quad . \tag{2.12}$$

2.4.4 Non-Homogeneous Defect Arrivals

We can extend the model to a more general technical system where defects may arrive as a NHPP, with properties given in Section 1.3.1. The NHPP requires a rate function $g(u)$ such that $g(u)du$ represents the expected number of defects arriving in the small interval $(u, u + du)$, when the system has been operating for time u, see Christer and Waller (1984a), Christer and Wang (1995). Hence, following the analysis in Section 2.3, the expected number of breakdowns, $B(t)$, will take on the convoluted form,

$$B(t) = \int_{u=0}^{t} g(u) F(t-u) du \quad , \tag{2.13}$$

given in Christer and Waller (1984a). The occurrence of failures, once again, will be NHPP due to the underlying process of defect arrivals and the independence between delay times and defect arrivals, see Ross (1983). The ROCOF is given by,

$$r(t) = B'(t) = \int_{x=0}^{t} g(x) f(t - x) dx \quad . \tag{2.14}$$

The number of breakdowns, B_t, to arrive in operating time t, is Poisson distributed, with mean $B(t)$, due to breakdown arrivals arising as an NHPP. The models for cumulative expected downtime and cost, derived in the chapter, will have identical forms for NHPP defect arrivals, but using the revised $B(t)$, given above. However, the asymptotic behaviour of breakdown occurrences and consequent downtime and cost per unit time, will depend on the defect arrival rate $g(u)$ and delay time p.d.f $f(h)$.

Many of the breakdown maintenance models presented above, or their variants will be used in the subsequent sections and chapters on inspection modelling. These models are not management models in the sense that they have no decision variable to control the failure pattern of the system. What the models do provide is estimates and forecasts of the stochastic consequence of a particular practice, which has value to management. It has been shown that modelling the failure characteristics of a repairable system can be achieved using the technique of delay time. Our objective, next, is to model the consequences of the inspection option, that is build a management model.

2.5 Periodic Based Inspection (PBI)

In this section, it is proposed to construct models assuming a policy of inspection on a periodical basis. The models will aim to predict cost and downtime consequences for alternative inspection periods. Inspection will at first be considered perfect and then relaxed to include the case of imperfect inspections.

A *perfect inspection* is a planned maintenance stoppage of the system whereby all defective components present in the system are located and corrected to a statistically 'as new' condition within the time allocated for inspection. The time constraint is imposed initially for simplifying convenience and will later be relaxed. It will be assumed that a perfect inspection returns the entire system to a 'statistically-as-new' condition, i.e it is analogous to a renewal, replacement or overhaul. An inspection is also assumed not to generate any defects. The following additional assumptions to the FBM set will be applied, see Christer and Waller (1984a):

(a) There is a constant time T between successive inspections irrespective of the cumulative breakdown repair time within each inspection interval (i.e T will not always be the operating time between inspections).

(b) Breakdowns impose a small amount of downtime compared to the inspection period T, with expected value d_b, $d_b \ll T$.

(c) The downtime required for each inspection is a constant d_I.

(d) The expected cost of an inspection is c_I.

(e) The expected cost of a repair to a defective component at an inspection is c_d ($< c_b$, the expected cost of a breakdown).

(f) If the system is down for breakdown repair, and is due for inspection, the component which has caused the breakdown has the repair completed within the time allocated for inspection, d_I.

The effect of inspection is shown by example in Fig. 2.4, where it is evident that perfect inspections reduce the number of breakdowns. However, over frequent inspections will incur an increased downtime and cost penalty. Hence, a compromise needs to be sought to identify the most appropriate inspection interval. The assumptions of the model were influenced by the research at a canning line plant, Christer and Waller (1984b), whereby daily inspections were planned, breakdowns were repaired within the inspection period and defective components were rectified at inspections. Management adopted the predicted optimum policy which was interestingly close to the current inspection practice.

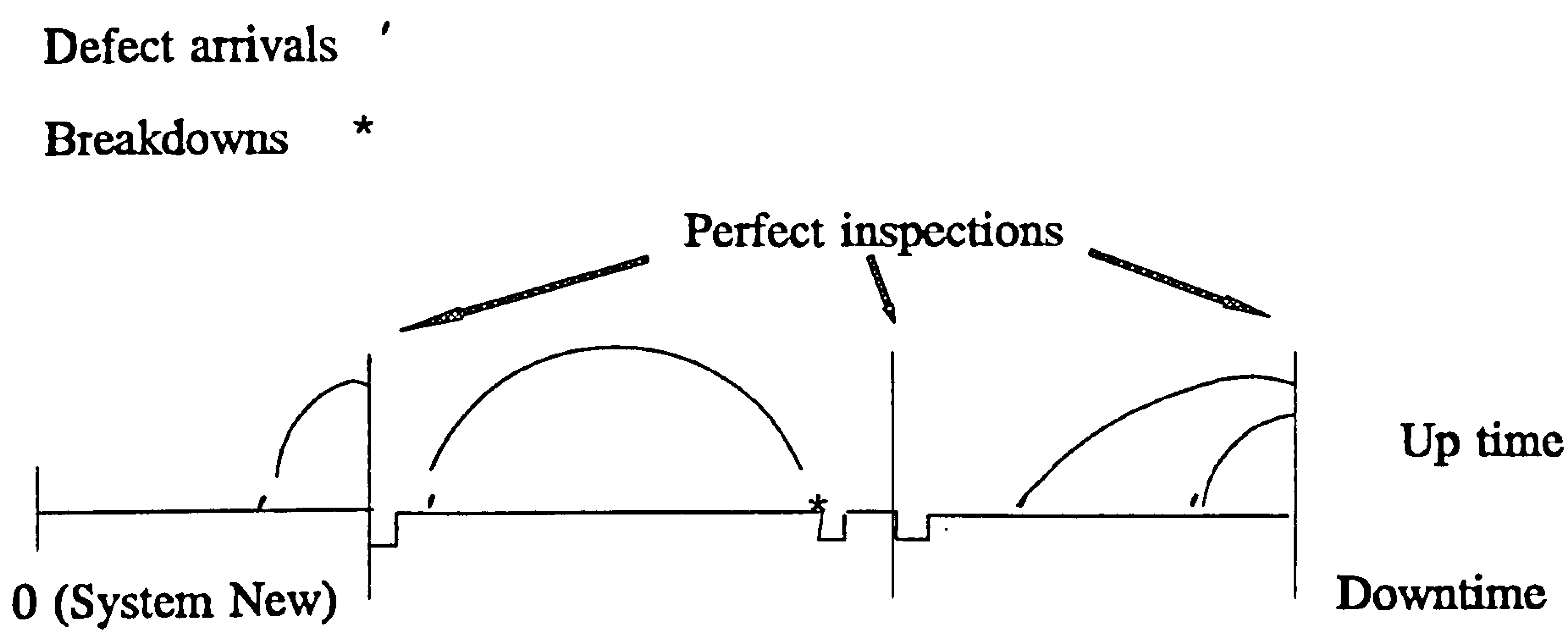

Fig. 2.4. The reduction of the number of breakdowns due to perfect inspections.

2.5.1 The Probability a Defect causes a Breakdown

Under the inspection policy proposed, it is clear that defects which arise will be either identified at inspections or cause breakdowns. Defects are assumed to arise independently of each other. Hence, we can define a probability, $b(T)$ say, that a defect will cause a breakdown given a defect has arisen in an inspection interval $(0, T)$, see Christer and Waller (1984a). This is a basic function in inspection modelling. Defects arriving as an HPP will be considered first.

In the case of instantaneous repairs $\{d_b = 0\}$, the probability, $b(T)$, can be calculated by the ratio of the expected number of breakdowns, $B(T)$, to the expected number of defects, kT, in time T. Hence,

$$b(T) = \frac{1}{T}\int_{x=0}^{T} F(x)\,dx \quad . \tag{2.15}$$

As $F(x) \leq F(T)$ for $x < T$, it can be seen that $b(T) \leq F(T)$. The probability, $b(T)$, can more formally be obtained by considering the initiation time, u, given only one defect has arisen in $(0, T)$. The distribution of u will be uniform over interval $(0, T)$, due to HPP defect arrivals. If the defect has a delay time in the small interval $(h, h + dh)$, an event occurring with probability $f(h)dh$, then it would cause a breakdown if the initiation time satisfies $u < T - h$, which has probability $(T - h)/T$. Hence, combining these probabilities and summing over all possible u, we have,

$$b(T) = \frac{1}{T}\int_{h=0}^{T}(T - h)f(h)dh \quad . \tag{2.16}$$

which was originally formulated in Christer and Waller (1984a). The functions (2.15) and (2.16) provide approximations for the probability a defect leads to a breakdown, when $d_b > 0$ and $d_b \ll T$, and can be shown to be equivalent by integrating equation (2.16) by parts. The function is an approximation due to the stochastic accumulation of breakdown repair time over each interval $(0, T)$. The characteristic shape of the function $b(T)$ is shown in Fig. 2.5. It can be seen to increase from zero, monotonically, to 1 when T tends to infinity, as expected. Due to $F(0) = 0$ and $b(T) \le F(T)$, then $b(T) \to 0$ when $T \to 0$. Essentially, the $T = 0$ implies that defects are immediately detected when arisen and subsequently repaired, which implies no failure occurrences and so $b(0) = 0$. The policy of FBM described in Section 2.4 is equivalent to $T = \infty$. The differential of $b(T)$ is given by,

$$b'(T) = \frac{1}{T^2}\int_{x=0}^{T}\{F(T) - F(x)\}dx \quad , \tag{2.17}$$

which is non-negative, implying the function is non-decreasing.

2.5.2 Modelling Downtime

The total expected downtime incurred over time interval $(0, T)$ can be approximated by the function (2.9), if assumption (b) holds, namely $d_b \ll T$, and the expected number of breakdowns $B(T)$ is small. These conditions are required due to the random loss of operating time over each inspection interval. Due to inspections being perfect and the cycle length constant at $T + d_I$, we have that the approximate expected downtime per unit, $d(T)$, over each cycle $(0, T + d_I)$, is given by,

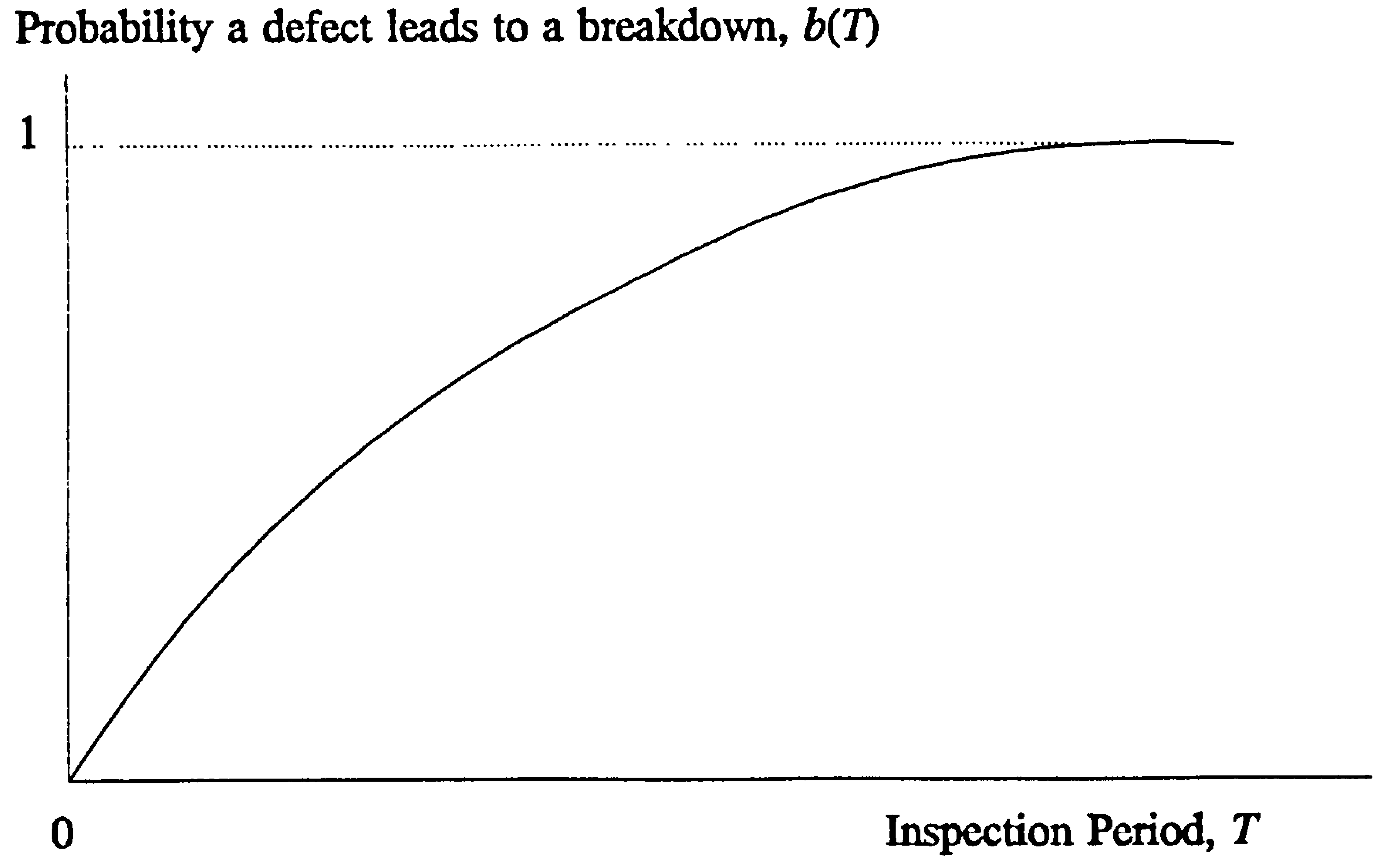

Fig. 2.5. The probability, $b(T)$, a defect arises as a breakdown.

$$d(T) = \frac{D(T) + d_I}{T + d_I} = \frac{d_b B(T) + d_I}{T + d_I} = \frac{kTd_b b(T) + d_I}{T + d_I} , \qquad (2.18)$$

which was originally formulated in Christer and Waller (1984a). The asymptotic value of function (2.18), as $T \rightarrow \infty$, can be seen to be kd_b since $b(T)$ tends to unity. In comparing this value with the limit for the actual asymptotic downtime per unit time (2.11), it can be seen that an additional condition for this model to be asymptotically valid is that kd_b should be small so that the value $k.d_b/(1 + k.d_b)$ is close to kd_b. If the condition is not satisfied then the optimal inspection period may be in error and either the stochastic model of downtime, given in the appendix, or the downtime model for an alternative inspection policy (Section 2.6) would need to be derived.

The value of T which minimizes the objective function (2.18) can be found graphically, or by numerical search or when analytically valid by differentiating with respect to the decision variable T and setting the result to zero to obtain the equation,

$$d_I F(T) + \int_{h=0}^{T} h f(h)\, dh - \frac{d_I}{kd_b} = 0 \ . \qquad (2.19)$$

The differential of the L.H.S of equation (2.19) is $(T + d_I)f(T)$, which is positive for $T > 0$ and $f(T) > 0$. Hence, it follows that the L.H.S is a monotonically increasing function. Thus, a finite solution T^*, if it exists, will be unique if $f(T) > 0$.

Inspection will not always be an optimal choice and a finite solution to equation (2.19) will not always arise since its existence is dependent on the selected parameters. The function on the L.H.S of equation (2.19) is negative at $T = 0$. Hence, for a solution, T^*, to this equation, a necessary and sufficient condition for a unique solution is that the L.H.S must be positive as $T \rightarrow \infty$. It then follows that for a finite solution to equation (2.19) to exist the following condition must hold,

$$\mu + d_I\left(1 - \frac{1}{kd_b}\right) > 0 \ . \tag{2.20}$$

If this condition is not satisfied by the estimated parameters, then FBM or another form of planned maintenance scheme will provide the lowest expected downtime per unit time for the system in question. When considering PBI, $d(T)$ would be a monotonically decreasing function with a limit of $k.d_b$ as $T \rightarrow \infty$. If condition (2.20) is satisfied then the solution inspection interval, T^*, obtained from equation (2.19) is the recommendation for optimizing downtime assuming a policy of inspection on a regular periodical basis.

2.5.3 Modelling Cost

We now turn our attention to the task of modelling cost under the assumptions given in Section 2.5. In order to model costs either cumulative or on an expected time basis, we need to calculate the expected number of defects which can be located and repaired at each inspection. Denote A_T as being the number of defect arrivals in $(0, T)$ and S_T as the number of defects in the system, which have arisen but not yet caused a breakdown, and therefore identified by perfect inspection at time T. Also let B_T be the number of breakdowns in each inspection interval. Therefore, as each defect would either be in the system or have caused a breakdown,

$$A_T = B_T + S_T \ \Rightarrow \ E(A_T) = E(B_T) + E(S_T) \ . \tag{2.21}$$

Due to HPP defect arrivals, the expected number of defects to arrive in time T,

$E(A_T) = kT$. Thus the mean number of defects in the system at time T since last inspection, $S(T) = E(S_T)$ say, is given by,

$$S(T) = kT - B(T) = kT(1 - b(T)) = k\int_{x=0}^{T}(1 - F(x))dx \quad , \tag{2.22}$$

which, using expression (2.3), tends to the product of the defect arrival rate and the delay time mean, $k\mu$, as T increases, . This is analogous to the immigration-death population model (or $M/M/\infty$ queue) where k is the immigration rate and $1/\mu$ is the death rate, see Cox and Miller (1965, p.168). The limit, $k\mu$, is the expected size of the population (in this case defects) as T increases to infinity. The limit is also the expected number of defects to be found on first inspection when switching from a contingency breakdown policy, i.e FBM, to a perfect inspection policy, shown in Christer (1982).

Medhi (1982, p.321) shows that in the related queuing situation, S_T is Poisson distributed with mean given by equation (2.22) when breakdown repair times are negligible. Ross (1983) shows that S_T and B_T are independent. These properties will aid in model parameter estimating and testing the fit of the model.

The expected total cost for breakdowns, $C(T) = c_b B(T)$, would here apply over each inspection cycle, $T + d_I$. The expected cost of repairs carried out at inspections is $c_d S(T)$ where as before $S(T)$ is the expected number of defects detected at T. It is assumed that the cost of an inspection is c_I. Hence, if we define $c(T)$ to be the expected cost per unit time over each cycle, $(0, T + d_I)$, then for the case of effectively instantaneous repairs,

$$c(T) = \frac{c_I + c_b B(T) + c_d S(T)}{T + d_I} \quad . \tag{2.23}$$

Clearly, the model $c(T)$, (2.23), would serve as an approximation for the case of non-instantaneous repairs, provided assumptions (b) and (f) hold, namely $d_b > 0$ and $d_b \ll T$. The approximation improves as the approximate downtime of breakdown repairs per unit time over interval $(0, T)$, namely $d_b B(T)/T$, decreases.

The asymptotic value of function (2.23), that is kc_b, can be found in the FBM case by noting that $B(T)/T$ tends to k and $S(T)/T$ tends to 0, as $T \rightarrow \infty$. In order to investigate conditions for a unique finite optimal solution, the value of T, if it exists, which minimizes this function can be found by differentiating, simplifying and setting to zero the equation,

$$d_I F(T) + \int_{h=0}^{T} h f(h) dh - \frac{(c_I - c_d k d_I)}{k(c_b - c_d)} = 0 \ . \qquad (2.24)$$

The differential of the L.H.S of equation (2.24) is $d_I f(T) + T f(T)$, implying the function is monotonically increasing , and that an optimum inspection period will be unique.

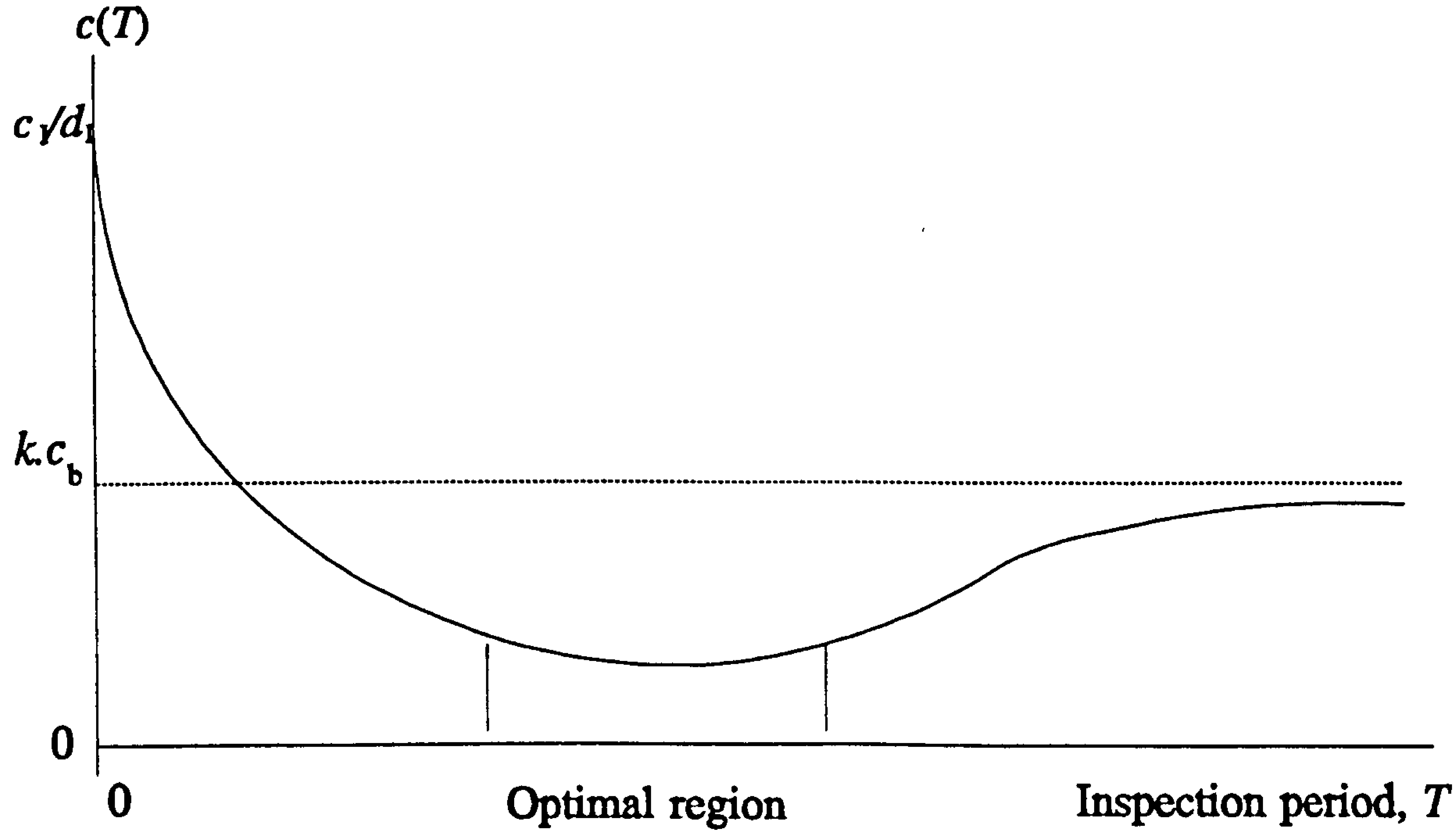

Fig. 2.6. The expected cost per unit time, $c(T)$, against T.

A similar shape of curve to the downtime model is seen to exist for the $c(T)$ model, see Fig. 2.6. In comparing equation (2.24) with equation (2.19), we see they differ only in the constant term. Hence, a unique solution to equation (2.24) exists if the L.H.S is negative at $T = 0$ and positive at $T = \infty$. At $T = 0$, we evidently require,

$$c_I - k c_d d_I > 0 \ , \qquad (2.25)$$

since we expect $c_b > c_d$. The necessary condition as $T \to \infty$ implies,

$$d_l + \mu - \frac{(c_l - kc_d d_l)}{k(c_b - c_d)} > 0 \ . \tag{2.26}$$

2.5.4 Perfect Inspection Models with Non-Homogeneous Defect Arrivals

Here, we now relax the HPP assumption of constant rate defect arrivals and let $g(u)$ be the instantaneous rate of defect arrivals at time u from the 'as new' condition. The defects are assumed arise in time as a non-homogeneous Poisson process (NHPP). Therefore the total expected number, $K(T)$ say, of defects to arrive in interval (0, T) is given by,

$$K(T) = \int_{u=0}^{T} g(u)\,du \ . \tag{2.27}$$

The arrival time, u, assumed for a random defect, given to arise over interval (0, T) has the p.d.f, $q(u;\ T)$ say, given by,

$$q(u;T) = \frac{g(u)}{K(T)} \qquad for \quad 0 < u < T \ , \tag{2.28}$$

which is a relationship shown by Ascher and Feingold (1984, p.32), and given in Christer and Redmond (1990). Essentially, if, B say, failures have been observed in time interval (0, T), then the B ordered breakdown arrival times, $y_1, y_2, ..., y_B$ say, are the order statistics of a sample, size B, taken from the p.d.f (2.28).

In this case, the probability that a defect arises as a failure now takes on the form,

$$b(T) = \int_{u=0}^{T} q(u;T)F(T-u)\,du = \int_{h=0}^{T} Q(T-h;T)f(h)\,dh \ , \tag{2.29}$$

where $Q(u;\ T) = K(u)/K(T)$ is the c.d.f of u. The form of the models for downtime and cost have identical structures (2.18) and (2.23) respectively, but now having $B(T)=K(T)b(T)$ and $S(T) = K(T) - B(T) = K(T)(1 - b(T))$, given in Christer and Waller (1984a). Due to $Q(u;\ T) \leq 1$, it can be seen that $b(T) \leq F(T)$ for the case of NHPP defect arrivals as well as HPP defect arrivals. It is noted that in the special case of HPP when

$q(u; T)$ is uniformly distributed over $(0, T)$, equation (2.29) reduces to equation (2.16) as required.

2.5.5 Imperfect Inspection Models with Homogeneous Defect Arrivals

It is common that inspections may not reveal all defects present in a system, especially for large complex systems. The quality of inspections depend on inspection practices imposed includes inspection techniques used, inspection training and the nature of any supervision. A method to model imperfect inspections is to allow a probability β for each defect present at an inspection to be detected. Under assumptions given in Section 2.5, Christer and Waller (1984a) have shown that, $b(T; \beta)$, the probability that a defect (generated by a HPP) arises as a breakdown, is given by,

$$b(T;\beta) = 1 - \frac{1}{T}\int_{u=0}^{T} \sum_{n=1}^{\infty} \beta(1-\beta)^{n-1}(1 - F(nT-u))du \ . \qquad (2.30)$$

The probability is calculated under the assumption that an indefinite number of inspections will be carried out so that a defect will eventually either arise as either a breakdown or be identified and repaired at inspection. Since the other aspects of the model (2.23) are otherwise the same, the function can be used in the downtime and cost models previously derived.

The function, $b(T; \beta)$ was derived by considering a single defect assumed to arrive in interval $(u, u + du)$ over an inspection interval $(0, T)$, an event with probability du/T. The probability that the defect will be detected on the n'th inspection after defect arrival is equal to $\beta(1-\beta)^{n-1}\{1 - F(nT - u)\}$, i.e the event that delay time $h > nT - u$, the defect is detected on n'th inspection and not detected on $n - 1$ previous inspections. Then, to form $b(T; \beta)$, the two probabilities above are formed for general n, summed over all possible values of n, and integrated over all possible u ε $(0, T)$ and the complement taken.

Clearly as expected, the function, $b(T; \beta)$ monotonically increases from 0 to 1 as T increases from 0 to ∞, in the case of perfect inspections. For a given inspection period T, $b(T; \beta)$ would intuitively decrease from 1 to $b(T; 1)$ as β increases from 0 to 1.

However, this property cannot easily be extracted simply by partially differentiating function (2.30) w.r.t β.

It is possible to estimate the expected downtime and cost per unit time over a finite time period from the 'statistically-as-new' condition. This can be achieved by deriving the expected number of breakdowns, $B_n(T)$ say, arriving in the n'th inspection interval from new, and the expected number of defects detected at the n'th inspection, $S_n(T)$ say. As will be seen, the probability $b(T; \beta)$ can also be derived from the function $B_n(T)$ by letting $n \to \infty$. The property that the partial differential of $b(T; \beta)$ w.r.t β is negative can also be shown. The functions $B_n(T)$ and $S_n(T)$ would also prove to be valuable in the estimation of modelling parameters.

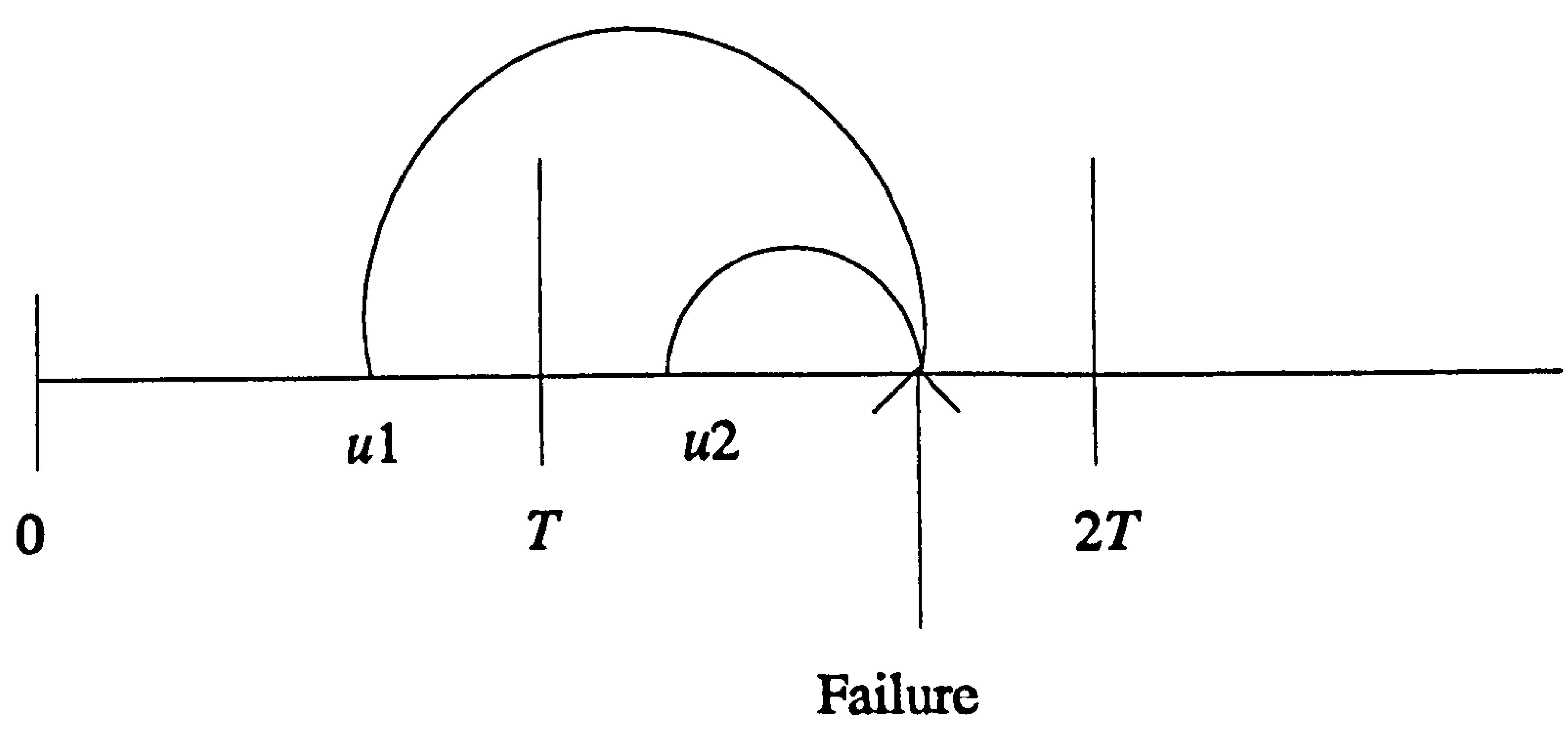

Fig. 2.7. An example of a breakdown occurrence under imperfect inspections.

It is clear that $B_1(T) = B(T)$, the expected number of breakdowns in the interval (0, T) from new, function (2.2), and that $S_1(T) = \beta S(T)$, where $S(T)$ is given by function (2.22). For the second interval, a failure can arise from either a defect arisen in the first interval or a defect arisen in the second interval, see Fig. 2.7. The expected number of breakdowns arising from defects which arrive in the second interval is again $B_1(T)$, and the expected number of defects detected at the inspection is $\beta S_1(T)$ from such defect arrivals. For a defect which arises in the first interval say at time u, we require its delay time, h, in interval $(T - u, 2T - u)$ and to be not detected at T, in order to cause a breakdown in the second inspection interval $(T, 2T)$. The probability of this event is $(1 - \beta)\{F(2T - u) - F(T - u)\}$ and the expected number of defect arrivals in $(u, u + du)$ is kdu. Therefore integrating the combined probability for all $u < T$ and adding $B(T)$ for the expected number of breakdowns arising from defect arrivals in the second interval, it follows that, using expression (2.2) for $B(T)$, $B_2(T)$ is given by,

$$\begin{aligned} B_2(T) &= B(T) + k(1-\beta)\int_{u=0}^{T}\{F(2T-u)-F(T-u)\}du \\ &= k\sum_{i=1}^{2}(1-\beta)^{i-1}\int_{u=0}^{T}\{F(iT-u) - F((i-1)T-u)\}du \quad , \end{aligned} \tag{2.31}$$

where $F(h) = 0$ for $h < 0$. To formulate $B_n(T)$, the expected number of defects breakdowns arriving in the n'th inspection interval due to defects arisen in the interval $(0, nT)$, we notice that the expected number of breakdowns in the n'th interval attributed to defects arriving in inspection intervals other than the first is $B_{n-1}(T)$. Therefore, for a defect arising in the first interval at time u we require its delay time, h, to be interval $(nT - u, [n - 1]T - u)$ and not to be detected at $n - 1$ inspections to cause a failure in the n'th inspection interval. Using previous analysis, it follows that $B_n(T)$ is given by,

$$\begin{aligned} B_n(T) &= B_{n-1}(T) + k(1-\beta)^{n-1}\int_{u=0}^{T}\{F(nT-u)-F((n-1)T-u)\}du \\ &= k\sum_{i=1}^{n}(1-\beta)^{i-1}\int_{u=0}^{T}\{F(iT-u)-F((i-1)T-u)\}du \qquad (2.32) \\ &= k\left\{\beta\sum_{i=1}^{n-1}(1-\beta)^{i-1}\int_{u=0}^{T}F(iT-u)du + (1-\beta)^{n-1}\int_{u=0}^{T}F(nT-u)du\right\} \quad , \end{aligned}$$

by expanding. It can be seen from the last formulation, that as $n \rightarrow \infty$ and using the geometric result that $\sum_{n=1}^{\infty}(1 - \beta)^{n-1} = 1/\beta$,then $B_n(T)/kT \rightarrow b(T; \beta)$, function (2.30), as

expected. Hence, using the second expression of equation (2.32), as another alternative form for $b(T;\beta)$, it can be verified that,

$$b(T;\beta) = \frac{B_\infty(T)}{kT} \Rightarrow \frac{\partial b(T;\beta)}{\partial\beta} < 0 \ . \tag{2.33}$$

This again is as expected, and indicates that the more perfect an inspection, the less chance that a defect arrival will cause a breakdown.

Next, consider the derivation of $S_n(T)$. Given a defect arrives at time u in the first inspection interval, then it would be detected on the n'th inspection if its delay time satisfies $h > nT - u$, it is not detected at the $n - 1$ previous inspections and it is detected on the n'th inspection. The probability of this event is $\beta(1 - \beta)^{n-1}\{1 - F(nT - u)\}$. Following the analysis for breakdown arrivals, we have,

$$\begin{aligned} S_n(T) &= S_{n-1}(T) + k\beta(1-\beta)^{n-1}\int_{u=0}^{T}\{1-F(nT-u)\}du \\ &= k\beta\sum_{i=1}^{n}(1-\beta)^{i-1}\int_{u=0}^{T}\{1-F(iT-u)\}du \ . \end{aligned} \tag{2.34}$$

It can be seen that, as expected, the limit of $S_n(T)/kT$, as $n \to \infty$, equals $1 - b(T;\beta)$, the asymptotic probability that a defect is detected at an inspection. Having derived these models, we are in a position to formulate models of cost or downtime over a finite time horizon when $\beta \neq 1$. E.g, the expected cost per unit time, say $c(T, m)$, over the time interval for m inspections of interval T and inspection downtime d_I, is given by,

$$c(T,m) = \frac{mc_I + \sum_{n=1}^{m}\{c_b B_n(T) + c_d S_n(T)\}}{m(T + d_I)} \ . \tag{2.35}$$

If interest was in reducing cost of a new system over a planned finite service time, say P, then the option exists to optimize over values for m such that $P = m(T + d_I)$.

2.6 Use Based Inspection (UBI)

This type of policy can provide models for downtime and cost, which may give lower expected downtime and cost per unit time than performing PBI. It is a policy whereby inspection is undertaken after a prescribed amount of operating time, t say, has elapsed, i.e t is the decision variable. The number of breakdowns arising before each inspection can vary in number as in the case of PBI. The assumption, $d_b \ll t$, can also be relaxed so that models of downtime and cost will be stochastic. For example, UBI could be applied to vehicles where the mileage is being recorded.

Consider a model for downtime when inspections are perfect. If the inspections are carried out indefinitely, the total real time between two consecutive inspections, T, will be a random variable, i.e $T = t + D_t$, where D_t, is the cumulative breakdown repair completion time since last inspection when the system has operated for a total time t. The theoretical expectation of D_t , $E(D_t) = d_b B(t)$. If inspections take a constant time d_I to perform, then the expected downtime per unit time, $d(t)$, over an infinite time horizon for the case of perfect inspections, would be given by,

$$d(t) = \frac{d_b B(t) + d_I}{t + d_b B(t) + d_I} . \tag{2.36}$$

The asymptotic value is $kd_b/(1 + kd_b)$ as $t \to \infty$, which is the same value (2.11) as obtained in Section 2.6 for the HPP defect arrival case. This confirms the theoretical asymptotic expected downtime per unit time expression for the FBM policy.

In comparing this model with the PBI model (2.18), it is noted that the denominator in the expression (2.36) is larger. Therefore this model will lie beneath the periodic perfect inspection model when plotted on the same axes. Hence, it follows that if an optimal solution exists for expression (2.36), then it would be associated with a lower expected downtime per unit time consequence compared with the optimal solution obtained in the periodic case. However it must be remembered that the PBI model is under the assumption, $d_b \ll T$, in which case, providing the inequality is valid, the difference in downtime per unit time may only be slight.

Turning our attention to cost consequences of the UBI policy, the expected cost per unit time model measured over an infinite time horizon, would take on the form,
where, as before, $S(t) = kt - B(t)$, the expected number of defects to be found at inspection after operating time t. In the case of HPP defect arrivals, the asymptotic value

$$c(t) = \frac{c_1 + c_b B(t) + c_d S(t)}{t + d_b B(t) + d_1} \quad , \tag{2.37}$$

of $c(t)$ is $kc_b/(1 + kd_b)$, formulated in Section 2.4.3, which is the same value (2.12) as the FBM policy, obtained using renewal reward theory.

The models can also be extended to imperfect inspections by replacing $B(t)$ by $k.t.b(t;\, \beta)$, where $b(t;\, \beta)$ is given by function (2.30).

2.7 Exponential Delay Time Model

In a number of cases, it has been found that delay times can be acceptably modelled by the exponential distribution with mean μ and the process of defects arriving by a HPP of rate, k say. In this case, the form of $b(T)$ is given by,

$$b(T) = 1 - \frac{\mu}{T}(1 - e^{-T/\mu}) \quad . \tag{2.38}$$

As can be seen, the probability $b(T)$ is independent of the value of k, since the distribution of one defect given to arise over an interval $(0, T)$ is uniform. Extending the result to the case of imperfect inspection, a closed form for the probability, $b(T;\, \beta)$, can be calculated using properties of the geometric series, namely

$$b(T;\beta) = 1 - \frac{\beta\mu(1 - e^{-T/\mu})}{T(1 - (1 - \beta)e^{-T/\mu})} \quad . \tag{2.39}$$

As a numerical example of $b(T)$, the case with $\mu = 5$, $\beta = 1$ and $\beta = 0.7$ is given in Fig.2.8. A numerical example of the cost consequences model (2.23) is given in Fig.2.9 for the PBI policy. The modelling parameters selected are:

$c_b = 0.5$ (Cost of a breakdown repair),

$c_1 = 0.3$ (Cost of inspection),

$c_d = 0.2$ (Cost of a defect repair),

$d_1 = 0.5$ (Downtime of inspection),

$k = 0.5$ (Defect rate),

$\mu = 5$ (Delay time mean),

$\beta = 1$ and 0.7 (Inspection perfectness).

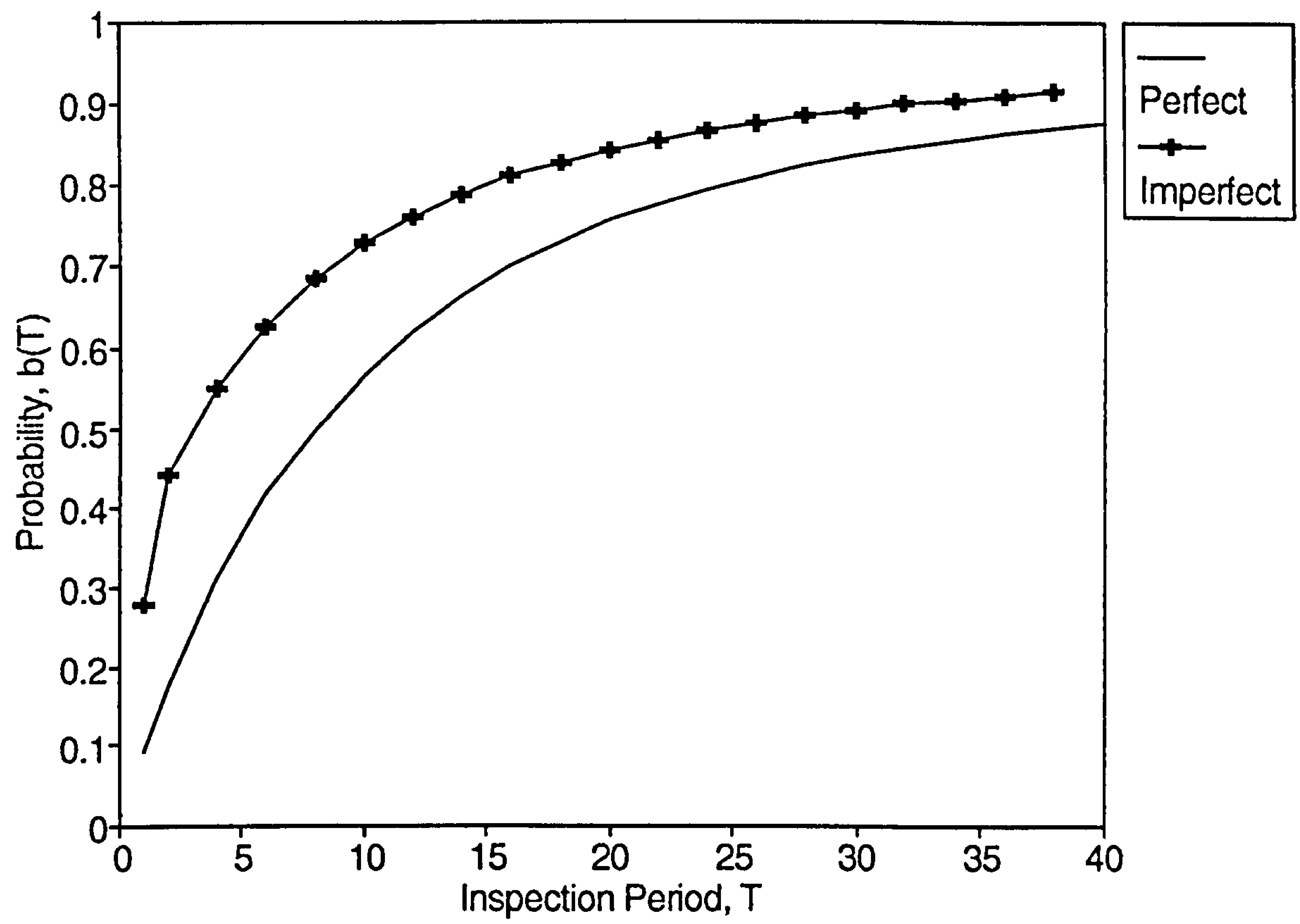

Fig. 2.8. The probability, $b(T)$, assuming exponential delay times.

The optimum inspection period is at $T = 5.1$ for the perfect inspection model and at $T = 4.7$ for the imperfect inspection model in Fig. 2.9. This implies that with the assumed costs for imperfect inspections, more frequent inspections would need to be carried out in order to optimize cost. This is reasonable to expect.

The exponential delay time and HPP defect arrival case can also be described by a Markov process in continuous time. Let breakdown repair times be negligible and define S_t to be the number of defects present in the system after time t, given the system is defect free at time 0. Consider the case when $S_t = n$ ($n \geq 0$) and the change of state in small time interval δt. Now $P\{S_{t+\delta t} = n + 1 \mid S_t = n\} \approx k\delta t$, i.e a defect arrival. A breakdown arrival event in δt would be attributed to any one of the n defects, each with probability $\delta t/\mu$ to cause the breakdown. A breakdown repair would decrease S_t. Hence, $P\{S_{t+\delta t} = n - 1 \mid S_t = n\} \approx n\delta t/\mu$.

It then follows that $P\{S_{t+\delta t} = n \mid S_t = n\} \approx 1 - (k + n/\mu)\delta t$. Hence, a state transition matrix can be formulated and an analysis into the finite and limiting behaviour of the

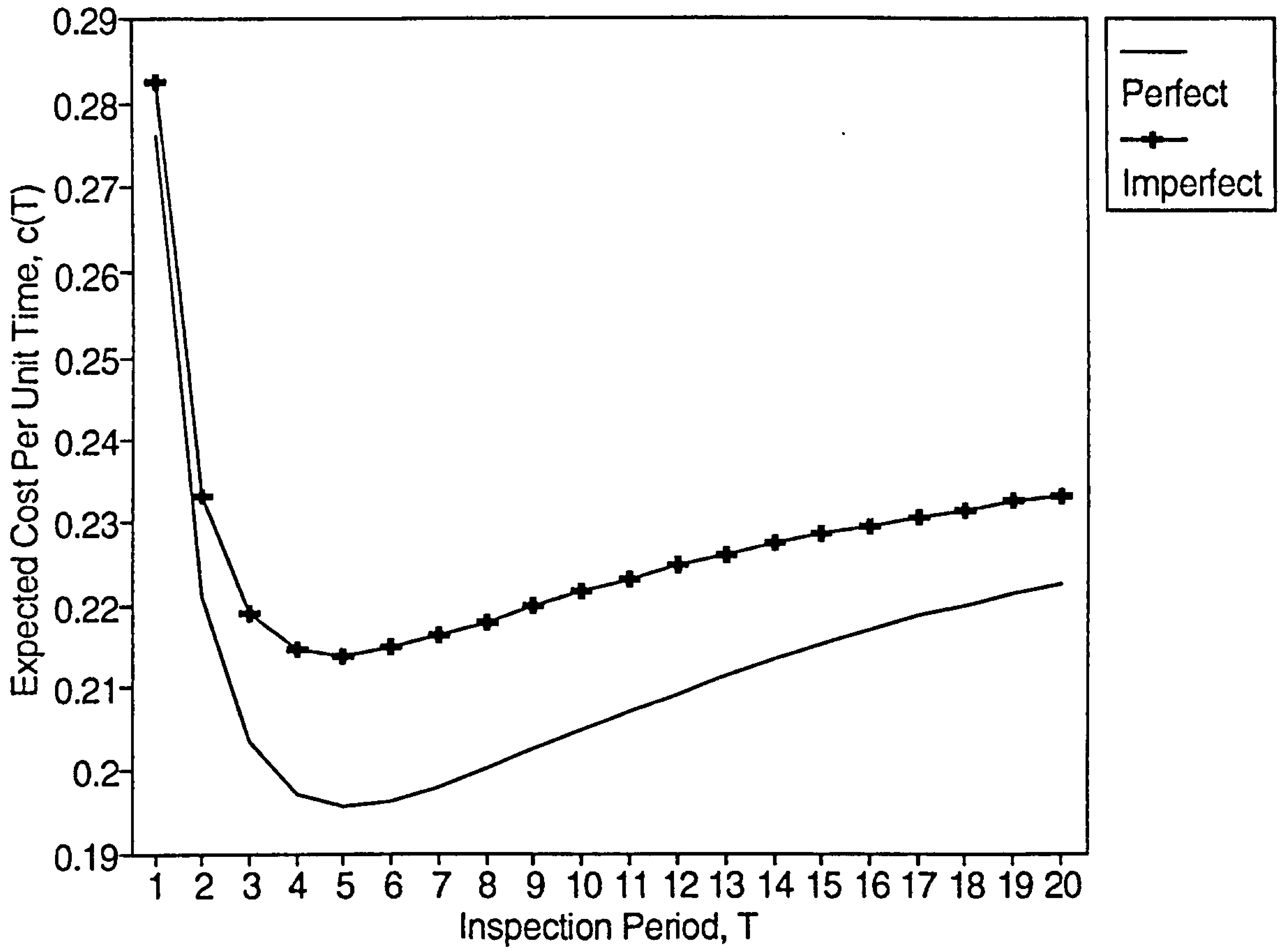

Fig. 2.9. The expected cost per unit time, $c(T)$, assuming exponential delay times.

process can be undertaken, i.e S_t is Poisson distributed with mean $S(t) = k\mu(1 - e^{-t/\mu})$, see Feller (1970a, p.460). However, the Markov model is only applicable in special cases and does not encompass the more realistic general situations. Work is being carried out elsewhere on the applicability of Markov models to delay time problems.

2.8 Conclusion

It has been seen that the concept of delay time can be used in modelling maintenance of a complex system. The NHPP model for the arrival process of breakdowns of a repairable system, endorsed by Ascher and Feingold (1984), can incorporate the concept of delay time by allowing the ROCOF to be a convolution of the defect rate and delay time p.d.f under the assumption of independence. In this way, the expected number of defects detected at inspections can also be modelled. However, the assumption of an NHPP breakdown arrival rate for a system will need to be tested in a specific case.

The case of imperfect inspection with NHPP defect arrivals has not been considered here and is an area for further research which is underway elsewhere. The inspection point for this case is not a system renewal implying that the defect arrival rate $g(u)$ cannot be considered identical in each inspection interval. This increases the modelling complexity.

Clearly, the downtime and cost of other inspection policies can also be investigated. For example, a policy could be to inspect after the n'th failure occurrence or when a particular length of operating time t has elapsed. The decision variables would be n and t. The UBI policy is when $n = \infty$. The case, $n = 1$, implies an age based replacement policy, for the case of perfect inspections, with the p.d.f of time to first failure, x say, given by, $r(x)e^{-B(x)}$, due to the process of failures following an NHPP.

Under restricted circumstances, it has been shown that the system can be modelled by a Markov process in continuous time. This model could also be expanded to the more realistic case when there are a finite number of defect prone components within a system.

A criticism of these models is that ageing of the system after each inspection has not been modelled. Ageing can be modelled by assuming non-identical defect rates, $g(u)$, over each inspection interval. It is also possible to allow the delay time of a defect to be dependent on u and the inspection interval in which the defect occurred, see Christer and Wang (1992). The process of breakdowns then would not necessarily be an NHPP.

The type of model selected is directly dependent on assumptions as to how the system is operated and used, the type and quality of maintenance, and the deterioration processes over time. The purpose of this chapter has been to introduce the basic nature of the delay time concept and the variety of models that may be constructed. Many of the models presented here will be used in conjunction with the results of the subsequent chapters on revision methods and the estimation of the parameters for the delay time distribution, in the light of observational and subjective data.

Chapter 3

Parameter Estimating and Updating for Delay Time Models

3.1 Introduction

In this chapter, procedures are constructed for estimating the parameters necessary to formulate the models derived in Chapter 2. These will be constructed and based upon the experience gained and the data collected in operating repairable systems over time. Two types of data will be focused on, namely subjective and objective. Subjective data can arise from engineers' estimation of the delay time of specific defects at breakdowns and inspections. Thus, data of this type is expected to be in error. However, the collection of this data has been shown to be possible and prior delay time distributions have been estimated in specific cases, see Christer and Waller (1984b), Chilcott and Christer (1991), Christer and Desa (1992). The objective data for estimating the delay time distribution is based upon observations of times of breakdowns and defect detections. This data will aid the estimation of delay time parameters and the testing of the fit of the subsequent maintenance model. This objective data, can be in error, but for now, we will assume this data to be accurate.

A maintenance model formulated with a substantial subjective input to delay time parameter estimates could not guarantee to automatically model the "status quo" characteristics of the system. That is subjective data may not imply that which is currently observable. Management interest may be in cost, downtime or proportion of defects which arise as failures under a current inspection practice. Eitherway, updating procedures are given to force the subjectively based model to agree with "status quo" observation. This could be considered as a "model tuning process". We will find that there is not necessarily a unique option for updating. However, a selection criteria is given based on other information, which may be available, over the system data collection period.

A case study example is given to highlight the sensitivity of the updated models in the case of downtime criteria. A decision consequence of changing operating practice to alternative inspection policy is discussed, and appropriate tests for the fit of the model recommended.

The work will be substantially based on the paper, Christer and Redmond (1992).

3.2 Assumptions and Data

The system, considered here, will be assumed to have been operated under the policies, periodic based inspection (PBI) or use based inspection (UBI) over a data collection period, with the following observational information, usually available in operating practice:

(a) The current practice is of inspecting the system on a constant time period, T_0 say. The value represents the real time between inspections for PBI; or this could represent the operating time before inspections with UBI.

(b) Inspections are initially assumed perfect and defective components detected at each inspection are replaced or repaired to 'as-new'.

(c) The observed number of inspections carried out, over the survey period is M say, each with downtime d_I and average cost, c_I.

(d) The total number of breakdowns observed over each inspection interval, i, is B_i say, $1 \leq i \leq M$.

(e) The incident time of each breakdown which arrived, say $\{y_j\}$, $1 \leq j \leq B$, where $B = \Sigma B_i$, is the total number of breakdowns arisen over the survey period.

(f) The total number of defects identified and repaired at each inspection is S_i say, $1 \leq i \leq M$.

(g) The average downtime for each breakdown repaired, d_b.

(h) The average repair cost of each breakdown repaired, c_b.

Measures of delay time, h, are not generally available from direct observation and need to be estimated in specific cases. For instance at a breakdown, the maintenance engineer could be asked to estimate how long ago (HLA) the defect could first have reasonably been expected to have been noticed by a given inspection procedure, see Fig. 3.1. This would provide an estimate of h in a specific case. Evidence of imperfect inspections would be indicated if h spans a previous inspection point. In making this assessment, the specific case is before the engineer along with any other evidence or clues that may exist, and in that sense the question is well defined. Of course, the answer will depend upon the engineer's understanding of the system and his relevant experience, that is, be his subjective estimate. By accumulating such delay time measures it has been seen that an estimate of the delay time p.d.f $f(h)$ associated with defects arising can be obtained, Christer and Waller (1984b,c).

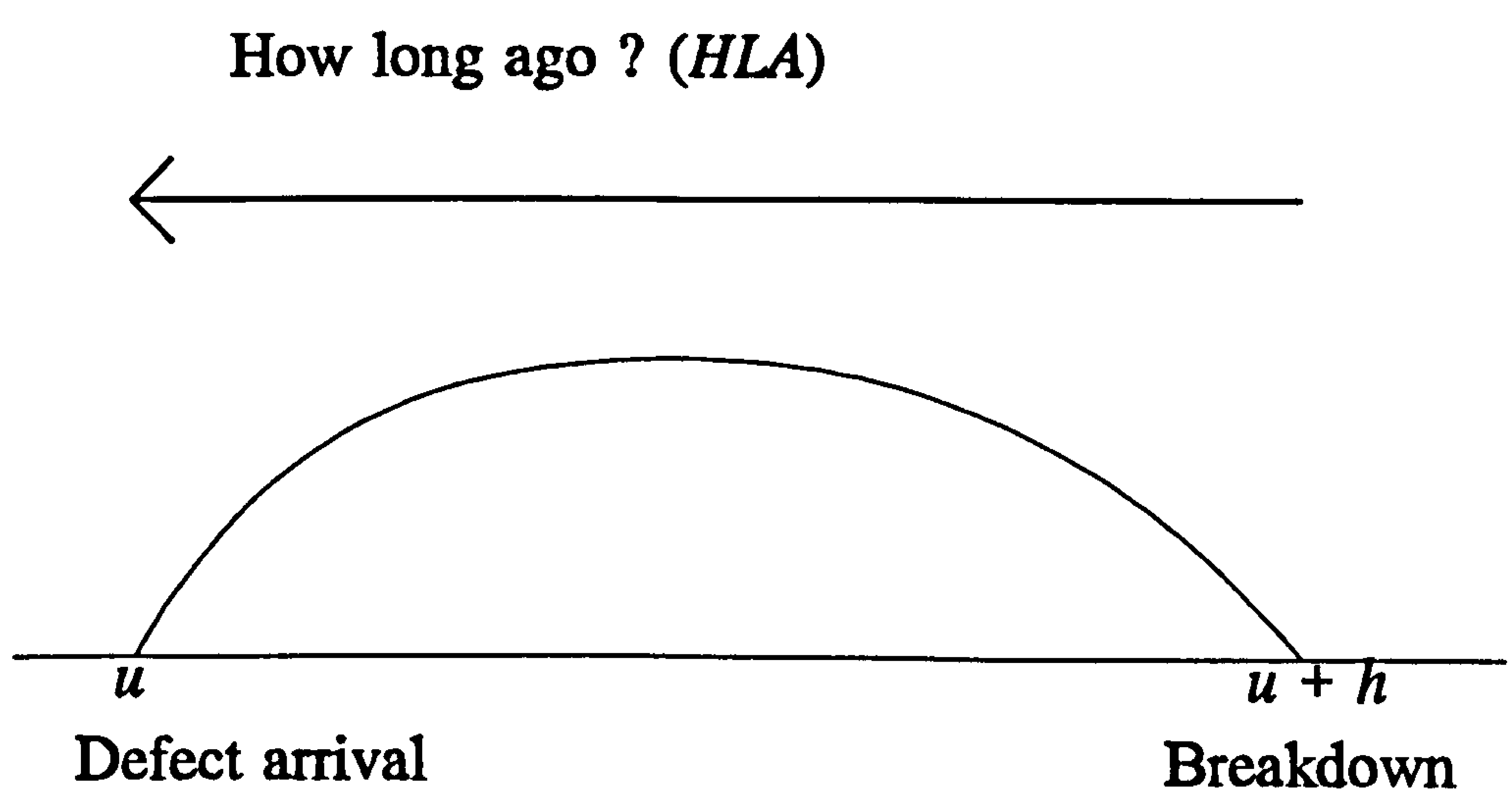

Fig. 3.1. Estimating delay time at breakdowns.

There are other ways of obtaining an estimate of $f(h)$. For instance, if a defect is identified at an inspection, the engineer could be asked both how long ago (*HLA*) the defect could be first have been observed by an inspection, l_1 say, and also if left unattended, how much longer (*HML*) the defect would last until a repair was necessary because of failure, say l_2. The accumulation of such measures $\{l_1 + l_2\}$, and pooling with any breakdown based delay time measures, would lead to the estimation of a distribution which is directly related to $f(h)$.

A situation of censored data may arise. For example, the delay times of defects detected at inspections may not be possible to estimate due to the necessary prediction of the future time to breakdown, i.e the inspection time + *HML*. This will be discussed in the next chapter. For the present, we will assume that a subjectively based estimate of the delay time distribution may be available. Essentially, the availability of subjective data, that is the willingness of those involved to collaborate with data collection experiments and surveys, will influence the form of the delay time distribution selected. Considerable care must be taken when collecting such estimates to ensure the questions being asked are properly understood. It may also be necessary to invite the engineer to provide a range estimate of *HLA* and *HML*. Eitherway, by perhaps by permitting and taking the mean or otherwise, such as an optimistic and pessimistic measure, we assume a single estimate of h (= *HLA* + *HML*) is available for each defect or breakdown considered.

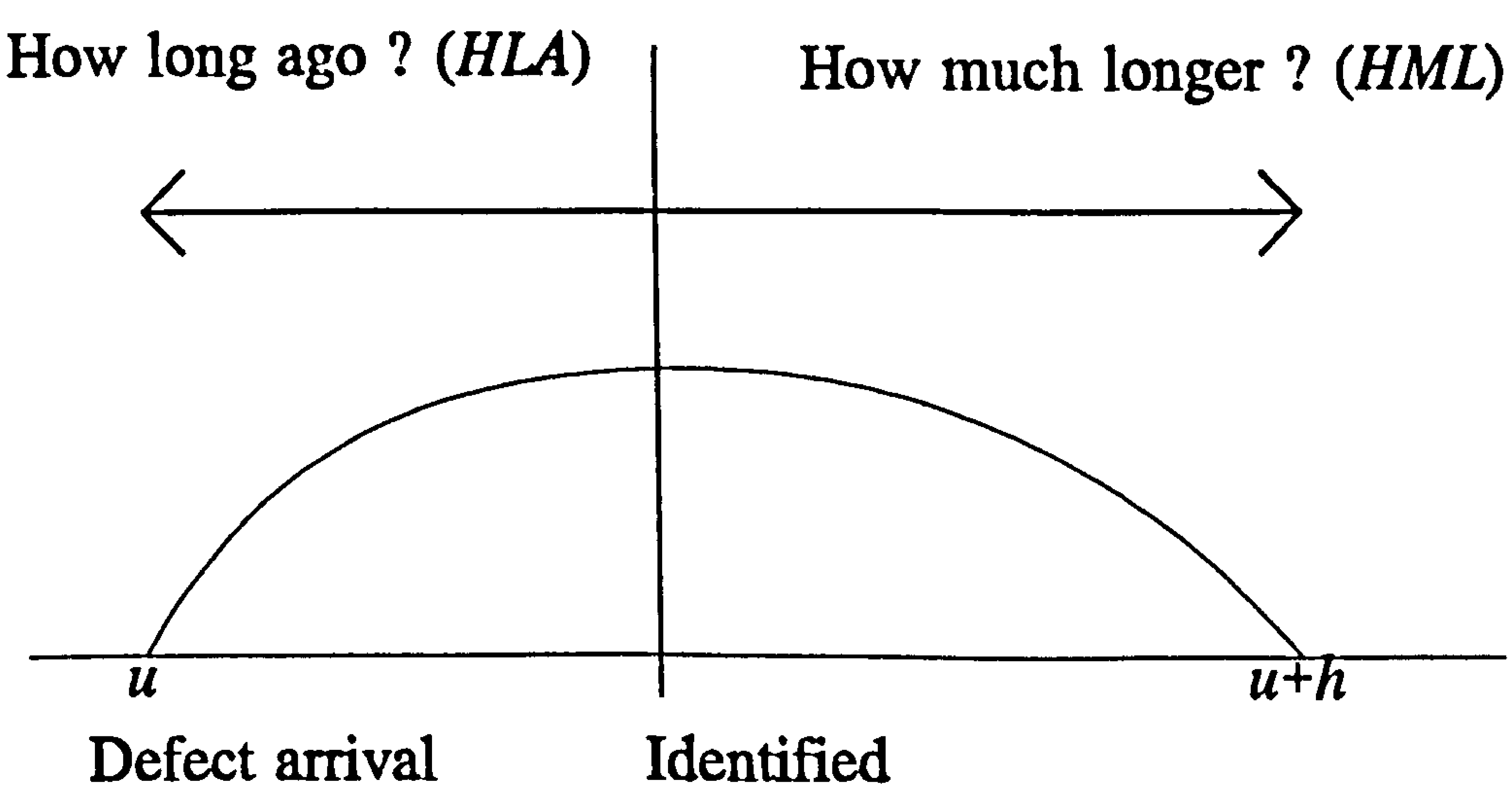

Fig. 3.2. Estimating delay time for defects identified at inspections.

The system will be assumed to have been operated over a survey period of M inspections of constant period T_0, subject to the assumptions of Sections 2.4 and 2.5 namely:

(a) At time 0, the system is in a new or 'statistically new' state, that is defect free.

(b) Defects arise, independently at a constant rate (HPP), with rate parameter k, only over the operating (or up) time. A prior test on estimates of u have been assumed, here, for the HPP assumption. Essentially estimates of u would be obtained from the estimates of *HLA*.

(c) Delay times, h, are independent of defect arrival time, u, and are distributed with the p.d.f $f(h)$ and c.d.f $F(h)$.

(d) A breakdown imposes a small amount of downtime compared to inspection period, T, with expected value d_b, $d_b \ll T$.

(e) The expected value of breakdown cost is c_b and the expected value of defect repair cost is c_d.

(f) Inspections are carried out with perfectness, β, where β is the probability that a defect present at an inspection is detected and consequently repaired.

The condition, $d_b \ll T$, is not necessary in the case of the UBI policy, but for the current simplified model is necessary in the PBI case.

3.3 Need to Update Prior Model

It should be noted that the downtime models (2.18) or (2.36) are driven by specific observable and measurable parameters such as; downtimes, d_I and d_b, by modelling assumptions such as $\beta = 1$ or $\beta \neq 1$, and, in this case, by the subjectively based distribution $F(h)$. Information relating to the observed level of overall downtime being experienced under the existing operating conditions has not been specifically used. It is a surprising fact that here we have been able to produce a downtime model for different inspection periods which has been derived without considering the downtime levels associated with the current practice, since conventionally the latter would be a starting point for the former. That current experience appears not to have had an explicit influence on the model is due, so far, to the mechanistic method of delay time analysis adopted. Of course, we are utilising subjective assessments each of which is presumably coloured by current experience.

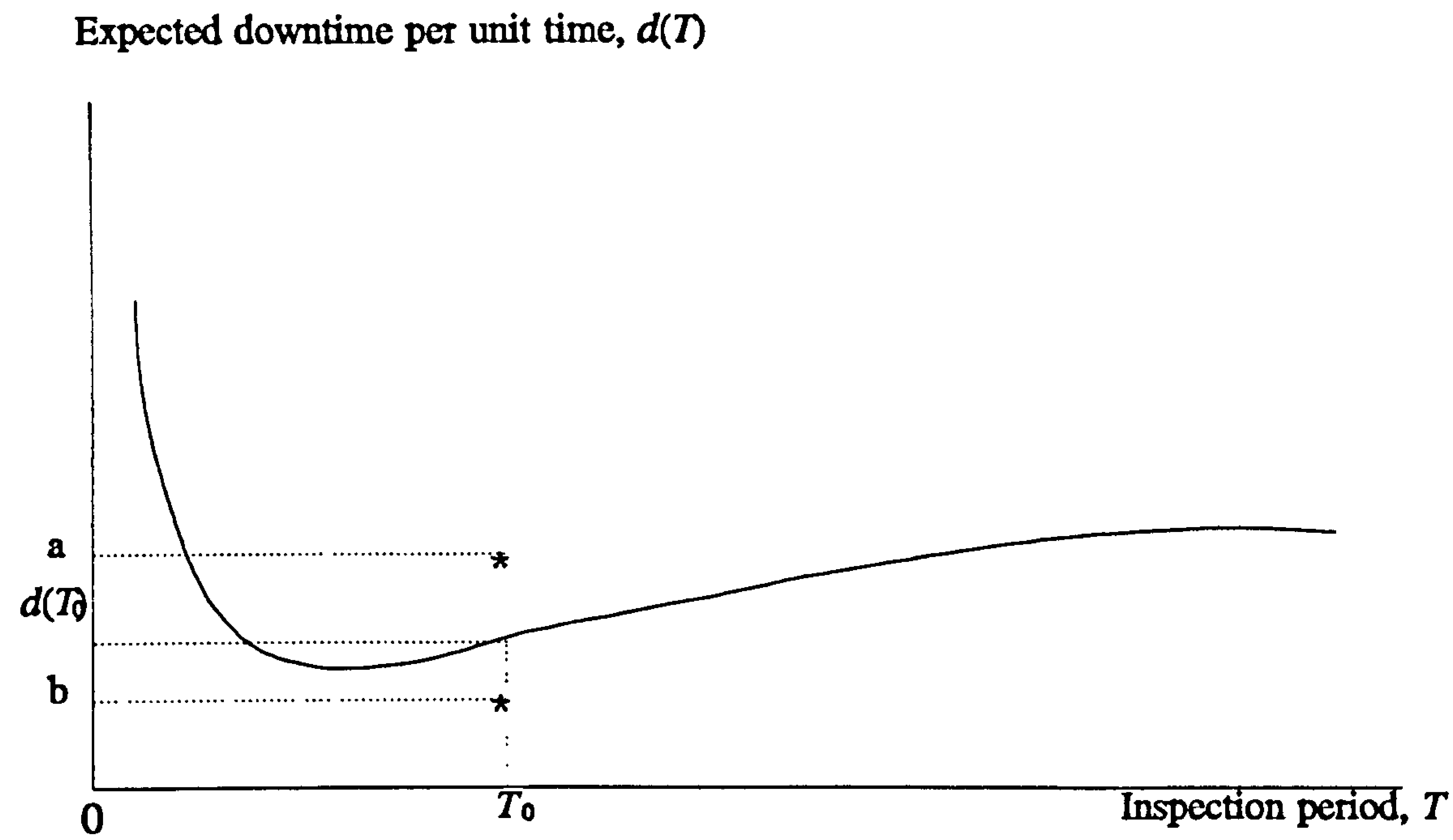

Fig. 3.3. Comparison of estimated model, $d(T)$, and possible observations.

The existing expected downtime levels should, and do, directly influence the model through estimates of d_1 and d_b. However, these parameters, the delay time distribution and the delay time model should also influence the model since it must be capable of predicting the status quo situation, that is current practice. If, for example, an inspection practice is to operate with a period T_0 and the operating experience is of an average plant downtime per unit time d_0^*, then the curve of Fig. 3.3 should pass through the point (T_0, $d(T_0)$). The chance of this subjectively derived point coinciding with the observed point (T_0, d_0^*) is remote. What is expected to occur is typified by Fig. 3.3 in which the observed point (T_0, d_0^*) could be in a position such as (a), or perhaps (b). Effectively, d_0^*, is likely to be a periodic based estimate of a stochastic variable, and the models of downtime presented here, are average value models of a stochastic variable. If d_0^* is above (below) the curve then the delay time p.d.f chosen has most likely been estimated with a higher (lower) mean value. The estimated defect rate, k or $g(u)$, modelling parameters, β, for the probability of detecting a defect at inspection, and estimates of the mean values of inspection and breakdown downtime, d_1 and d_b, will also influence the position of d_0^* relative to the estimated model.

Ideally, we would wish the subjectively derived curve $d(T)$ to pass through the known point, and an updating or revision is necessary either to the prior distribution estimate $F(h)$ or to the structure of the model where the β value is changed, or to both $F(h)$ and to the value of β. The rest of the chapter will be devoted to investigating techniques for this updating process.

3.4 Updating Procedure

Any method of updating a prior delay time model will obviously depend upon the existing information available for comparison of theory and practice. It will both be realistic and useful to suppose that the information which will exist is the number of defects which cause breakdowns, B, arising within a system over a series of M inspections and the total number of defects spotted at inspections, $S = \Sigma S_i$ say. An estimate of $b_0^* = B/(B + S)$, the probability that a fault leads to a breakdown can be made under the current inspection period T_0. The value b_0^* would be an unbiased estimate of the theoretical probability of a defect arising as a breakdown for perfect inspections of period T_0, but may be biased in the case of imperfect inspections, due to any defects which may have gone undetected at the last inspection of the data collection survey. An approximate $100(1 - \alpha)\%$ confidence interval for the true value of this proportion is stated in Chatfield (1970, p.364),

$$b_0^* \pm z_{\alpha/2}\sqrt{b_0^*(1 - b_0^*)/(B + S)} \quad , \tag{3.1}$$

when b_0^* is not close to 0 (i.e no breakdown occurrences) or 1 (no inspection practice), and $P\{Z < z_{\alpha/2}\} = \alpha/2$ when Z is standard normal. We then wish to model $b(T)$ to pass through the point (T_0, b_0^*), assuming that b_0^* lies inside a sufficiently small confidence interval, and $b(T_0)$ lies significantly outside, see Fig. 3.4. We will assume initially that defects arise as an HPP.

To carry out the task of updating the prior model $b(T)$, we need to let the delay time p.d.f $f(h)$ be dependent on a set of parameters, $\underset{\sim}{\lambda}$ say. The requirement to update and model given the known point is met by formulating the likelihood function for the observed number of breakdowns and defect detections for the survey period. It follows from Section 2.5 that the number of recorded breakdowns, B_i and defects detected, S_i, over the i'th inspection interval, are independently Poisson distributed with means $kT_0b(T_0; \underline{\lambda})$ and $kT_0\{1 - b(T_0; \underline{\lambda})\}$ respectively, when inspections are perfect. Hence, the likelihood function over the survey period, given the observed B_i breakdowns and S_i inspection identified defects in the i'th inspection cycle, $L(\underline{\lambda})$ say, is given by,

$$L(\underline{\lambda}) = \prod_{i=1}^{M} \left(\frac{e^{-kT_0b(T_0;\underline{\lambda})}(kT_0b(T_0;\underline{\lambda}))^{B_i} e^{kT_0(b(T_0;\underline{\lambda}) - 1)}(kT_0(1 - b(T_0;\underline{\lambda}))^{S_i}}{B_i!S_i!} \right) . \quad (3.2)$$

Here, $b(T; \underline{\lambda})$ is given by function (2.16) using the parameterized form of delay time p.d.f $f(h)$. By simplifying equation (3.2) and partially differentiating with respect to $\underline{\lambda}$ the corresponding log likelihood function, it follows that we need to solve,

$$b(T_0;\underline{\lambda}) = B/(B + S) = b_0^* , \quad (3.3)$$

to obtain the maximum likelihood estimates (MLE) of λ. It also follows that the MLE for the rate of defect arrivals, k, when allowing k to be a parameter, is as expected,

$$\hat{k} = \frac{B + S}{MT_0} . \quad (3.4)$$

Hence the proposed process to model the known point of current practice by equating $b(T_0; \underline{\lambda})$ to b_0^* is equivalent to satisfying a requirement for a maximum likelihood fit of objective data. Techniques to generate a set of parameters in order to revise or update the prior model of $b(T)$ are now considered.

3.5 Linear Delay Time Update

Here we consider the case when each estimate of delay time h for a defect is linearly related to its actual delay time h' by,

$$h' = \alpha h + \omega , \quad (3.5)$$

where α and ω are parameters to be estimated.

There are two restrictions on the parameter values of α and ω. First, it is assumed that $\alpha > 0$, i.e.,

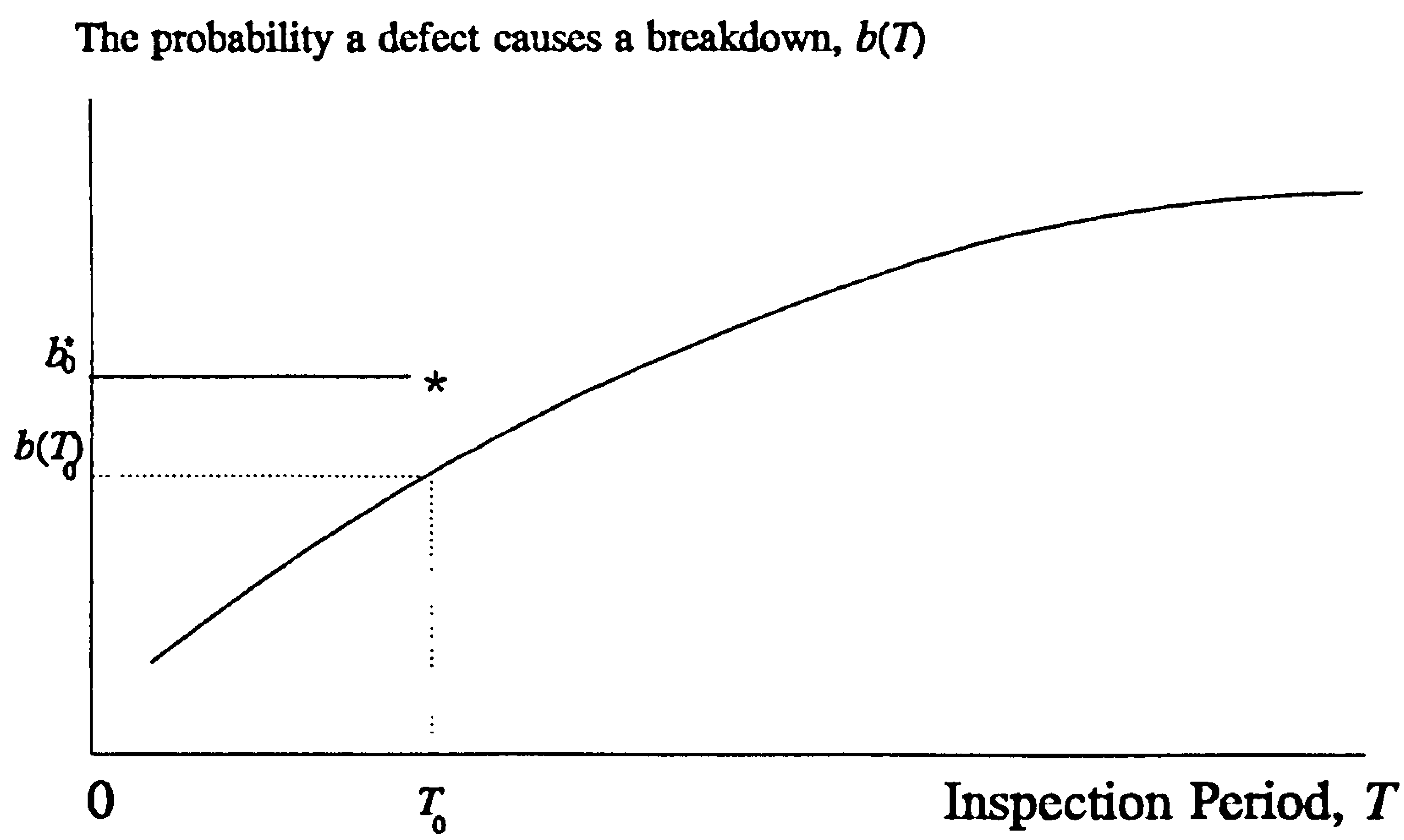

Fig. 3.4. Comparison of model, $b(T)$, with known observation point.

$$\alpha > 0 \ , \tag{3.6}$$

otherwise large estimates of h would transform to small actual values h' and vice-versa. Secondly, h' must be non-negative so if h_0 is the smallest value of h for which the prior distribution satisfies $F(h) > 0$ for $h > h_0$, then we require,

$$\alpha h_0 + \omega \ \geq \ 0 \ . \tag{3.7}$$

It follows that for any x, the probability $P\{h' \leq x\} = P\{h \leq (x - \omega)/\alpha\}$. Consequently the parametric form for the c.d.f of h' is given by,

$$P\{h' \leq x\} \ = \ F\left(\frac{x - \omega}{\alpha}\right) \ , \tag{3.8}$$

where $F(h)$ is assumed zero for $h \leq 0$, which implies c.d.f (3.8) is zero when $0 \leq h' \leq \alpha h_0 + \omega$.

Let the initial model $b(T)$ be constructed with an estimate β_0 for the probability of perfect inspection, β, and let $b(T;\ \alpha,\ \omega)$ be the updated parametric form for $b(T)$ when replacing $F(h)$ by the transformed expression (3.8). Clearly we seek a set of parameters $(\alpha,\ \omega)$ such that the status quo condition is satisfied, that is,

$$b(T_0;\alpha,\omega) \;=\; b_0^* \;. \tag{3.9}$$

Essentially we are seeking maximum likelihood estimates of the scale, α, and location parameter ω of an assumed delay time p.d.f family based on numerous observations. We now consider the possible existence of and uniqueness of solutions to equation (3.9).

3.5.1 Special Case: Scale Parameter, α, Update

Consider, first, the case when $\omega = 0$. It will be shown that a unique solution for α exists. Now, if $\beta_0 = 1$, the case of perfect inspection, to satisfy condition (3.9) with b(T) given by function (2.16) and using the parameterised c.d.f (3.8), we seek a value α satisfying,

$$\int_{x=0}^{T_0} \frac{(T_0 - x)}{T_0\alpha} f\left(\frac{x}{\alpha}\right) dx \;=\; b_0^* \;. \tag{3.10}$$

Using the substitution $x = \alpha h$ and re-arranging, we have,

$$\int_{h=0}^{T_0/\alpha} \frac{(T_0/\alpha - h)}{T_0/\alpha} f(h)\, dh \;=\; b_0^* \;, \tag{3.11}$$

where the left hand side of (3.11) is the model of $b(T)$, function (2.16), with T replaced by T_0/α. For a solution to equation (3.11), therefore, we require an α such that,

$$b\left(\frac{T_0}{\alpha}\right) \;=\; b_0^* \;. \tag{3.12}$$

The model $b(T)$ increases monotonically from zero to one as T increases from h_0 to infinity. Thus, since $0 < b_0^* < 1$, there exists a unique inspection interval T_1, such that,

$$b(T_1) \;=\; b_0^* \;. \tag{3.13}$$

It then follows that the unique solution for α is given by,

$$\alpha = \frac{T_0}{T_1} \; . \tag{3.14}$$

The solution α can also be found graphically by scaling or shearing the model $b(T)$, that is, by translating each point $\{T, b(T)\}$ to $\{T, b(T(T_1/T_0))\}$.

In other words, in the case of perfect inspections, $\beta = 1$, a unique transform of the delay time $h' = \alpha h$ can always be made to satisfy a status quo observation on $b(T)$, where $\alpha = T_0/T_1$ and T_1 is the solution of equation (3.13). In the case when $\beta_0 < 1$, it can be seen that equation (3.9) is also appropriate for this case by using the substitution $x = \alpha h$ in function (2.30). It then follows, as before, that a unique solution for α exists.

It also follows that,

$$\begin{aligned} b_0^* > b(T_0) \Rightarrow T_1 > T_0 \Rightarrow \alpha < 1 \quad , \\ b_0^* < b(T_0) \Rightarrow T_1 < T_0 \Rightarrow \alpha > 1 \quad . \end{aligned} \tag{3.15}$$

The main point here is that there exists a unique transform of the initial prior distribution which will modify the model of $b(T)$ to satisfy a known status quo observation for $\beta_0 \leq 1$. An example of this formal updating is given below.

Example. The following example is based upon a case study for a canning line, Christer and Waller (1984b). For this situation we let,

T_0 = 24 hrs,
$f(h) = 0.0447.\exp(-0.0447h)$, (exponential),
β_0 = 1, (perfect inspections),
d_b = 0.698 hrs,
d_I = 0.525 hrs,
b_0^* = 0.390,
k = 0.101 hrs^{-1}.

In the study it was found that, $b(T_0) = 0.387$, which was not felt significantly different from b_0^* to warrant updating. This is envisaged to be a rare occasion because the model

$b(T_0)$ is subjectively derived, whereas the number 0.387 reflects observation. For demonstration purposes, we consider two extreme cases here, one of over estimating and one of under estimating, namely, we suppose two levels of observed downtime,

(a) $b_0^* = 0.2$

and

(b) $b_0^* = 0.6$

and update the model of $b(T)$ using the transform $h' = \alpha h$. Case (a) represents an observation significantly less than the prior model value $b(T_0) = 0.387$, and case (b) is correspondingly significantly greater. Hence, the equation for α, (3.10), takes on the form,

$$\int_{x=0}^{24} \frac{(24 - x)}{24} \frac{0.0447}{\alpha} \exp(-0.0447x/\alpha)\, dx = b_0^* , \tag{3.16}$$

which on integrating the L.H.S. of equation (3.16), simplifies to,

$$1 - \frac{\alpha(1 - \exp(-1.0728/\alpha))}{1.0728} = b_0^* . \tag{3.17}$$

This equation can be shown to have the following respective solution for case (a) and case (b),

(a) $\alpha = 2.304$,

(b) $\alpha = 0.481$.

To demonstrate the effect of these updates, the initial model of $b(T)$ is shown in Figs. 3.5 and 3.6, with the scale update model for cases (a) and (b) respectively. The updated curves for $b(T)$ clearly pass through the known point (24, 0.2) and (24, 0.6) as required, that is the curves satisfy the status quo condition (3.9).

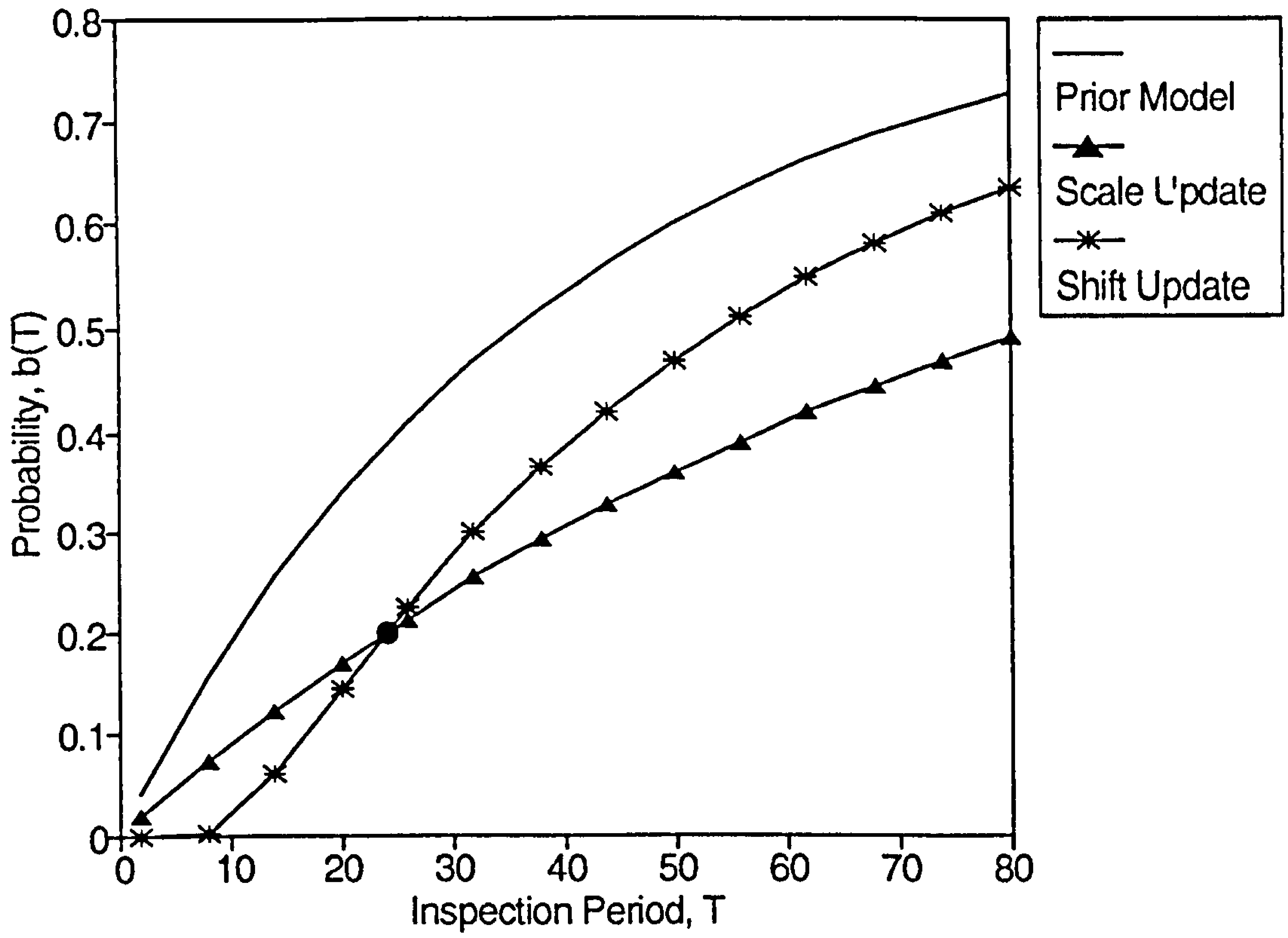

Fig. 3.5. Updating the model, $b(T)$, by the scaled-α and shift-ω methods for $b_0^* = 0.2$.

3.5.2 Special Case: Shift Parameter, ω, Update

Returning to the general discussion with $h' = \alpha h + \omega$, we consider the case $\beta_0 \leq 1$ and $\alpha = 1$. It will be shown that a unique solution to condition (3.9) exists for ω if and only if,

$$b_0^* \leq b(T_0; 1, -h_0) \quad , \tag{3.18}$$

where $\beta_0 \leq 1$. The transformation method here, is equivalent to shifting the p.d.f along the time axis until the model $b(T)$ passes through the status quo point. For the case of perfect inspections, providing the observation $b_0^* > 0$, it is clear that $\omega < T_0 - h_0$, with h_0 equal to the smallest prior delay time. This condition is necessary because no breakdowns would arise when $\omega \geq T_0 - h_0$ as all delay times would be greater than T_0.

To satisfy condition (3.7) when $\alpha = 1$ we must select ω such that,

$$\omega \geq -h_0 \quad . \tag{3.19}$$

As ω is increases to $T_0 - h_0$, $b(T_0;\ 1,\ \omega)$ tends to zero because the minimum delay time is increased beyond bound, which implies no breakdowns will occur in the system. In the case of perfect inspections, $\beta_0 = 1$, we consider the partial derivative of $b(T_0;\ 1,\omega)$ with respect to ω from equation (2.15),

$$\frac{\partial b}{\partial \omega} = \frac{-F(T_0 - \omega)}{T_0} \ . \tag{3.20}$$

Thus,

$$\frac{\partial b(T_0;1,\omega)}{\partial \omega} < 0 \qquad for \qquad T_0 > h_0 + \omega \ . \tag{3.21}$$

Clearly, therefore, the maximum value of $b(T_0;\ 1,\ \omega)$ is when $\omega = -h_0$, which corresponds to the shortest possible actual delay time, and $b(T_0;\ 1,\ \omega)$ monotonically decreases to zero as ω increases from $-h_0$ to $T_0 - \omega$. It follows that there exists a unique solution ω if and only if condition (3.18) is satisfied, namely $b_0^* \leq b(T_0;1,-h_0)$, due to the upper bound on $b(T_0;\ 1,\ \omega)$. It can be seen using equation (2.30) that the result generalises to the case $\beta_0 \leq 1$. $b(T_0;\ \beta_0,\ \omega)$ decreases to zero when ω tends to ∞, as there is evidently no maximum bound on ω.

Returning to the above case study example when $h_0 = 0$ and $\beta_0 = 1$, condition (3.19) implies $\omega \geq 0$. Since we need $b_0^* \leq b(T_0)$ an update based on the ω parameter can only be performed in case (b), $b_0^* = 0.2$, and in this case equation (3.9) takes the form,

$$\int_{x=\omega}^{24} \frac{(24 - x)}{24} 0.0447 \exp(-0.0447(x - \omega))\, dx = 0.2 \ , \tag{3.22}$$

which has the unique solution $\omega = 7.55$. A graph of this scaled updated model for $b(T)$ is also given in Fig.3.5.

3.5.3 General Linear Case

Here we let α and ω be unrestricted. It will be shown that for each $\alpha > 0$, there exists a unique ω which satisfies condition (3.9) if and only if,

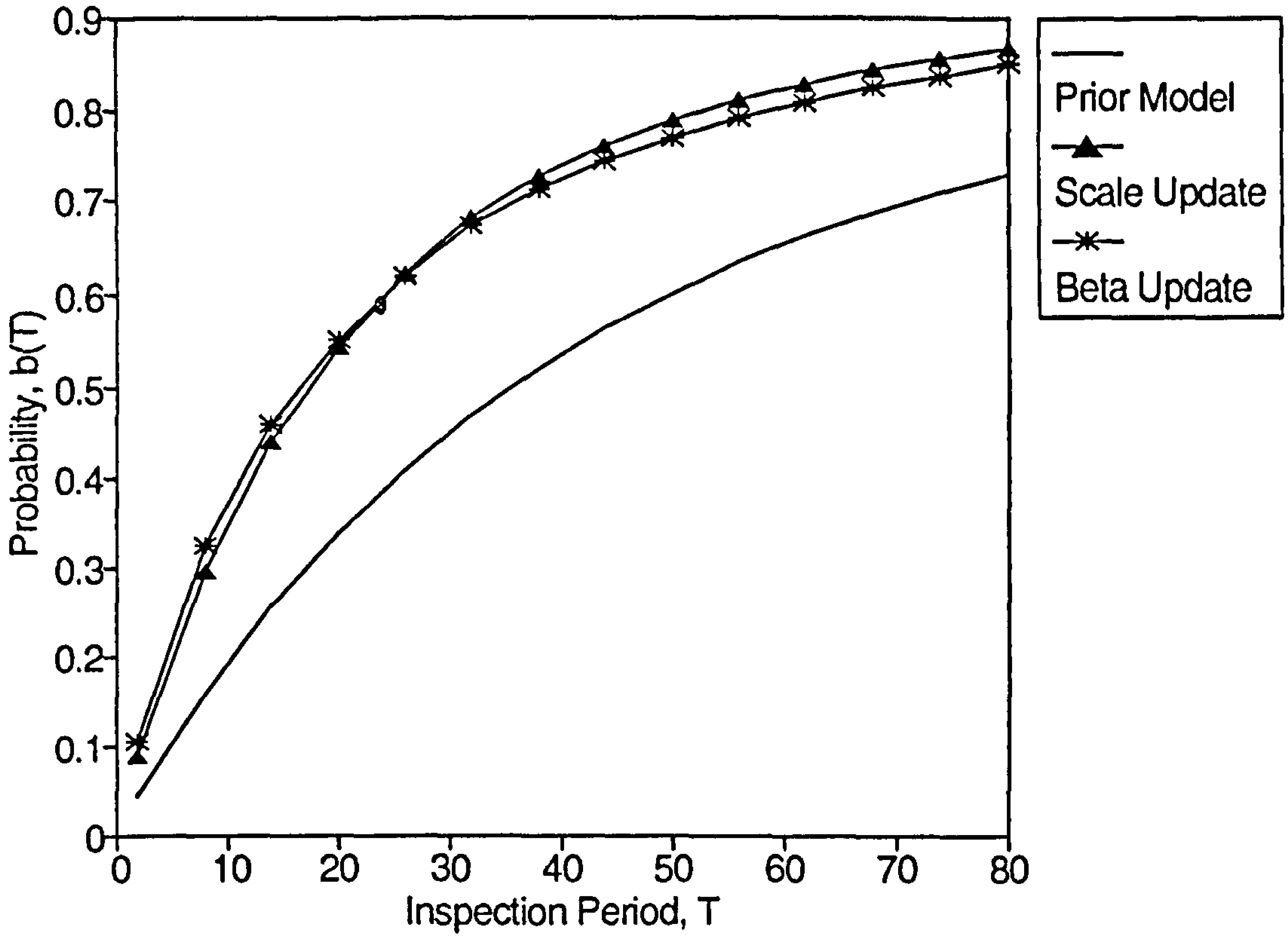

Fig. 3.6. Updating the model, $b(T)$, by the scaled-α and revised-β methods, $b_0^* = 0.6$.

$$b_0^* \leq b(T_0;\alpha,-\alpha h_0) \quad , \tag{3.23}$$

where $\omega \geq -\alpha h_0$ and $\beta_0 \leq 1$.

To satisfy condition (3.7) we require $\omega \geq -\alpha h_0$, which gives the maximum value of $b(T_0;\alpha, \omega)$ with respect to ω. Due to $\alpha > 0$, the partial derivative (3.20) of $b(T_0; \alpha, \omega)$ with respect to ω will still be negative. In the case $\beta_0 = 1$, the maximum bound of ω would be $T_0 - \alpha h_0$ because all transformed delay times, h', would be greater than T_0, implying no breakdowns would arise (assuming $b_0^* \neq 0$). For the case $\beta_0 < 1$, $b(T_0;\alpha, \omega)$ tends to zero as ω tends to ∞. Hence, condition (3.23) above is both necessary and sufficient.

It is interesting to show, here, that for the case of perfect inspections, a relationship exists between the parameters (α, ω) selected and the updated delay time mean. If μ is the mean of the prior distribution $F(h,)$ then the mean of the transformed distribution, μ' say, is given by,

$$\mu' = \alpha\mu + \omega \ . \tag{3.24}$$

In the case $\beta_0 = 1$, to satisfy the known point we can also solve, using function (2.15) for $b(T)$, the equation,

$$T_0(1 - b_0^*) = \int_{x=0}^{T_0}\left[1 - F\left(\frac{x - \omega}{\alpha}\right)\right]dx \ . \tag{3.25}$$

The R.H.S is the integral of the survivor function, which tends monotonically to μ', as T_0 tends to ∞. Therefore, the L.H.S would be less than the delay time mean value. Hence, we obtain the relationship,

$$\mu' = \alpha\mu + \omega > T_0(1 - b_0^*) \ . \tag{3.26}$$

which provides a lower bound for the delay time mean, assuming that b_0^* is known to a sufficient degree of accuracy and modelling assumptions are valid.

3.6 Model Parameter Variation

In this case we now allow β to vary in the prior maintenance model, which is denoted by the probability function (2.30), $b(T;\beta)$, and we seek a value β such that,

$$b(T_0;\beta) = b_0^* \ . \tag{3.27}$$

That is, we now seek to satisfy the status quo condition on the assumption that it was the original modelling that was at fault in assuming that inspections were or were not perfect, i.e selecting the wrong value for β_0.

The minimum value of $b(T_0;\beta)$ corresponds to $\beta = 1$, the perfect inspection model. When $\beta = 0$ we have $b(T;\beta) = 1$ for all T, and as shown with result (2.33) in Section 2.5.5,

$$\frac{\partial b(T_0;\beta)}{\partial\beta} < 0 \ , \tag{3.28}$$

for any value of β.

We have, therefore, that $b(T_0;\, \beta)$ is monotonically decreasing as β increases from zero to one and, therefore, there exists a unique solution to the status quo condition (3.27) if, b_0^* satisfies,

$$b_0^* > b(T_0;1) \ . \tag{3.29}$$

In the above case study example, a solution can only be found for case (b) when $b_0^* = 0.6$ since b_0^* lies above the initial perfect inspection model value of 0.387 when $\beta_0 = 1$. Hence, equation (3.27) for β takes the form,

$$1-\Big(\int_{y=0}^{24} \sum_{n=1}^{\infty} \frac{\beta(1-\beta)^{n-1}}{24} \exp(-0.045(24n-y))dy\Big) = 0.6 \tag{3.30}$$

which using function (2.39), has the unique solution $\beta = 0.551$.

A graph of this β-update of the $b(T)$ model is given in Fig.3.6, and is seen in this case to give an updated result for $b(T)$ which is very similar to that resulting from the α transformation method. Case (a) cannot be updated in a β variation transform since the necessary condition (3.29) for a solution is not satisfied.

3.6.1 Combining the Methods

Here we consider the more general case in which β, α and ω are all permitted to vary in the model form for the probability of a defect resulting in a breakdown, denoted by $b(T;\, \beta, \alpha, \omega)$, and we seek a parameter set (β, α, ω) such that the status quo condition (3.9) is satisfied, that is,

$$b(T_0;\ \beta,\alpha,\omega) = b_0^* \ . \tag{3.31}$$

The solution set (β, α, ω) is clearly not unique. We have already seen above, for any (α, β) pair there exists a unique solution for ω satisfying condition (3.31) provided,

$$b_0^* \leq b(T_0;\beta,\alpha,-\alpha h_0) \ , \tag{3.32}$$

with different (α, β) pairs leading to different members of the solution set (β, α, ω). In particular, when $\omega = 0$, for each β there exists a unique α, such that condition (3.31) is satisfied, shown in Section 3.5.1. As β tends to zero, α will tend to infinity because a more imperfect inspection means delay times need to be longer for condition (3.31) to hold. The minimum value of α will be the solution to equation (3.11) when $\beta_0 = 1$, the case of perfect inspections.

When $\alpha = 1$, we have from the above analysis that as β tends to zero, ω tends to infinity. However, the range of β will be bound and the maximum value β which will be the solution to equation (3.31) when $\omega = -h_0$ and $\alpha = 1$. This is due to condition (3.7).

The main point is that the solution set (β, α, ω) for updating the delay time model is non-empty and non-unique. Our task in any practical situation will be to select one member from this set, that is, produce an updated model.

3.7 Criteria for Choosing Method of Updating and Estimating

As indicated above, it is clear that in the general case there is a non-unique set γ, say, of the vector (β, α, ω) capable of updating the prior model $b(T)$ to satisfy the status quo condition (3.9). Many forms of criteria of choice of update can be used here depending on the objective information available from the current inspection practice and the occurrence of breakdowns. If all we have, for example, is the observation b_0^*, then we could select the parameters (β, α, ω) to update the model on the basis of minimizing the Euclidean distance between the initial parameters ($\beta = \beta_0$, $\alpha = 1$, $\omega = 0$) and the update choice from the set (β, α, ω), i.e, we select (β^*, α^*, ω^*) such that,

$$(\beta^*,\alpha^*,\omega^*) \; = \; \underset{(\beta,\alpha,\omega)\subset\gamma}{\text{Min}} \{(\alpha - 1)^2 + \omega^2 + (\beta - \beta_0)^2\} \; . \tag{3.33}$$

This might be seen as some form of 'purely' measure based upon the original subjective assessments. A weighted form of this criteria could also be used if it was felt some of the α, ω or β measures were more important. For instance, the term in β could be replaced by $0.1(\beta - \beta_0)^2$, say, to bias against a large shift from the recognised quality of current inspection practice β_0.

3.7.1 Method of Moment Parameter Selection

Another possibility for choosing a solution (β^*, α^*, ω^*) from the satisfying set (β,α,ω) arises from the observations of times of breakdowns, was mentioned in the papers Christer and Redmond (1992). Accepting current practice, we can calculate the time since the last inspection for each breakdown, y say. The moments of the breakdown time, e.g mean and variance, can be estimated and set equal to parametric models of such statistics. Then, the parameters (β, α, ω) can be selected to satisfy the set of equations produced. In order to achieve this, we need to first derive the p.d.f of times of breakdown.

Breakdowns arise as a NHPP, shown in Section 2.4. Therefore, the conditional p.d.f of the breakdown time of a given random breakdown, $p_b(y;\,T)$ say, is given by,

$$p_b(y;T) \;=\; \frac{r(y)}{B(T)} \qquad for \quad 0 \le y \le T \ , \tag{3.34}$$

where $B(T)$ is the expected number of breakdowns in interval (0, T) and $r(y)$ is the ROCOF for the system after time 0. In the case of perfect inspections, $r(y) = B'(y)$. The corresponding result to the breakdown time distribution (3.34) can be found in Parzen (1962, p.145) and Ascher and Feingold (1984, p.32).

For perfect inspections ($\beta_0 = 1$) and HPP defect arrivals, $r(y) = kF(y)$, and using function (2.15) the p.d.f p_b (y; T), equation (3.34), then takes on the form,

$$p_b(y;T) \;=\; \frac{F(y)}{Tb(T)} \;=\; \frac{F(y)}{\int_{y=0}^{T} F(y)dy} \ . \tag{3.35}$$

The function $p_b(y;T)$ can be seen to be non-decreasing function over the interval $y \in (0, T)$. Hence, it is anticipated that bounds can be attached to the mean value of y. The p.d.f with the least mean value would be the uniform distribution when all delay times of defects are zero. Hence, defining the mean breakdown time by $M(T)$, we have,

$$M(T) \;=\; \frac{1}{Tb(T)}\int_{y=0}^{T} yF(y)\,dy \;\ge\; \frac{T}{2} \ . \tag{3.36}$$

Also since $F(y) \leq 1$, we have $M(T) \leq T/(2b(T))$. Due to the p.d.f (3.35) being numerically equal to the delay time c.d.f divided by its integral over the interval $(0,T_0)$, a method of obtaining a prior delay time c.d.f could also be obtained from observation of failure times. E.g if $p\hat{}(y)$ is the sample p.d.f of breakdown times, an estimate of the c.d.f of delay time is $F\hat{}(y) = T_0 b_0^* p\hat{}(y)$, for $0 \leq y \leq T$, assuming that $p\hat{}(y)$ is monotonically increasing. Consequently the model $b(T)$ and subsequently $B(T)$ can be estimated for $0 \leq T \leq T_0$.

In Section 3.5, it was shown that there exists a non-unique set (α, ω) to update the model $b(T)$. If an estimate of the mean time of breakdown is available, m_0^* say, then it follows, using the above p.d.f (3.35), that we need to solve,

$$\frac{1}{b(T_0;\alpha,\omega)\,T_0}\int_{y=0}^{T_0} yF(\alpha y + \omega)\,dy = m_0^* \,, \tag{3.37}$$

in addition to solving the condition (3.31). We are in effect seeking values for α and ω such that the mean time of breakdown and the proportion of defects arising as breakdowns satisfy the observed current inspection practice, T_0. Clearly, we require $m_0^* > T_0/2$ for the existence of a solution. This proposed method is demonstrated using simulated data in Chapter 5. Clearly, the method could also be extended to imperfect inspections by using the sample estimate of the variance of breakdown times to give three equations to solve for β, α and ω.

3.7.2 Maximum Likelihood Parameter Selection

The maximum likelihood method of estimating parameters, in general, gives more efficient estimates, i.e they become closer to the true theoretical value of the parameters rapidly as the sample size increases, see Edwards (1972). If N_b breakdowns have occurred over the interval $(0, T)$ and $\{y_j\}$, $1 \leq j \leq N_b$, are the failure times measured from 0, then due to NHPP breakdown arrivals, the likelihood, L say, is given by,

$$L_1 = \exp(-B(T))\prod_{i=1}^{N_b} r(y_i) \quad , \tag{3.38}$$

shown in Cox and Hinkley (1974) for the NHPP. The functions $r(y)$ and $B(T)$ can then be parameterized in terms of (β, α, ω). For the case of imperfect inspections the ROCOF would need to be formulated for each inspection interval over the survey period. This can be achieved by deriving the expected number of breakdowns, $B_n(y;\, T)$ say, occurring in the interval (0, y) after the n'th inspection. The ROCOF for the n'th interval from when the system was new can then be obtained by differentiating this function w.r.t y. If times of failures have not been recorded then an alternative likelihood can be formed from the observed number of breakdowns within each inspection interval. This likelihood would be constructed using function (2.32), that is the expected number of breakdowns, $B_n(T)$, occurring in the n'th inspection interval.

Over the series of inspections, the likelihood of each interval would be combined, and this can then, also, be multiplied with the likelihood of the number of defects recorded at each inspection, based on the Poisson distribution. The combined likelihood for perfect inspections, using the notation in Section 3.2, is then given by,

$$L_2 = \frac{\left(S(T_0)^{\sum_{i=1}^{M} s_i} e^{-MS(T_0)} \right)}{\prod_{i=1}^{M} s_i!} e^{-MB(T_0)} \prod_{j=1}^{B} r(y_j) \quad , \tag{3.39}$$

where $B(T)$ is the expected number of breakdowns in an inspection interval (0, T) and $S(T)$ is the expected number defects to be detected at each inspection. The function (3.39) can then be simplified to (omitting constant factors),

$$L_3 = K(T)^S e^{-MK(T)} (1 - b(T))^S \prod_{j=1}^{B} r(y_j) \quad , \tag{3.40}$$

where $K(T)$ is given by function (2.7). In the case of assumed HPP defect arrivals then $r(y) = kF(y)$, the likelihood factorizes to give,

$$L_4 = (1 - b(T))^S \prod_{j=1}^{B} F(y_j) \quad . \tag{3.41}$$

The parameters can then be selected to maximize over the resulting likelihood (or log-likelihood) function. This process will be demonstrated using simulated data in Chapter 5. When inserting the selected parameters into the function $b(T)$, the status quo condition will not necessarily be modelled exactly. Clearly, if modelling assumptions are correctly postulated the estimated value $b(T_0)$ should lie within confidence limits of b_0^*. Modelling assumptions will need to be revised in the case when this is not so.

The method to use objective data has been used for estimating the delay time distribution of a single-component system, see Christer and Wang (1992), Baker and Wang (1992).

3.8 Statistical Tests of Fit

A list is given here on the ways of testing the selected model, assuming that parameters (β, α, ω) have been estimated using any of the methods outlined in Sections 3.5, 3.6 and 3.7:

(a) Satisfying the known point, b_0^*.

(b) Chi-square test or Kilmogorov-Smirnov test, based upon the number of breakdowns in each interval being Poisson distributed.

(c) As in (b), but for the detected defects at inspections.

(d) As in (b), but where the times of failures have the p.d.f $p_b(y; T)$.

(e) Satisfying the sample mean time of failure, m_0^*.

(f) Satisfying the sample variance of time of failure, v_0^* say.

If in a practical situation, all tests (a)-(f) are positive, then the belief is reinforced that modelling assumptions and updating procedures are valid for the system in question.

If many tests are negative then we could infer that defects may not be HPP, in which case a time dependent form of the rate of occurrence of defects should be sought. Also, linear updating may not be appropriate. E.g, the delay times defects which caused breakdown under an assumed perfect inspection policy may be more accurately estimated than delay times captured at inspections. The p.d.f type for delay time may also be wrongly selected.

3.9 Other updating methods

The section outlines other updating methods for further research. For example, a non-linear delay time transformation could be applied, such as,

$$h' = \alpha h^{\gamma} + \omega \quad . \tag{3.42}$$

In this case, if h was initially estimated as being exponential, then the true delay time h' would be distributed as a three parameter Weibull.

Alternatively to this method, one could select different standard parametric distributions for the delay time distribution, e.g Weibull and Gamma, use maximum likelihood estimation for the parameters given the observations, and then carry out the goodness-of-fit tests outlined in Section 3.8.

In general, it could be assumed that there exists a conditional p.d.f of the actual delay time, h', given a subjective estimate h, say $f_c(h'; h)$. It then follows that if $f(h)$ is the prior p.d.f of delay time subjective estimates, then the p.d.f of delay time, say $f^{*}(h')$, is given by,

$$f^{*}(h') = \int_{h=0}^{\infty} f_c(h';h) f(h)\, dh \quad . \tag{3.43}$$

For the case of the delay time non-linear transform, the c.d.f form of $f_c(h'; h)$ would be the Heaviside function $H(h' - (\alpha h^{\gamma} + \omega))$.

3.10 Decision Consequence

Having discussed the updating and testing of fit, we now consider the decision consequence of the various updating procedures. That is, how would the optimal decision for $d(T)$, say, change with the update and how different is the actual associated $d(T)$ value. We shall consider this in the context of the above case example.

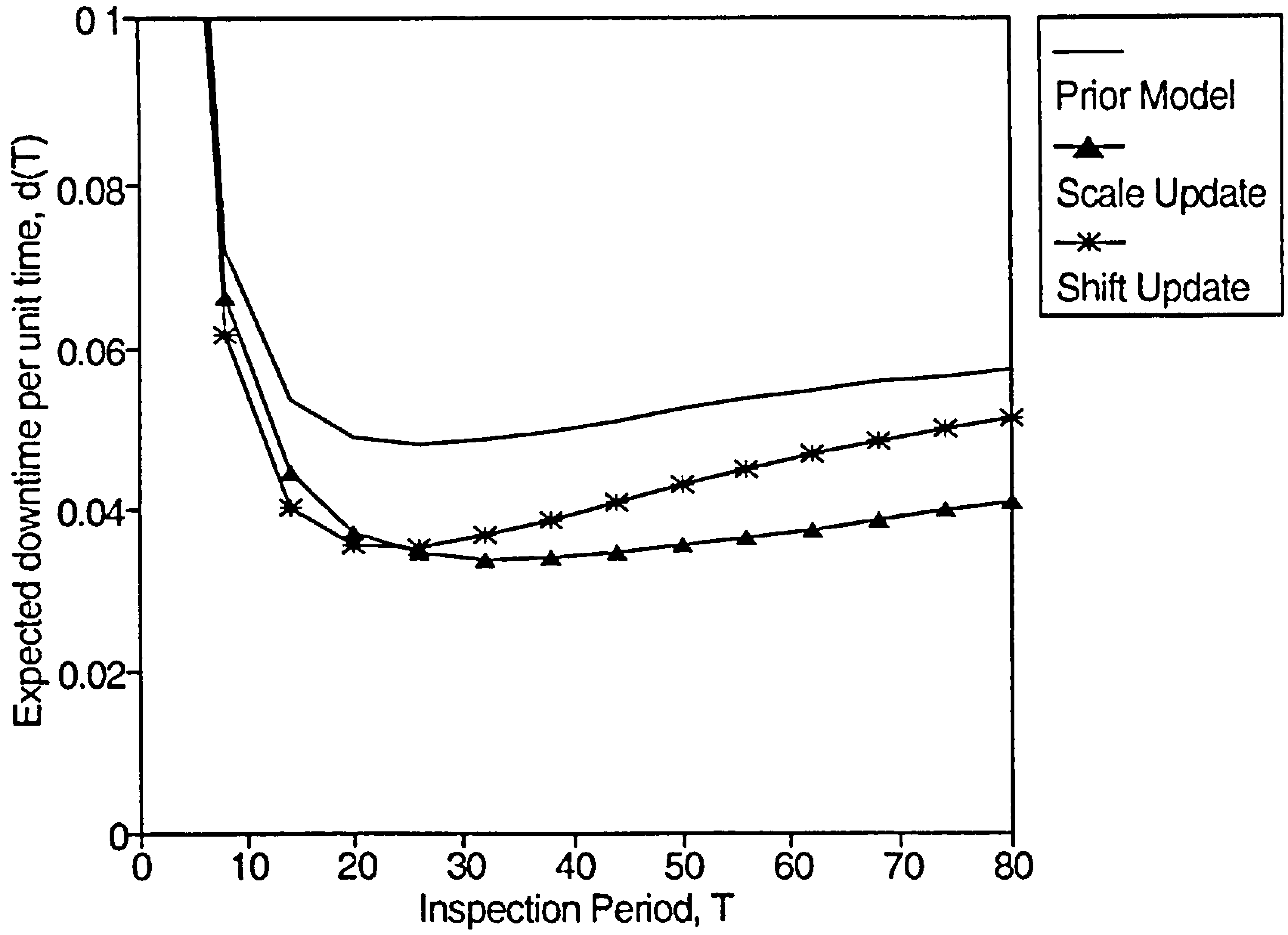

Fig. 3.7. The decision consequence, $d(T)$, for update methods α and ω.

First, we shall look at case (a) when $b_0^* = 0.2$. In Fig. 3.7 the graphs for $d(T)$, equation (2.18), using the delay time distribution update methods based on α-scale and ω-shift are shown. The point at which the update methods intersect, would be the estimated downtime per unit time d_0^* for the current practice T_0, where the values of b_0^*, k, d_b and d_I are estimated over the same time survey time period, that is $M(T_0^* + d_I)$, for which d_0^* is estimated i.e,

$$d_0^* = \frac{kT_0 d_b b_0^* + d_I}{T_0 + d_I} . \tag{3.44}$$

The initial model has its minimum around $T = 24$ hours, for the actual current practice. Whilst the minimum for the α-scale update model is greater, around $T = 36$ hours, it is evident that the ω update model is slightly less. Considering the two methods, it can be seen the optimal region is probably between 20 and 40 hours. Although the downtime

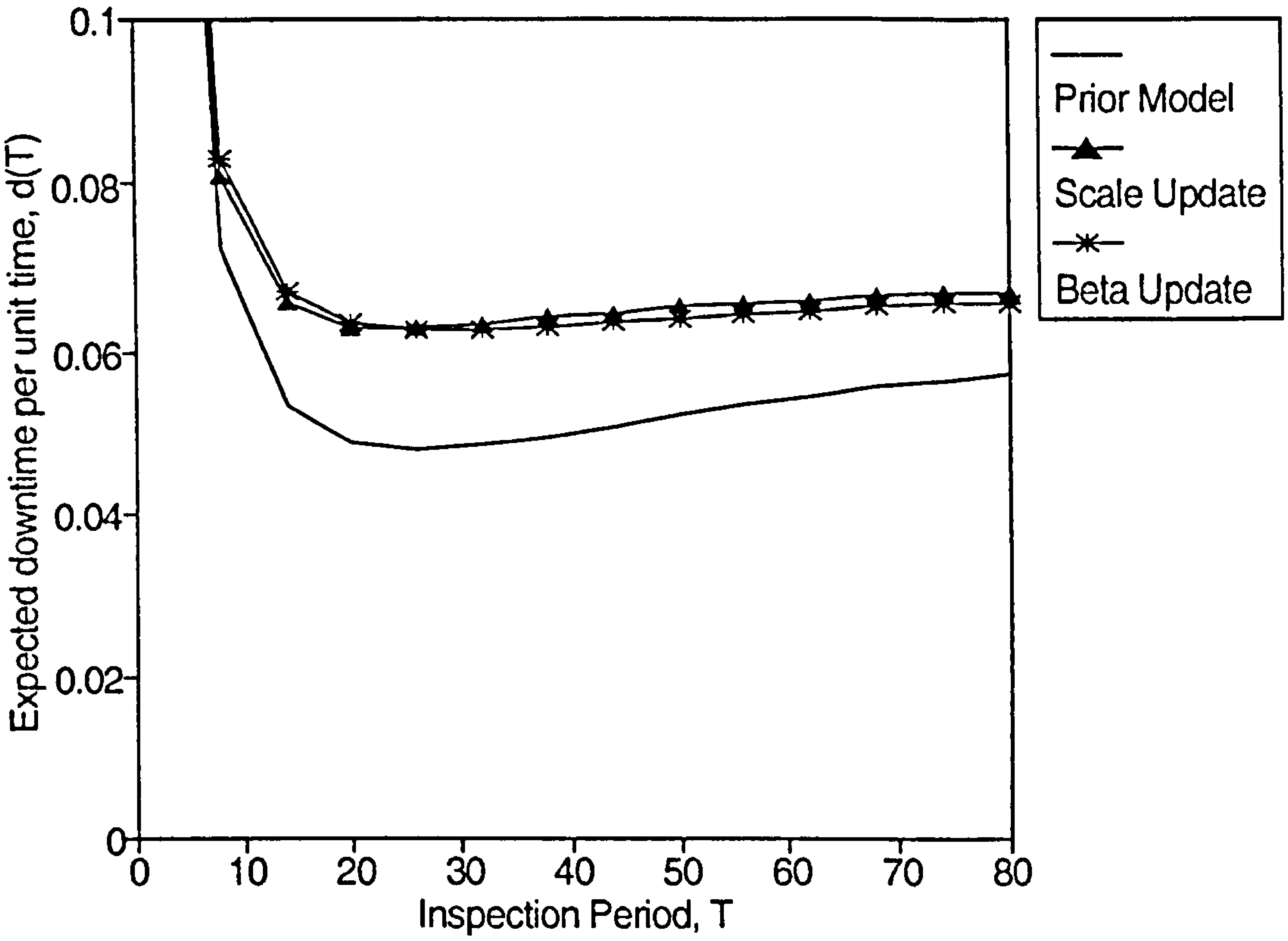

Fig. 3.8. The decision consequence, $d(T)$, with the update methods α and β.

consequences over this range from 20 to 40 indicate that considerable care needs to be taken in selecting an updating method. Secondly, we shall look at case (b) when $b_0^* = 0.6$. In Fig. 3.8 the graphs for $d(T)$ using the update methods based on α and β are shown. It can be seen that minimal change of the optimum value of $d(T)$ has resulted from these updates. Also, the change in $d(T)$ is very slight as T ranges from 10 hours to 120 hours and the value of $d(T)$ is close to the limiting value of $b(T)$ for large T, which is $kd_b = 0.07$ in this case. The value of kd_b, as shown in Chapter 2, represents the approximate expected downtime per unit time experienced under a breakdown maintenance scheme, and so for case (b) inspections may be considered, in the example, not to have much impact on reducing downtime as T increases beyond 24 hours.

3.11 Conclusion

Several formal techniques of updating delay time models have been presented. These have been based on the existence of subjective data to decide a prior or type of delay time distribution. The prior is then parameterized under a linear transform and the uniqueness and existence of a solution to modelling the "status quo" discussed. It has been found that a unique solution exists under a simple scale transform and a set of solutions under the more general linear case. The effects of changing the model, that is for $\beta = 1$ to $\beta \neq 1$ and vice-versa, or simply varying β, as another updating option has also been investigated which highlights the variety of updating options.

The effects of the change in the downtime model and consequently the optimum inspection period has been demonstrated for various updating techniques. Further research could lie in predicting the behaviour of the optimum for updating options, modelling parameters and delay time p.d.f types.

Another method of parameter estimation has been proposed based on observed times of breakdown and the defects detected at inspections. For this method, the prior distribution type can be assumed and the parameters are then determined using the method of moment or maximum likelihood technique. It has been seen that the observed failure times can be also be used in a test for fitting a model and in deciding upon a delay time prior when no delay time data is available.

Delay times may not only be biased subjectively, but also through estimates being drawn from a censored data set. Delay times, for example, may only be estimated from the failures which occur over a data collection survey. Hence, an observational bias enters the problem. The existence of this type of bias will be discussed in the next chapter along with methods for its resolution.

Chapter 4

Bias in Delay Time and Initiation Time Parameter Estimates for Censored Data

4.1 Introduction

In this chapter, the case of having a censored data sample with which to estimate a delay time distribution is discussed. As previously outlined, one situation which could arise is that delay times and initiation times of defects may only be readily estimated from either breakdown events or when defects are detected at inspections. Another situation could be a non-balanced mixture of these two extremes in that we may not be able to obtain an estimate of delay time and initiation time for each defect which has arisen over a survey period.

In the case of censored data, a bias in the estimated distribution of delay times or initiation time would exist. This will be established by deriving the respective conditional p.d.f of delay time and initiation time associated with defects which arise as breakdowns, and those which are detected at inspection. The p.d.fs will be derived for both perfect and imperfect inspection policies. A maximum likelihood estimation technique and appropriate tests of model fit are then recommended to cope with the observational bias introduced.

Much of the work of this chapter is based on the paper, Christer and Redmond (1990).

4.2 Bias in Delay Time Estimates

Practical applications of delay time analysis have so far exhibited the usual characteristics of any initial exploration or application of an applied scientific idea. The approach has been pragmatic in style in an attempt to obtain initial order-of-magnitude effects and feedback prior to refinement. Delay time estimates, h, have, for example, been derived by estimating both the delay time associated with a breakdown, $h_b = HLA$ (How long ago the defect first arisen) say, and that with the inspection repair of a defect, $h_d = HLA + HML$ (How much longer if left to deteriorate), see Figs. 4.1 and 4.2. These two data sources have then been pooled and the prior p.d.f, $f(h)$ estimated and appropriate analysis undertaken, see Christer and Waller (1984b).

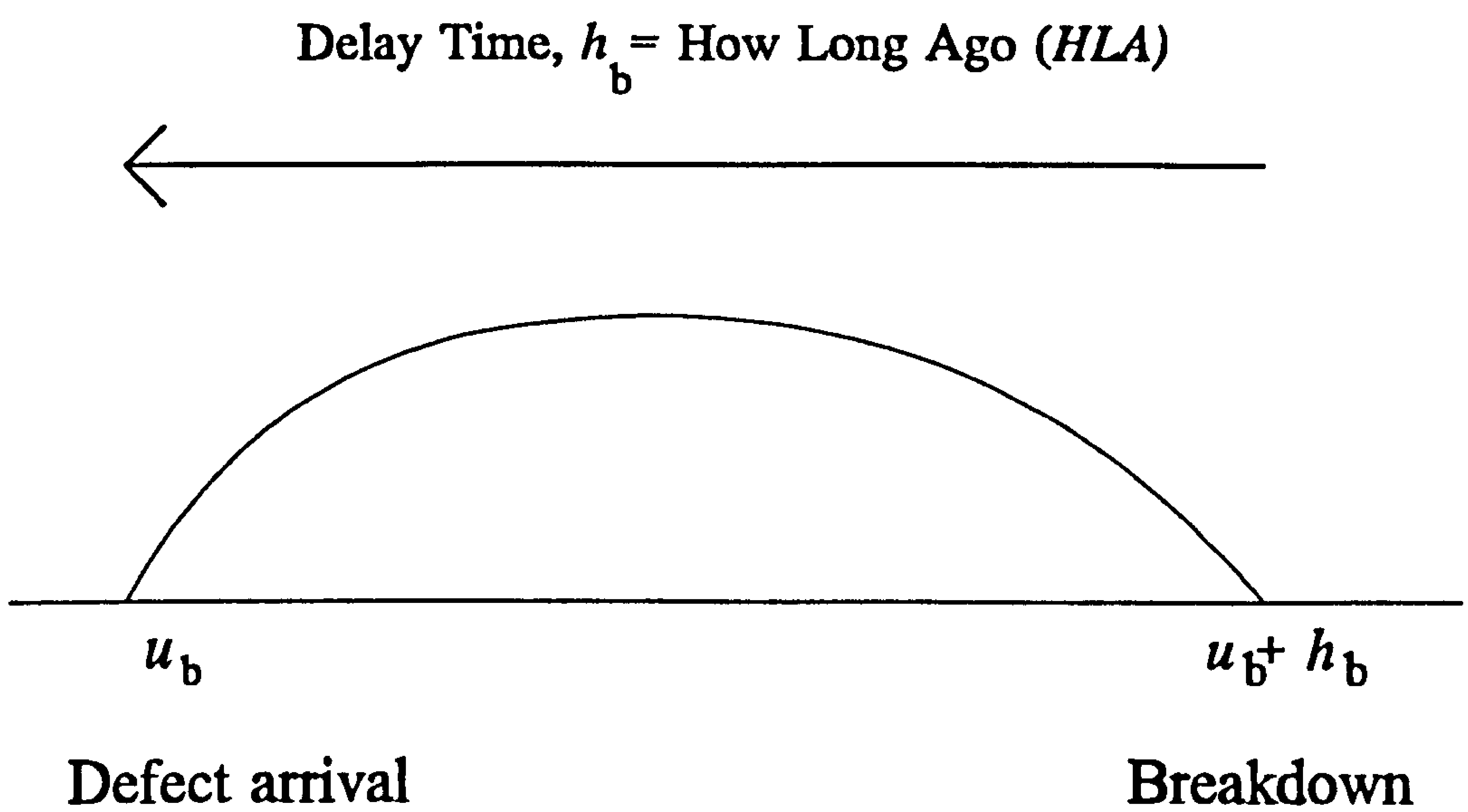

Fig. 4.1. Delay time estimated at breakdown.

In following this initial process, it is recognized that apart for a possible bias in individual estimates of delay time associated with the individual providing the estimate, an additional statistical bias may enter into the modelling.

To demonstrate this we initially assume perfect inspections and estimates of $h = h_b$ are only obtained at breakdowns. Here, a set of estimates of h_b will generally produce parameter estimates which underestimate the p.d.f of h, due to the delay times, h_b, all being constrained to be less than T. Further, a set of estimates of h_d, will produce an overestimate of the p.d.f of h, due to sampling at T which gives rise to a higher chance of estimating a longer delay time. This effect ties in with length-biased sampling discussed by Cox (1957, p.65) and with the waiting time paradox, see Feller (1970b).

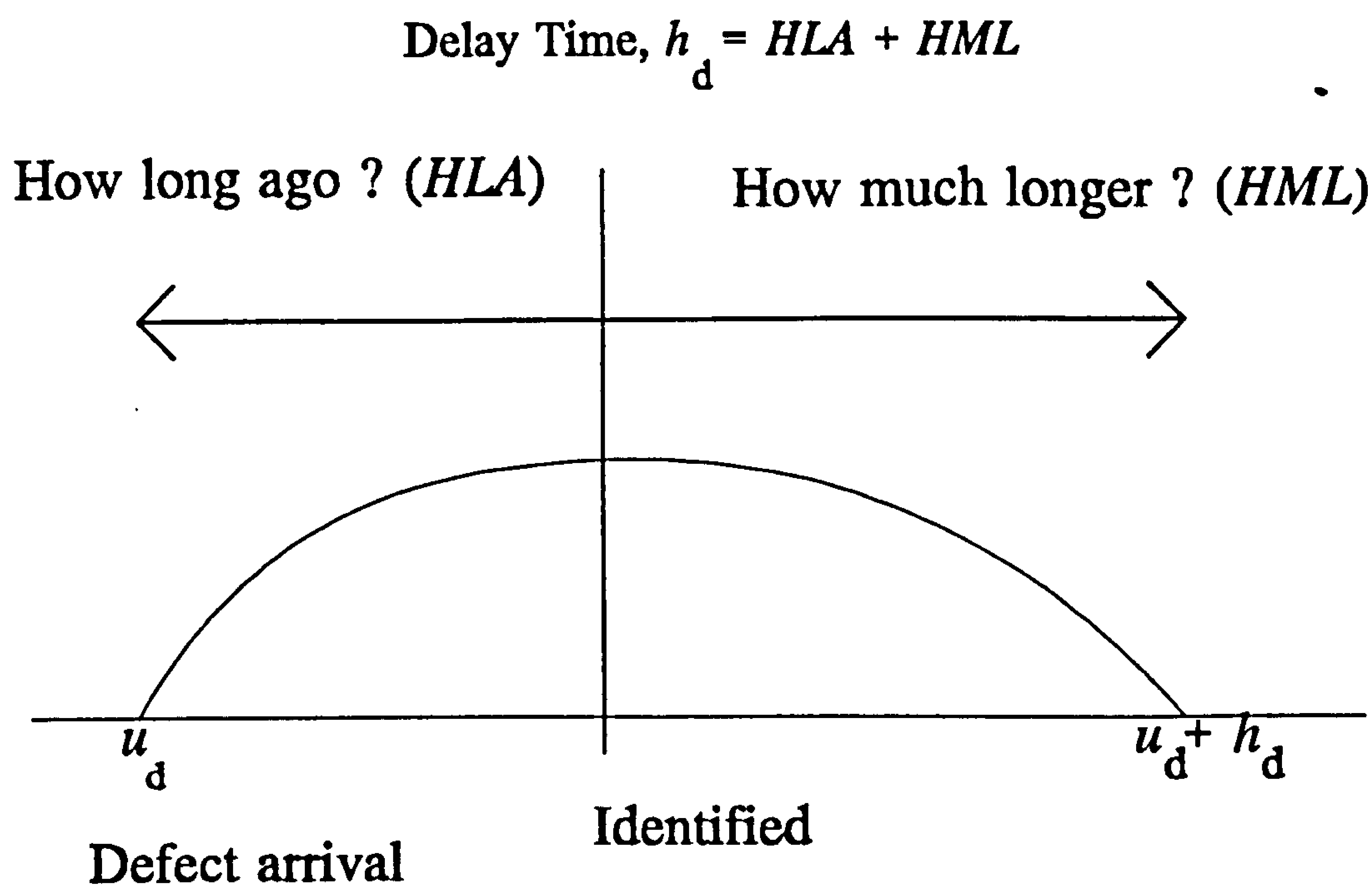

Fig. 4.2. Delay time estimated at inspections.

We imply here, that if μ_b is the expected value of h_b and μ_d is the expected value of h_d, then we infer that $\mu_b \leq \mu$ and $\mu_d \geq \mu$, where μ is the expected value of h. Strict inequality will apply in most cases. These hypotheses will be proved in the following sections. We shall deal first with the distribution of delay times based upon estimates captured at breakdowns, h_b, and then with delay times based upon inspection based estimates, h_d.

4.2.1 Delay Times associated with Breakdowns

Initially, assume a perfect periodic inspection policy T with defects having delay times distributed with p.d.f $f(h)$ and the conditional initiation times, u, having p.d.f $q(u;T)$, $u \varepsilon (0, T)$. It is also assumed that breakdowns are repaired with negligible time. We then wish to calculate the p.d.f of the delay times captured at breakdowns, h_b, say $f_b(\xi;T)$. All the delay times would evidently be less than T. Hence the p.d.f will have its domain over the interval $(0, T)$. We are, in effect, dealing with a conditional p.d.f, the condition being that a defect causes a breakdown.

Consider the probability of the event, $\{h_b \varepsilon (\xi, \xi + d\xi)\}$, for any ξ, that is, $f_b(\xi; T)d\xi$ for small $d\xi$. If we now let b be the event that a defect leads to a breakdown when inspection is on period T, and let H be the event that the delay time, h, of a defect satisfies, $h \varepsilon (\xi, \xi + d\xi)$, it follows that,

$$P\{h_b \ \varepsilon \ (\xi, \xi + d\xi)\} = P\{H|b\} \ . \qquad (4.1)$$

Using Bayes' theorem, we have for $P\{b\} > 0$ the relation,

$$P\{H|b\} = \frac{P\{H,b\}}{P\{b\}} = \frac{P\{H\}P\{b|H\}}{P\{b\}} \ . \qquad (4.2)$$

In the case $P\{b\} = 0$ there is no data set $\{h_b\}$ to consider, and for the current case, therefore, we assume $P\{b\} > 0$. We have by definition that the probability of H is $f(\xi)d\xi$. Therefore, if we let $d\xi$ tend to zero in equation (4.1), using result (4.2), we can obtain the formula,

$$f_b(\xi;T) = \frac{f(\xi)P\{b|h=\xi\}}{P\{b\}} \ . \qquad (4.3)$$

The probability of b is $b(T)$, which is given by function (2.29),

$$b(T) = \int_{h=0}^{T} Q(T - h; T)f(h)dh \ , \qquad (4.4)$$

where $Q(u; T)$ is the c.d.f of u. It remains to calculate, $P\{b|h = \xi\}$, the probability that a defect leads to a breakdown given delay time $h = \xi$. If $\xi > T$ the defect would be detected by an inspection, assuming inspections are perfect. If $\xi < T$ the defect would cause a breakdown if the initiation time $u < T - \xi$, which is an event with probability $Q(T - \xi; T)$. Hence we obtain the p.d.f over the domain $(0, T)$,

$$f_b(\xi;T) = \frac{f(\xi)Q(T - \xi;T)}{b(T)} \qquad for \quad 0 < \xi < T \ . \qquad (4.5)$$

A c.d.f form of this p.d.f, $F_b(\xi; T)$ say, was derived in Christer and Redmond (1990), namely,

$$F_b(\xi;T) = \begin{cases} \frac{1}{b(T)}\left(F(\xi)Q(T - \xi;T) + \int_{u=T-\xi}^{T} q(u;T)F(T - u)du\right) & \xi < T \ , \\ 1 & \xi \geq T \end{cases} \qquad (4.6)$$

and is equivalent to the integral of function (4.5).

We can investigate the mean μ_b of $f_b(\xi; T)$ by noticing that the function $Q(T - \xi;T)$ monontonically decreases from 1 to 0 as ξ ranges from 0 to T. Therefore, there exists a unique value $\xi = \theta$ such that $Q(T - \theta; T) = b(T)$. Evidently, the p.d.f f_b would lie above f over the interval $(0, \theta)$ and below f over the interval (θ, T), see Fig.4.3 for an example, whereby θ is the delay time value at the intersection of the three p.d.fs. It follows that the c.d.f of h_b will lie above the c.d.f of h over the interval $(0, \theta)$. Hence, the associated reliability function (r.f) of h_b will lie below the r.f of h over this interval. It can be readily seen that over the interval (θ, T), the r.f of h_b will also lie below the r.f of h. It is a well known result that the mean of a non-negative random variable is the integral of its r.f. Hence, we obtain the expected result,

$$\mu_b < \mu \ , \tag{4.7}$$

that is the delay time parameter estimates associated with failures are biased. Clearly, as $T \to \infty$, $f_b \to f$, that is estimates of h_b becomes asymptotically unbiased with T. An example of bias is provided by the special case when delay times are exponential and defects arise uniformly as a HPP. Let $a = 1/\mu$ be the exponential parameter, then we obtain from equation (4.4),

$$f_b(\xi;T) = \frac{a^2(T - \xi)e^{-a\xi}}{aT + e^{-aT} - 1} \qquad for \quad 0 < \xi < T \ . \tag{4.8}$$

As anticipated, it can be seen that $f_b \to f$ as $T \to \infty$ by replacing e^{-aT} by its Maclaurin series. A numerical example is given in Fig.4.3 for $T = 10$ and $a = 0.2$. The mean μ_b in this case is equal to 2.38, whereas $\mu = 5$. It is evident here that the biased p.d.f of h can be radically different from the true distribution.

4.2.2 Delay Times associated with Inspected Defects

In this case, we seek the p.d.f, say $f_d(\xi; T)$, of delay times, h_d, which span the inspection point T, again assuming the inspection at T is perfect. That is we require the conditional p.d.f of delay time given a defect is detected and repaired at the inspection T. Let d be the event that a defect is detected at an inspection, then by following the process of analysis leading to equation (4.3), it can be shown that for $P\{d\} > 0$,

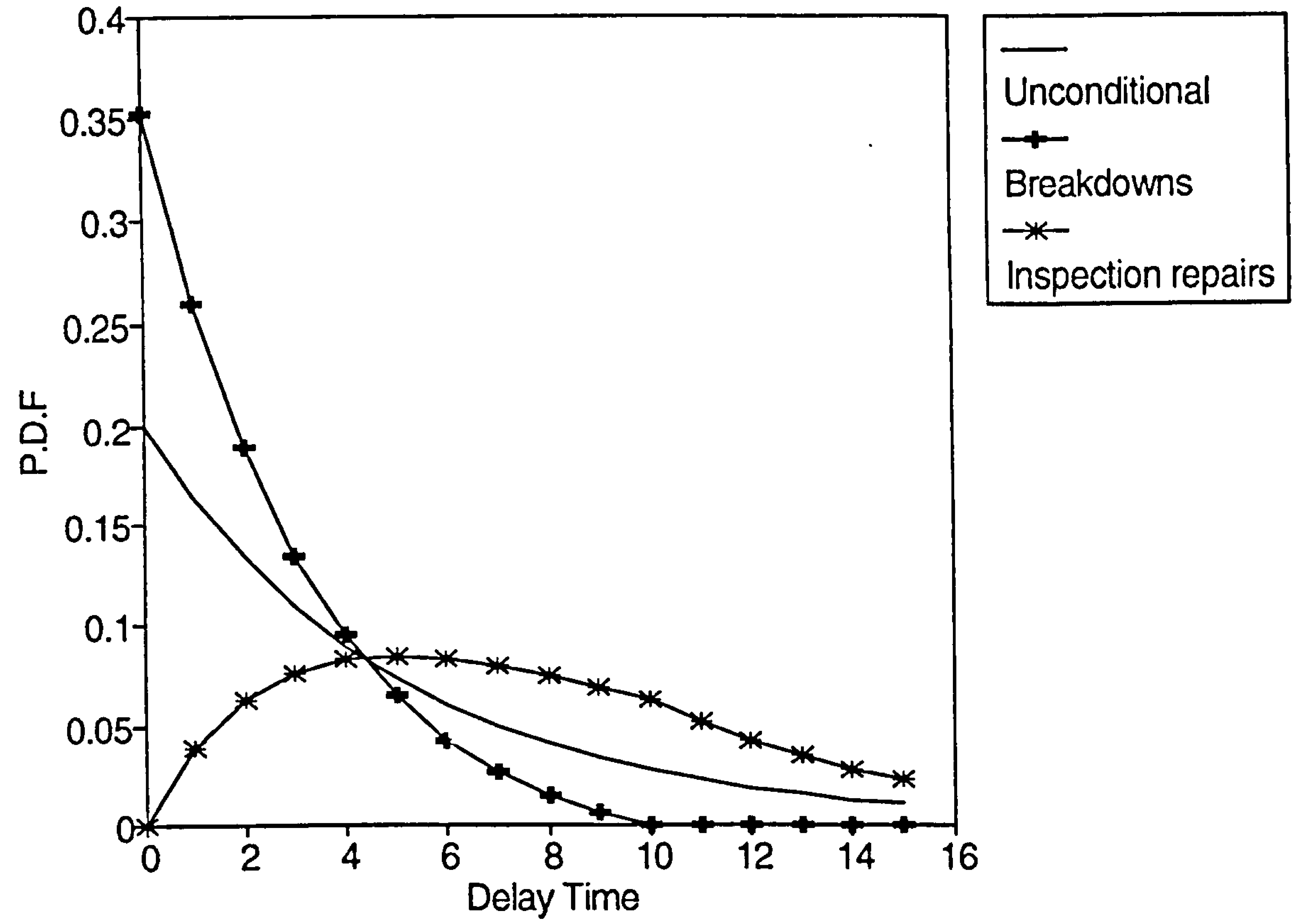

Fig. 4.3. An example of unconditional and conditional p.d.fs of delay time.

$$f_d(\xi;T) = \frac{f(\xi)P\{d|h=\xi\}}{P\{d\}} , \quad if \quad P\{d\} > 0 \ . \tag{4.9}$$

We have for a defect that $P\{d\} = 1 - b(T)$, and therefore need to consider the probability $P\{d \mid h = \xi\}$. For the case, $\xi > T$, we have that $P\{d \mid h = \xi\} = 1$, i.e the defect will be detected given its delay time is greater than T and the inspection is perfect. When $\xi<T$, then the defect will be detected if its initiation time satisfies $u > T - \xi$, which is an event with probability $1 - Q(T - \xi; T)$. Hence, the required p.d.f of detected delay times is given by,

$$f_d(\xi;T) = \begin{cases} \dfrac{f(\xi)(1 - Q(T - \xi;T))}{1 - b(T)} & for \ \ 0 < \xi < T \\ \dfrac{f(\xi)}{1 - b(T)} & for \ \ \xi \geq T \ . \end{cases} \tag{4.10}$$

As $T \rightarrow 0$, it can be seen $f_d \rightarrow f$, that is h_d becomes asymptotically unbiased as inspections become more frequent. This behaviour is to be expected. A c.d.f form of f_d, say $F_d(\xi; T)$, was derived in Christer and Redmond (1990) namely,

$$F_d(\xi;T) = \int_{u=\max(0,T-\xi)}^{T} q(u;T)\frac{[F(\xi) - F(T-u)]}{1 - b(T)}du \ , \qquad (4.11)$$

and is equivalent to the integral of function (4.10). Returning to the special case example in Section 4.2.1, we find that the p.d.f (4.10) takes the form,

$$f_d(\xi;T) = \begin{cases} \dfrac{\xi a^2 e^{-a\xi}}{1 - e^{-aT}} & \text{for } 0 < \xi < T \\ \dfrac{T a^2 e^{-a\xi}}{1 - e^{-aT}} & \text{for } \xi \geq T \ . \end{cases} \qquad (4.12)$$

As anticipated, $f_d \to f$ as $T \to 0$, which is established using the Maclaurin series for e^{-aT}. The expected value of h_b is,

$$\mu_d = \frac{2}{a} - \frac{T}{(e^{aT} - 1)} \ , \qquad (4.13)$$

which has a limit of $1/a$ when $T \to 0$ and $2/a$ when $T \to \infty$, and corresponds to failure based maintenance. Again, use of the Maclaurin series is required to establish these limiting results. The limit when $T \to \infty$ compares to the limiting p.d.f of the length-biased sampling example given in Cox(1957, p.65). The p.d.f $f_d \to a\xi f(\xi)$ as $T \to \infty$. The differential of equation (4.10) w.r.t T, is given by,

$$\frac{d(\mu_d)}{dT} = \frac{e^{aT}(aT - 1) + 1}{(e^{aT} - 1)^2} \ . \qquad (4.14)$$

This can be seen to be positive for all values of $T > 0$. Therefore the expected value, μ_d, monotonically increases from $1/a$ to $2/a$, as T increases. In the above example, it is evident that $\mu_d > \mu$. That is delay times captured at inspections give rise to an overestimate of h. In the general case, we can confirm this by noting that,

$$E(h) = E(h|b)P\{b\} + E(h|d)P\{d\} \Rightarrow$$

$$\mu = \mu_b b(T) + \mu_d(1 - b(T)) \ . \qquad (4.15)$$

In Section 4.2.1 it was shown that $\mu_b < \mu$. Hence re-arranging the above equation to obtain μ_b, it follows that,

$$\mu_b = \frac{\mu - \mu_d(1 - b(T))}{b(T)} < \mu \Rightarrow \mu_d > \mu. \tag{4.16}$$

Hence, in the general case the delay times of defects captured only at inspections are biased with an expected value greater than the delay time mean. A numerical example when $T = 10$ and $a = 0.2$ is also given in Fig.4.3 with $\mu_d = 8.34$. It can be seen that all three p.d.fs intersect at the same point. The delay time at this point ($\theta = 4.34$), is such that a defect with this delay time value will have probability $Q(T - \theta;T) = b(T)$ of causing a breakdown, and it can be seen that the delay time p.d.f in equations (4.5) and (4.10) at $\xi = \theta$ will equal $f(\theta)$.

4.3 The Bias in the Initiation Times

We shall proceed to derive the p.d.fs of the initiation time associated with failures and defects repaired at inspections. It will then be shown that the initiation time parameter estimates are also biased under censored data. This is reasonable to expect, because the initiation times of, for example, breakdowns, would be more likely to occur in the first half of the inspection interval under a perfect inspection policy. In the case of inspection identified and repaired defects, the initiation time would more likely to be in the latter half of the inspection interval.

Let $q_b(\zeta; T)$ be the p.d.f of initiation time, u_b say, for defects which result in failures, where $\zeta \in (0, T)$. It follows from analysis in Section 4.2.1 that the p.d.f would be given by,

$$q_b(\zeta;T) = \frac{q(\zeta;T)P\{b|u=\zeta\}}{b(T)} . \tag{4.17}$$

Essentially, h has been replaced by u and f by q in function (4.3) due to independence between u and h. Given the initiation time of a defect, $u = \zeta$, the defect will cause a breakdown if its delay time $h < T - \zeta$, which is an event with probability $F(T - \zeta)$. Therefore, the p.d.f of u_b is given by,

$$q_b(\zeta;T) = \frac{q(\zeta;T)F(T-\zeta)}{b(T)} \qquad \textit{for} \quad 0 < \zeta < T \ . \tag{4.18}$$

We shall now investigate the mean of u_b, say η_b. In Section 2.5.5, it was shown that for perfect inspections, $b(T) \leq F(T)$. Hence the p.d.f q_b lies above or at q at $\zeta = 0$. At $\zeta = T$, the p.d.f then must lie below or at q, since $F(T - \zeta)$ is a monotonically decreasing function. In the strictly decreasing case, there will exist only one point θ such that $F(T - \theta) = b(T)$. Hence, it follows from analysis on the conditional p.d.f f_b, in Section 4.2.1, that if the mean of the initiation time is η then,

$$\eta_b < \eta \ . \tag{4.19}$$

That is the initiation times of defects which cause breakdowns are biased with an expected value less than the population initiation time.

The initiation times of defects which are detected and repaired, say u_d, will also be biased. In order to calculate the p.d.f of u_d, it is evident from previous arguments that we require the probability that a defect will be detected given its initiation time, $u = \zeta$, that is $P\{d \mid u = \zeta\}$, for $\zeta \ \varepsilon \ (0, T)$. In this case, it is clear this is the probability that a delay time satisfies $h > T - \zeta$, which is given by $1 - F(T - \zeta)$. Let $q_d(\zeta; T)$ be the p.d.f of u_d, then as before it follows that,

$$q_d(\zeta;T) = \frac{q(\zeta;T)[1 - F(T-\zeta)]}{1 - b(T)} \ . \tag{4.20}$$

To show the expected bias exists, we can follow a similar argument to that used in the case of breakdown initiation times. Alternatively, if we let η_d be the expected value of u_d, then,

$$E(u) = E(u|b)P\{b\} + E(u|d)P\{d\} \ , \tag{4.21}$$

that is,

$$\eta = \eta_b b(T) + \eta_d(1 - b(T)) \ . \tag{4.22}$$

Hence, due to the fact that $\eta_b < \eta$, we obtain the result,

$$\eta_d > \eta \ , \tag{4.23}$$

that is the initiation times of defects which are repaired at inspections will be biased with a greater expected value.

4.4 Imperfect Inspections ($\beta \neq 1$)

Here we relax the condition of perfect inspections and seek the p.d.fs of delay time and initiation time, for breakdowns and inspection repairs. It will be assumed that the probability a defect is detected if present at an inspection is β and defects arise uniformly over time, i.e as a homogeneous Poisson process. Breakdown and inspection repairs carried out in different inspection intervals will have non-identical conditional p.d.fs of delay time and initiation time. However, here we shall seek the derivation of the asymptotic p.d.fs, i.e assuming data has been collected over a sufficiently large number of inspection intervals, so that the theoretical probability that a defect will arise as a breakdown is given by $b(T;\ \beta)$, function (2.30). The conditional p.d.fs of delay time will then be given by function (4.3) and (4.9). Hence in order to calculate the p.d.f of delay times which cause breakdowns, we require the probability that a defect will cause a breakdown, $P\{b \mid h = \xi\}$, given $h = \xi$ for imperfect inspections.

First, we consider the case $\xi < T$ in which the event that a breakdown will occur corresponds to the initiation time satisfying $u < T - \xi$, which has probability $(T - \xi)/T$, or when $u > T - \xi$ and the defect is not detected at the inspection at T, which has probability, $(1 - \beta)\xi/T$. Hence, summing these two probabilities,

$$P\{b|h{=}\xi\} \ = \ \frac{T - \beta\xi}{T} \ , \qquad 0 \leq \xi < T \ . \tag{4.24}$$

Next, consider the general case $\xi \in [nT, [n + 1]T)$ for $n \geq 1$. Again, a breakdown can occur in two ways, that is in the interval $(nT, [n + 1]T)$ or $([n + 1]T, [n + 2]T)$. For the former event, $u < (n + 1)T - \xi$ and the defect remained undetected on the previous n inspections. For the latter event, $u > (n + 1)T - \xi$ and the defect remained undetected on $n + 1$ sequential inspections. It follows that the probability of a defect with delay time $h = \xi \in [nT, [n + 1]T)$ arises as a breakdown, is given by,

$$P\{b|h=\xi\} = (1-\beta)^n\frac{[(n+1)T-\xi]}{T} + (1-\beta)^{n+1}\frac{(\xi-nT)}{T} \tag{4.25}$$

$$= \frac{(1-\beta)^n}{T}(T-\beta(\xi-nT)) \quad , \quad nT \le \xi < (n+1)T \ .$$

This has the same format as (4.24) in the case of $n = 0$ and we have (4.25) is valid for all n. Therefore, defining $f_b(\xi;\ T,\ \beta)$ as the p.d.f of delay times of breakdowns h_b, we obtain, using the equation (4.3) and writing n as $[\xi/T]$, the integer part of ξ/T,

$$f_b(\xi;T,\beta) = \frac{f(\xi)(1-\beta)^{[\xi/T]}(T-\beta(\xi-[\xi/T]T))}{Tb(T;\beta)} \ . \tag{4.26}$$

Returning to the discussion on bias, we can see that the p.d.f (4.26) for f_b is not equal to f. Further, at $\xi = 0$, $f_b > f$ when $b(T;\beta) < 1$, and as $\xi \to \infty$, $f_b \to 0$ below f. The function $P(b|h=\xi)$ is monotonically decreasing. Hence it follows the p.d.f f_b will intersect f only once. Thus, we obtain as in the case of perfect inspections,

$$\mu_b < \mu \ , \tag{4.27}$$

where μ_b is the mean of h_b and μ is the mean of delay time h.

We consider now the case of defects detected at inspections. To derive the p.d.f of delay time, we evidently require the probability a defect is detected and repaired at a non-perfect inspection of period T given the delay time $h = \xi$, $P\{d \mid h = \xi\}$. This is simply given by equation (4.25) as,

$$P\{d|h=\xi\} = 1 - P\{b|h=\xi\} = 1 - \frac{(1-\beta)^{[\xi/T]}}{T}(T-\beta(\xi-[\xi/T]T) \ . \tag{4.28}$$

Defining $f_d(\xi;\ T,\ \beta)$ as the p.d.f of delay times h_d, then it follows using equation (4.9) that,

$$f_d(\xi;T,\beta) = \frac{f(\xi)\ \{T-(1-\beta)^{[\xi/T]}(T-\beta(\xi-[\xi/t]T))\}}{T(1-b(T;\beta))} \ . \tag{4.29}$$

The expected delay time μ, μ_b and μ_d are clearly connected by the relationship,

$$\mu = b(T;\beta)\mu_b + (1 - b(T;\beta))\mu_d \quad . \tag{4.30}$$

Hence, due to the condition $\mu_b < \mu$, it follows that $\mu_d > \mu$.

4.4.1 Initiation times when $\beta \neq 1$

Assuming imperfect inspections, the parameter estimates of the initiation times of defects will also be biased. In order to derive the p.d.fs, we require the probability that a defect will be detected given its initiation time $u = \zeta$, for $\zeta \in [0, T)$. This has been derived in Christer and Waller (1984a),

$$\begin{aligned} P\{d|u = \zeta\} &= \sum_{n=1}^{\infty} \beta(1 - \beta)^{n-1}(1 - F(nT - \zeta)) \\ &= 1 - \beta\sum_{n=1}^{\infty}(1 - \beta)^{n-1}F(nT - \zeta) \quad . \end{aligned} \tag{4.31}$$

Letting $q_d(\zeta;\, T,\, \beta)$ and $q_b(\zeta;\, T,\, \beta)$ be the p.d.fs of the initiation times, u_d and u_b respectively, it follows from previous analysis using functions (4.17) and (4.20) that,

$$q_d(\zeta;T,\beta) = \frac{1 - \beta\sum_{n=1}^{\infty}(1 - \beta)^{n-1}F(nT - \zeta)}{T(1 - b(T;\beta))} \quad , \tag{4.32}$$

and

$$q_b(\zeta;T,\beta) = \frac{\beta\sum_{n=1}^{\infty}(1 - \beta)^{n-1}F(nT - \zeta)}{Tb(T;\beta)} \quad . \tag{4.33}$$

It can be seen the p.d.f q_b monotonically decreases over the interval $\zeta \in (0, T)$. Therefore due to q_b being a p.d.f, q_b must lie above q at $\zeta = 0$ and below q at $\zeta = T$. Therefore, the mean of u_b, will be less than the mean of u. The expected value of u for HPP defect arrivals is $T/2$. If η_b is the expected value of breakdown initiation times and η_d is the expected value of initiation times for inspection repairs, then it follows from previous analysis that we have the following relationships,

$$\eta_b < \frac{T}{2} \quad , \tag{4.34}$$

$$\eta_b b(T;\beta) + \eta_d(1 - b(T;\beta)) = \frac{T}{2} , \tag{4.35}$$

$$\eta_d > \frac{T}{2} . \tag{4.36}$$

We conclude that for both perfect and imperfect inspections, delay times and initiation time parameter estimates will be biased given a censored data set. The delay times and initiation times of defects repaired at inspection give rise to an overestimate of their respective mean values. For the case of breakdown repairs, the delay time and initiation time would provide an underestimate of their respective expected values. In practical situations it may be necessary to collect censored data, and we now propose methods to cope with a censored data set.

4.5 Correcting for Bias in Estimates of Delay Time and Initiation Time

Suppose that at least one of the following data sets of estimates have been obtained from a field subjective data collection experiment, observing a total of, B say, breakdowns and, D, inspection repairs of defects, namely,

1. Set $\{h_{b,i}\}$, of failure based estimates of delay times, $i = 1,\ldots,M_1$, where $M_1 \leq B$, i.e the situation allows for not having captured all delay time estimating opportunity available. It then follows the set $\{u_{b,i}\}$, of failure initiation times, corresponding to the i'th delay time, will also be available.

2. Set $\{h_{d,i}\}$, of inspection-repair based estimates of delay times, $i = 1,\ldots,M_2$, where $M_2 \leq D$, which implies not all inspection opportunities to estimate the delay time are assumed taken. The set $\{u_{d,i}\}$, of inspection-repair initiation times estimates, corresponding to the i'th delay time estimate is likewise available.

The initiation times and times of failure are assumed to be measured from the start of the inspection interval in which the defect or failure occurred. A prior p.d.f of delay time, $f(h, \underline{\lambda})$ say, will be assumed available, where $\underline{\lambda}$ is a set of parameters to be estimated. Likewise, the defect arrival rate will be parameterised as $g(u, \underline{\gamma})$. The prior format may be inferred from initial histogram plots of the data sets. A prior value of the perfectness of inspection, β_0 say, will also be assumed available.

4.5.1 Perfect Inspections, $\beta_0 = 1$.

The p.d.fs of the initiation and delay times have been derived in Sections 4.2 and 4.4. Hence, we can now formulate the likelihood for each data set:

$$L_1(\underline{\lambda},\underline{\gamma}) = \prod_{i=1}^{M_1} f_b(h_{b,i};T,\underline{\lambda},\underline{\gamma}) \quad , \qquad (4.37)$$

$$L_2(\underline{\lambda},\underline{\gamma}) = \prod_{i=1}^{M_2} f_d(h_{d,i};T,\underline{\lambda},\underline{\gamma}) \quad , \qquad (4.38)$$

$$L_3(\underline{\lambda},\underline{\gamma}) = \prod_{i=1}^{M_1} q_b(u_{b,i};T,\underline{\lambda},\underline{\gamma}) \quad , \qquad (4.39)$$

and

$$L_4(\underline{\lambda},\underline{\gamma}) = \prod_{i=1}^{M_2} q_d(u_{d,i};T,\underline{\lambda},\underline{\gamma}) \quad , \qquad (4.40)$$

Due to the perfect inspection assumption, it will be assumed that the delay times of breakdowns, $\{h_{b,i}\}$, are all less than T.

Maximum likelihood estimation could be applied to each likelihood function and parameter estimates can be compared. The maximum likelihood method proposed in Christer and Redmond (1990), is based on combining the delay time likelihoods L_1 and L_2. Indeed the initiation time likelihood functions, L_3 and L_4, could also be combined and parameter estimation carried out. The joint likelihood of only observing the breakdown data set (1), would not be the product of L_1 and L_3 due to the delay time and initiation time of each breakdown being dependent. To derive the joint likelihood, we first condition on $u_b = \zeta$ say. The p.d.f of the delay time h_b would then be $f(\xi)/F(T - \zeta)$,

$0 \le \xi \le T - \zeta$. Hence, combining this p.d.f with the p.d.f of u_b, function (4.16), it follows after simplification that the joint likelihood of observing breakdown data set (1) is given by,

$$L_b = \prod_{i=1}^{M_1} \left(\frac{f(h_{b,i};\underline{\lambda})\,q(u_{b,i};\underline{\gamma})}{b(T;\underline{\lambda},\underline{\gamma})} \right) . \tag{4.41}$$

Likewise the joint likelihood of observing the inspection repair delay time set (2) is given by,

$$L_d = \prod_{i=1}^{M_2} \left(\frac{f(h_{d,i};\underline{\lambda})\,q(u_{d,i};\underline{\gamma})}{1 - b(T;\underline{\lambda},\underline{\gamma})} \right) . \tag{4.42}$$

It can be seen that when the p.d.f of u is assumed to be known, e.g uniform, the likelihoods L_b and L_1 are equivalent, in respect to estimation, after omitting the factors $q(u)$ and $Q(u)$ respectively from each likelihood. Similarly, L_d and L_2 would be equivalent for this case. This is expected due to selecting the prior distribution of u. If data is available for both breakdowns and inspection repairs, then the combined likelihood, i.e the product $L_b L_d$, can also be used to provide parameter estimates.

It is evident that a number of likelihoods could be used to obtain parameter estimates. This is an advantage, because if we obtain similar maximum likelihood estimates from different likelihood functions then greater confidence exists in modelling assumptions, such as perfect inspections or it may be the assumption of HPP defect arrivals. If parameter estimates differ sufficiently the assumptions will need to be revised. Another estimation procedure could be to allow the inspection period T be a parameter to be estimated in the maximum likelihood process. The estimated T, say T^*, should then be approximately equal to T for the model assumptions to hold. However, a large amount of data may be required for this test to work, due to the increased number of parameters in the likelihood function.

It will now be shown that given an uncensored data set, i.e when $M_1 = B$ and $M_2 = D$, the maximum likelihood estimate of parameters for an initiation time and delay time distribution will be unbiased (in the asymptotic sense.) Assume B breakdowns have arisen in one inspection interval $(0, T)$ and D defects were repaired at the inspection.

Due to B and D being independently Poisson distributed the probability, say P, of observing this joint event is,

$$P = \frac{e^{-B(T)}B(T)^B}{B!}\frac{e^{-(K(T)-B(T))}(K(T)-B(T))^D}{D!}$$

$$= \frac{e^{-K(T)}K(T)^{B+D}}{B!D!}b(T)^B(1-b(T))^D \quad . \tag{4.43}$$

To obtain the joint likelihood, say L, of observing B breakdowns, D defect repairs and the delay time and initiation time sets (1) and (2), we need to multiply P by L_b and L_d. By writing $q(u)$ as $g(u)/K(T)$, then cancelling $b(T)$ and simplifying, we are left with,

$$L = \frac{e^{-K(T)}}{B!D!}\prod_{i=1}^{B} f(h_{b,i})g(u_{b,i})\prod_{i=1}^{D} f(h_{d,i})g(u_{d,i}) \quad , \tag{4.44}$$

where it can be seen that the pooled likelihood of observing both delay time sets, i.e,

$$\prod_{i=1}^{B} f(h_{b,i})\prod_{i=1}^{D} f(h_{d,i}) \quad , \tag{4.45}$$

factorizes. Also, the remaining part of the likelihood function L is the likelihood of observing $B + D$ defects with an assumed ROCOF $g(u; \gamma)$ over the interval $(0, T)$. Over a series of inspections, events over each interval are assumed independent in the case of perfect inspections. Hence, parameters estimates of initiation time and delay time distributions can be undertaken without considering bias when data is collected from a complete set of inspection and breakdown estimates.

4.5.2 Imperfect Inspections

The maximum likelihood techniques employed in the previous section could also be used if imperfect inspections and HPP defect arrivals were assumed. The p.d.fs derived in Section 4.4, which correspond to each data set, would be used with $\beta = \beta_0$. If β is unknown then this could also be allowed to be estimated by the maximum likelihood process. This method could also be used to confirm perfect inspections. Again T could be used as a parameter to confirm the model.

4.6 Additional Tests of Model Fit

Splitting the delay time and initiation time data into disjoint sets based on failures and inspection repairs can provide additional tests of fit to those given in Chapter 3, even with uncensored data. The procedure can also help in verifying modelling assumptions. Let the model parameter estimates be $(\underline{\lambda}^*, \underline{\gamma}^*)$ for perfect inspections with NHPP defect arrivals or $(\underline{\lambda}^*, \beta^*)$ for imperfect inspections with HPP defect arrivals. We could, for example, then proceed with distribution tests such as K-S or chi-square on all or some of the data sets of delay time and initiation, assuming enough data is available for each case.

4.7 Conclusion

It has been seen that a statistical bias can exist in the data leading to parameter estimates of delay time and initiation time for censored data. Breakdown based observations give rise to an underestimate of delay time and initiation time, whilst observations based upon defects identified and repaired at inspections give rise to overestimates. The bias has been shown to be dependent on inspection frequency and the perfectness of inspections.

Methods based on maximum likelihood have been proposed to correct for the bias have been proposed, which leads to the estimation of the actual initiation time and the delay time distributions.

In the case of censored data, it may be possible that parameters for the delay time distribution can be estimated to an acceptable degree of accuracy by updating procedures instead of performing the bias correction. Some form of iteration method could also be adopted. The task of parameter fitting will be further investigated in the next chapter on simulation.

Chapter 5

Simulation Study of the Delay Time Model

5.1 Introduction

A simulation study is undertaken to further investigate and verify the delay time models and proposed method of analysis of earlier chapters. Simulation programs, written in Pascal, have been used to simulate the delay time process given sets of input parameters and assumptions. Simulation algorithms are derived for the case of perfect inspections and instantaneous repair of breakdowns. Then, these modelling assumptions are relaxed to imperfect inspections and non-instantaneous breakdown repairs. The outputs of simulation experiments are analyzed and compared to the appropriate theoretical values of the models of the earlier chapters.

An investigation is undertaken into the accuracy and effectiveness of the parameter estimation procedures given in Chapters 3 and 4. Correction of bias is carried out on censored simulated data. The effects of not correcting for bias but using an updating method, as a further option, is also explored. An iteration method is developed which alternates between updating the scale parameter of a Weibull delay time distribution and only estimating the shape parameter using maximum likelihood. The estimation of delay time distribution parameters based upon only observational data (failure times and number of defects detected at inspections) is also demonstrated. Results are shown for simulated data sets and conclusions drawn.

5.2 Pilot Simulation of the Delay Time Model

A simulation of the delay time model would involve simulating over a time period for a set of inspections when defects would arise within each inspection interval. However, for a pilot simulation, we shall assume one defect to arrive in one inspection interval, say (0, T). This model along with its development could be used as a modelling module in a more complex situation. The outcome for a defect, i.e failure or detected at inspection, can then be repeated for N defect trials. Given this assumption, we can then calculate, for example, the proportion of failures arising for different values of T, which should asymptotically agree with the model $b(T)$, that is function (2.15), as the number of simulated defect trials, $N \to \infty$. We shall first deal with the perfect inspection case, Section 2.5.1, and then with the imperfect inspection case.

5.2.1 Perfect Inspections

The following assumptions will initially be assumed to apply for the system to be simulated:

(a) The conditional arrival time of a defect is uniformly distributed over the interval (0, T) since the last inspection.

(b) The delay time of the defect is Weibull distributed with scale parameter α and shape parameter γ; the c.d.f being, $F(h) = 1 - \exp(-(\alpha h)^{\gamma})$.

(c) Breakdowns are instantaneously repaired.

(d) The inspection at T is perfect.

(e) The simulation will be repeated N times.

A program, DTS1, written in Pascal, was used to perform this simulation. The method used to generate random samples of u and h was the inverse transform method, see Maisel and Gnugnoli (1972, p.150), given a pseudo-random number generator which generates uniformly distributed random numbers between (0, 1). A seed value can also be given to select different sequences of random numbers. The program requires as input the input parameters (α, γ, T, N) and produces files containing the simulated events. The events recorded are u and h values for both failures and inspected defects.

The algorithm used to simulate the inspection process is as follows:

1. Generate the defect arrival time, u_i, $1 \leq i \leq N$, given by,

$$u_i = r_i T \ , \tag{5.1}$$

where, r_i, is a sequence of independent unordered uniform random numbers on interval (0, 1).

2. Generate the defect delay time, h_i, for defect i, given by,

$$h_i = \frac{1}{\alpha}(-\ln(q_i))^{1/\gamma} \ , \tag{5.2}$$

where, q_i, is another sequence of uniform random numbers.

3. Generate indicator variables, f_i, such that,

$$f_i = \begin{cases} 1 & \text{if } u_i + h_i < T \\ 0 & \text{otherwise} \end{cases} \ , \tag{5.3}$$

where $f_i = 1$ indicates the defect caused a breakdown and $f_i = 0$ indicates the defect was detected at the inspection T.

4. Tabulate results in files.

The program stores data in three files:

1. u_i, h_i and f_i, $1 \leq i \leq N$ (i.e all defect trials).
2. u_i, h_i when $f_i = 1$, (failures only).
3. u_i, h_i when $f_i = 0$, (detections only).

The user is informed of the total number of breakdown occurrences, B_T say, given by,

$$B_T = \sum_{i=1}^{N} f_i \ , \tag{5.4}$$

which will be binomially distributed with parameters N and $b(T)$, where $b(T)$ is the probability a defect leads to a breakdown, given from equation (2.29) by,

$$b(T) = 1 - \frac{1}{T}\int_{h=0}^{T} \exp(-(\alpha h)^{\gamma})\,dh \quad . \tag{5.5}$$

The proportion of defects which have led to breakdowns, $b_T = B_T/N$ say, having expectation $b(T)$, is also signalled.

The program was then extended to allow the user the option to investigate a series of M equi-spaced inspection periods, T_j, $1 \le j \le M$, resulting in a single file containing; the inspection period, T_j, number of breakdowns, B_{Tj}, proportion of failures, b_{Tj}, and the sample standard error, err_j say, of b_{Tj}, given by,

$$err_j = \sqrt{b_{T_j}(1 - b_{T_j})/(N - 1)} \quad . \tag{5.6}$$

The sample standard error can then be used in constructing confidence intervals for the proportion of defects arising as failures. For example, a run was carried out with the following input parameters:

$\alpha = 0.3$, $\gamma = 1.2$, $T = (2, 4, \ldots, 20)$ and $N = 5000$.

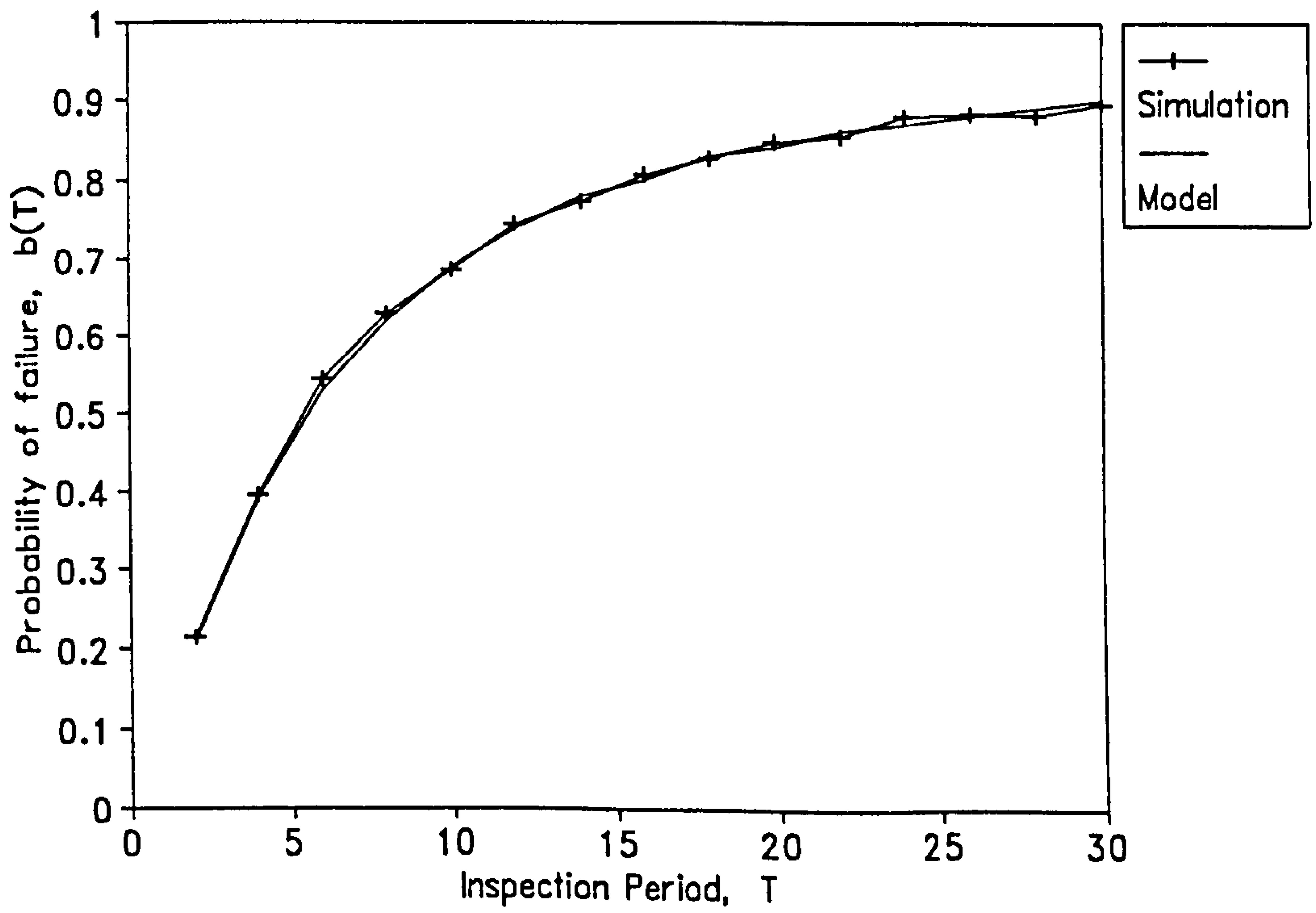

Fig. 5.1. Comparison of a simulation run to the model for $b(T)$.

A graph of the estimated proportion of breakdowns against the theoretical model (2.15) of $b(T)$, for the set of T values, is given in Fig. 5.1. The err_j values lie between 0.004 and 0.007 indicating that a 95% confidence interval for the theoretical probability of a breakdown {predicted to be $b(T)$} will have a width between 0.008 and 0.014. A statistical test to compare the theoretical and sample proportions, $b(T)$ and b_T, was carried out for all T values. The result was to accept the hypotheses that defects arise as failures with probability $b(T)$ for all T values under a 5% significance level.

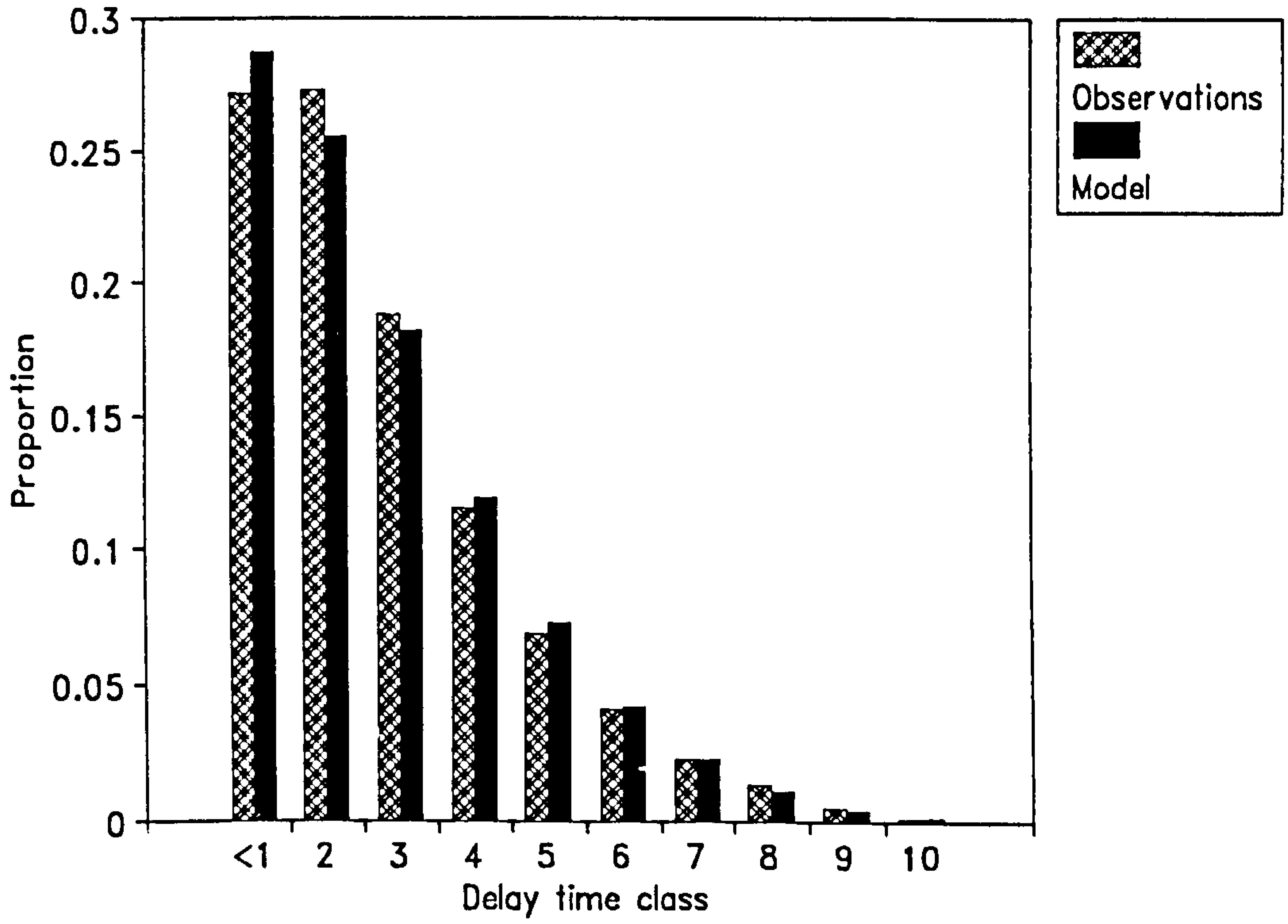

Fig. 5.2. Histogram of breakdown delay time data versus theoretical model.

Members of the conditional delay time and initiation time sets $\{h_i; f_i = 1\}$, $\{h_i; f_i = 0\}$, $\{u_i; f_i = 1\}$ and $\{u_i; f_i = 0\}$ will be distributed with the p.d.fs of h_b, h_d, u_b and u_d, respectively given by functions (4.5), (4.10), (4.18) and (4.20) respectively. In this way the simulation process has been used to generate data to verify the theoretical formulation of these p.d.fs. A comparison of the theoretical p.d.f and those generated from the simulated results for delay times are given as an example in Figs. 5.2 and 5.3 when $\alpha = 0.3$, $\gamma = 1.2$, $T = 10$ and $N = 1000$. In carrying out the chi-square goodness of fit test, the test statistics for the two cases, namely breakdown and inspected delay time, are 4.9 and 11.4 respectively. Adopting a 5% significance level, these values

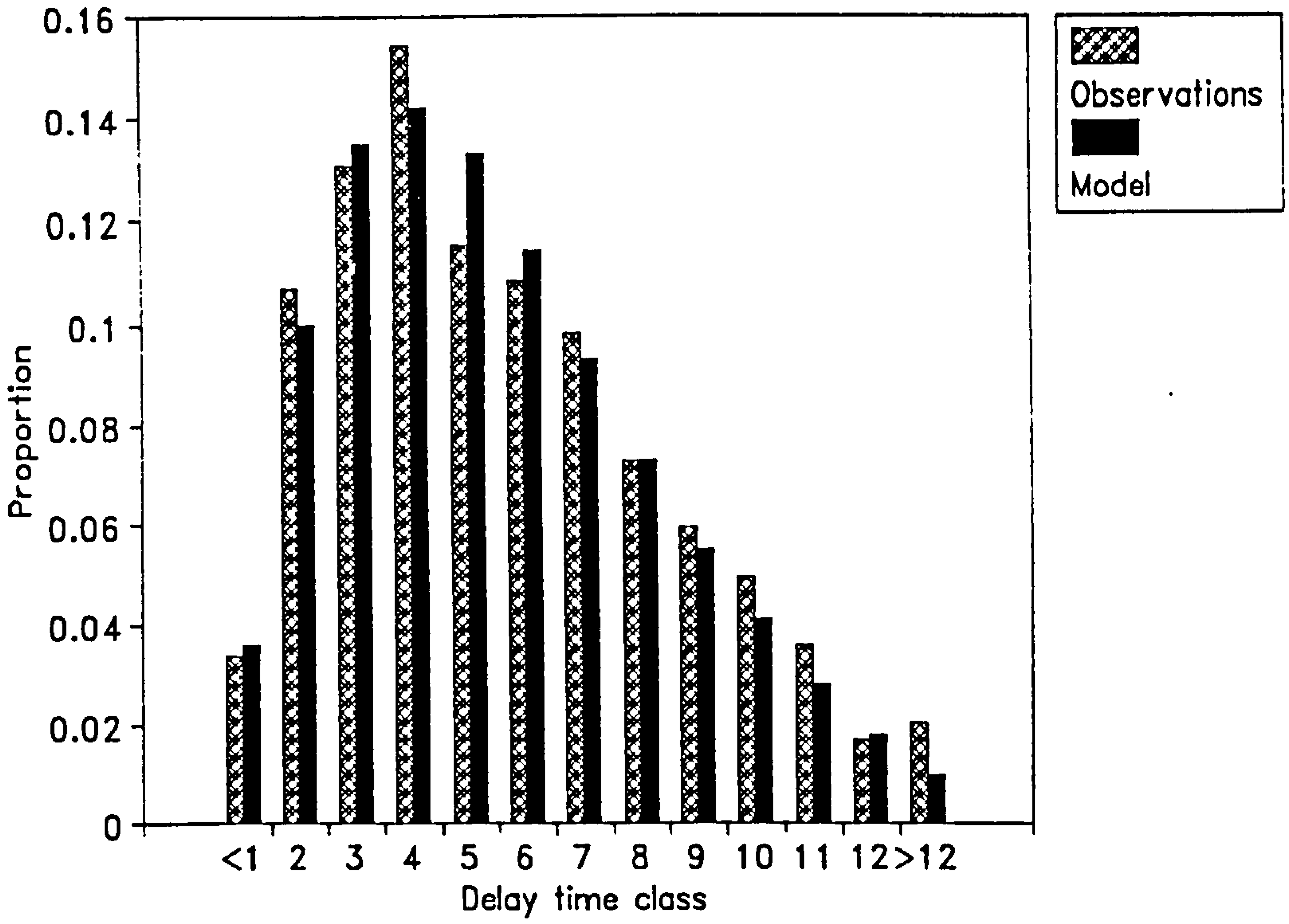

Fig. 5.3. Comparison of inspected delay time data versus theoretical model.

should be compared to $\chi_9 = 16.9$ and $\chi_{12} = 21.0$ respectively. Clearly, the test statistic are less in both cases. Hence, even if the physical evidence of Figs. 5.2 and 5.3 are not accepted, given the additional significance, we accept the hypotheses that the delay time simulation observations are distributed with p.d.fs (4.5) and (4.10) respectively.

5.2.2 Imperfect Inspections

Imperfect inspections can easily be dealt with by extending the previous method. Let β be the probability a defect is detected and repaired at an inspection, if it is present. Assume also that the process of periodic inspections continues indefinitely, so that a defect which arises in an interval $(0, T)$, will eventually arise either as a breakdown or be repaired at an inspection.

For a defect with arrival time, u_i, and delay time, h_i, $1 \le i \le N$, the defect will cause a failure if not detected within $[(u_i + h_i)/T]$ inspections which is an event with probability $(1 - \beta)^{[(u_i + h_i)/T]}$, where $[x]$ denotes the integer part of x. Hence, if we redefine the indicator variable, f_i, so that it is given by,

$$f_i = \begin{cases} 1 & \text{if} \quad r_i < (1 - \beta)^{[(u_i + h_i)/T]} \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

where r_i is uniform (0,1), it follows that $f_i = 1$ implies the defect causes a breakdown over the current inspection period or a future inspection period and f_i=0 implies the defect is repaired at an inspection.

A numerical example for $b(T)$ is given in Fig. 5.4 when $\beta = 0.7$, $\alpha = 0.3$, $\gamma = 1.2$ and $N = 1000$ and $T = 2, 4, ..., 20$. Close agreement between $b(T)$ and simulation estimates is attained. Members of the conditional delay time and initiation time sets $\{h_i; f_i = 1\}$, $\{h_i; f_i = 0\}$, $\{u_i; f_i = 1\}$ and $\{u_i; f_i = 0\}$ will be distributed with the p.d.fs of h_b, h_d, u_b and u_d, respectively, given by functions (4.26), (4.29), (4.32) and (4.33). Numerical examples for delay time are given in Figures 5.5 and 5.6 for $T = 10$. Close agreement is achieved which again verifies the formulation of the p.d.fs formulated in earlier chapters. In carrying out the chi-square goodness of fit test, the test statistics for the two cases, namely breakdown and inspected delay time, are 12.2 and 6.6 respectively. Adopting a 5% significance level, these values should be compared to $\chi_{10} = 18.3$. Clearly, the test statistics are less in both cases. Hence, we accept the hypotheses that the delay time simulation observations are distributed with p.d.fs (4.26) and (4.29) respectively.

The simulation procedure is a module for the simulation of an inspection practice of a system. The next section deals with the simulation of the delay time process over a series of inspections taking into account downtime, cost and assuming defect arrivals arise as a rate process over time.

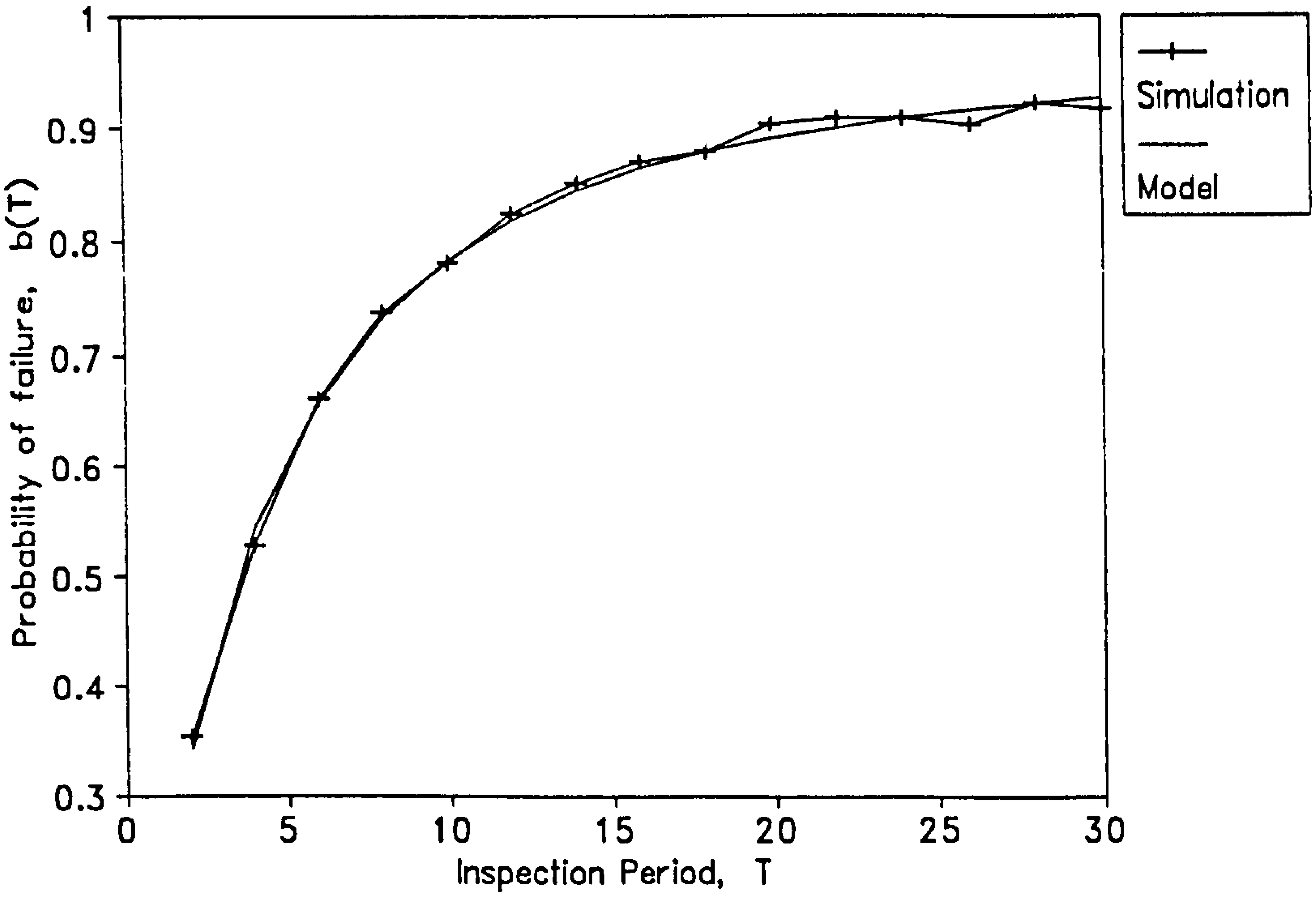

Fig. 5.4. Comparison of imperfect inspection model *b(T)* with simulation.

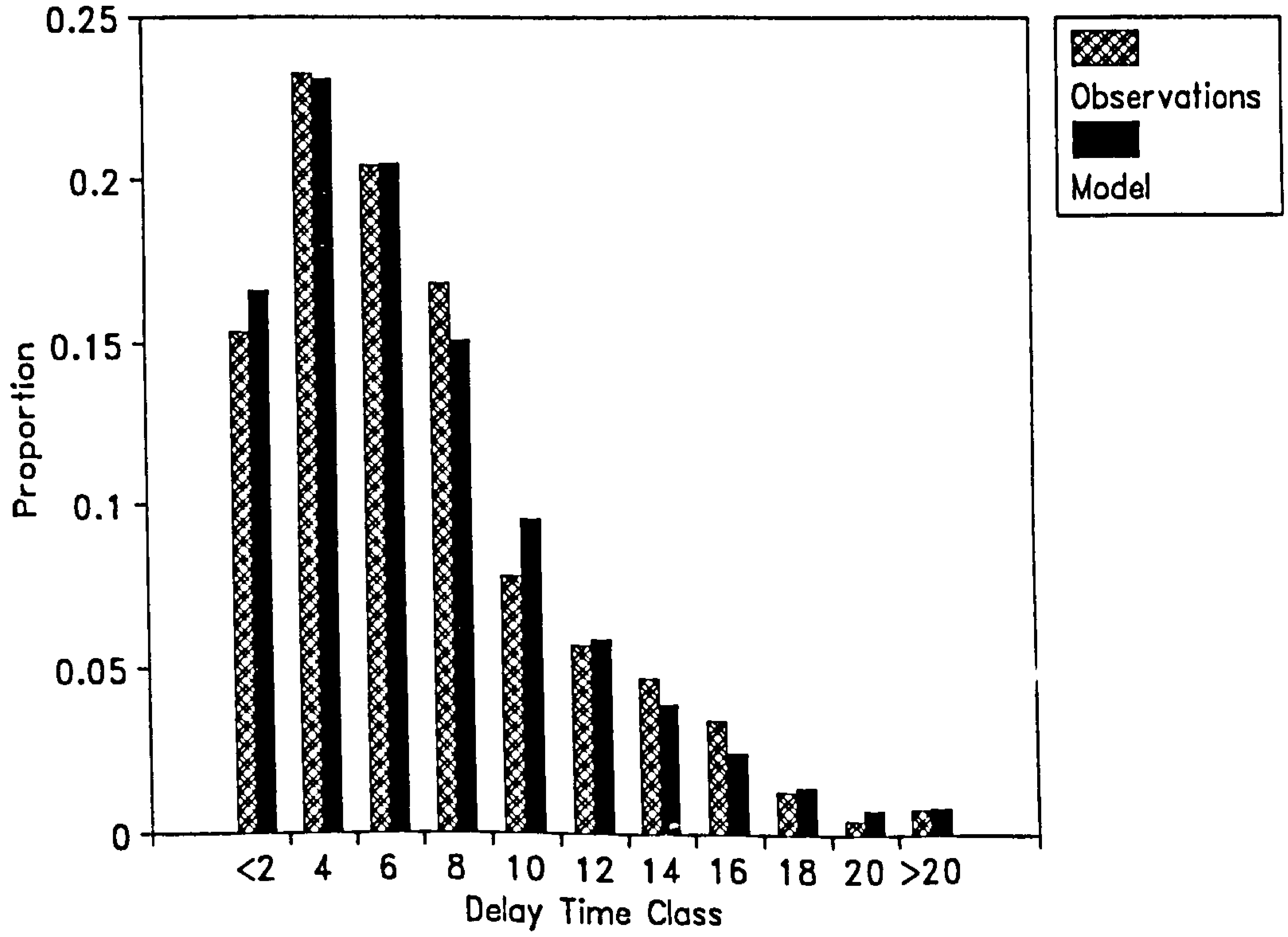

Fig. 5.5. Imperfect inspection p.d.f of breakdown delay time versus simulation.

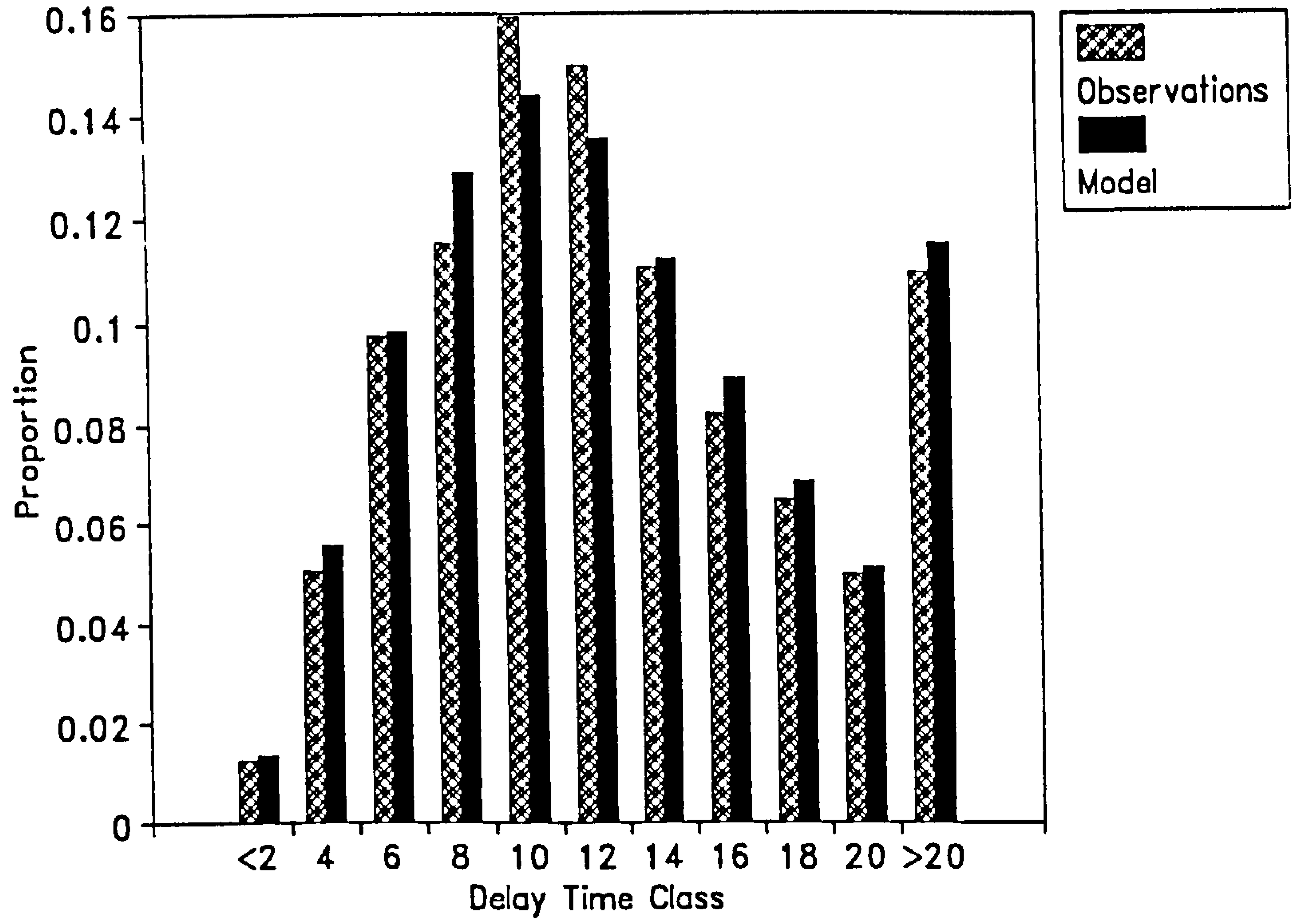

Fig. 5.6. Imperfect inspection model for inspected delay time versus simulation.

5.3 Simulation over a Series of Inspections

We give here a method to simulate the delay time process over a series of inspections. First, perfect inspections with instantaneous breakdown repairs are dealt with. Then, we relax these conditions to include the case of imperfect inspections and non-instantaneous breakdown repairs.

5.3.1 Perfect Inspections

We assume the following assumptions apply to a situation to be simulated over a series of inspections:

(a) The system is defect free at the start of simulation.

(b) N perfect inspections will be undertaken with period T.

(c) Defects arise as a homogeneous Poisson process (HPP) with rate parameter k

over each of the N inspection intervals.

(d) The delay time of each defect is Weibull distributed with scale parameter α and shape parameter γ.

(e) Breakdowns are instantaneously repaired.

It is clear that the number of defects to arrive in each inspection interval $(0,T)$ is Poisson distributed with mean kT. Hence, one approach to simulating the system could be to generate samples from the Poisson distribution for the number of defects to arrive in each of the N inspection intervals and then use the pilot simulation method for each inspection. There are many methods to generate Poisson samples. One method, which was the method adopted, is to use the result that the time between defect arrivals is exponentially distributed with mean $1/k$. If r_i is a sequence of uniform (0,1) random numbers then,

$$e_i = -\frac{1}{k}\ln(r_i) \quad , \tag{5.7}$$

is a sequence of exponentially distributed random variables of mean $1/k$. Consider the number of defects, A say, to arrive in an inspection interval. For A to be Poisson distributed with mean kT we need to sum the values e_i until the result exceeds T. Hence, let u_n say, be the n'th summation, given by,

$$u_n = \sum_{i=1}^{n} e_i \qquad , u_0 = 0, \quad n = 1, 2... \quad . \tag{5.8}$$

Then let A be given by,

$$A = \{n; \ u_n \le T, \ u_{n+1} > T\} \quad . \tag{5.9}$$

It follows that A is Poisson distributed with the required mean value. The pilot simulation method can then be used to calculate the number of breakdowns arriving over the interval $(0, T)$. However, for the case of HPP defect arrivals the generation of A has also generated a sample of defect arrival times, i.e the set $\{u_n\}$, $1 \le n \le A$, for $A > 0$. Essentially, this set is the order statistics of a set of A independent uniform random samples on the interval $(0, T)$.

Note: If $\{f_n\}$ is the set of indicator variables corresponding to $\{u_n\}$ and $\{h_n\}$, then in this case due to the index n corresponding to the n'th defect arising over $(0, T)$, the probability that the n'th defect arises as a breakdown, i.e $P\{f_n = 1\} \neq b(T)$. For example, the first defect arrival will have less chance of causing a failure than the second defect arrival. However, a defect chosen at random will have the probability $b(T)$ to cause a failure.

The total number of breakdowns, B, arising over interval $(0, T)$ is given by,

$$B = \begin{cases} 0 & if\ A=0 \\ \sum_{n=1}^{A} f_n & if\ A>0 \end{cases} \tag{5.10}$$

The number of defects detected and repaired at the inspection T, say S, is then given by,

$$S = A - B \ . \tag{5.11}$$

B will be Poisson distributed with mean $kTb(T)$ and S will be Poisson distributed with mean $kT(1 - b(T))$, independently of B.

The program written in Pascal, DTS2, on execution creates four files as output. The first file contains N lines of four data:

(a) Inspection interval number j, where $j = 1, 2...N$, that is j'th interval terminates at jT.

(b) Number of defect arrivals, A_j.

(c) Number of failure occurrences, B_j.

(d) Number of defects detected, S_j, where $S_j = A_j - B_j$ due to perfect inspections.

The second file contains seven data items on each defect which arrives over the simulation period, in the order of defect arrival time :

(a) Inspection interval number j.

(b) The defect arrival number, i, for the ith defect to arrive in inspection interval j.

(c) Delay time of defect, h.

(d) Arrival time of defect, u, from start of inspection interval j.

(e) An indicator variable, f, where $f = 0$ if defect was detected, (i.e. if $u + h > T$), and $f = 1$ if defect caused a failure (i.e if $u + h \leq T$).

(f) How long ago defect first arrived, HLA, where $HLA = T - u$, if $f = 0$, and $HLA = h$, if $f = 1$.

(g) How much longer, HML, the defect can be left before causing a failure where, $HML = 0$, if a failure, and $HML = u + h - T$, if inspected.

The third file contains only the data on defects which were inspected. The items of data will be the same for the defect file except for the indicator variable, f. The fourth file contains only the data on defects which caused a failure. The items of data are (a), (b), (c) and (d) from the first file plus the following :

(a) The time of failure, y_b, where $y_b = u + h$. The p.d.f of y_b is given by function (3.34).

(b) The failure arrival number, m, for the m'th failure in inspection interval j.

The following parameter values have been selected for an initial test,

$$k = 0.5,\ \alpha = 0.2,\ \gamma = 1.2,\ T = 10,\ N = 1000.$$

The theoretical mean and standard deviation of the sampled variables are tabulated in Table 5.1, along with the sample mean, standard deviation and standard error of mean from a simulation run. Some variables have a suffix implying a condition; b indicates breakdowns and d indicates detections. The total number of defects arising for this particular test was 4918 with 2823 failures. It can be seen that there is close agreement between theoretical parameters and sample estimates, validating the simulation procedure.

5.3.2 Imperfect inspections

A program, DTS3, was written to accommodate imperfect inspections. The probability, β, that a defect present at an inspection is detected is given on input to the simulation. The indicator variable, f, for a defect arising as a failure is set accordingly by,

$$f = \begin{cases} 1 & \text{if } (1-\beta)^{[(u+h)/T]} < r \text{ and } (u + h + jT) < NT \\ 0 & \text{otherwise} \end{cases} , \qquad (5.12)$$

where u is the defect arrival time from last inspection point. h is the delay time, j is the inspection interval in which the defect arisen, $1 \le j \le N$, and r is a uniform (0,1) random number. An additional file is output by the simulation on defects which are left in the system after the N'th imperfect inspection.

Variable	Theo. Mean	Sample Mean	Mean Error	Theo. St.Dev	Sample St.Dev
A	5.0000	4.9180	0.0708	2.2361	2.2384
B	2.8213	2.8230	0.0539	1.6797	1.7060
S	2.1787	2.0950	0.0476	1.4760	1.5061
h	4.7033	4.6582	0.0544	3.9361	3.8828
u	5.0000	4.9396	0.0415	2.8868	2.9089
f	0.5643	0.5740	0.0071	0.4958	0.4945
h_d	7.3146	7.2990	0.0932	4.2884	4.2644
u_d	6.7396	6.7873	0.0546	2.5092	2.4990
h_b	2.6867	2.6985	0.0365	1.9807	1.9380
u_b	3.6566	3.5684	0.0449	2.4011	2.3863
y_b	6.3432	6.2669	0.0441	2.4012	2.3427

Table 5.1 Comparison of theoretical parameters and sample estimates from simulation data.

5.3.3 Non-instantaneous Breakdown Repairs

A program, DTS4, was written to simulate downtime and cost consequences under the assumption of non-instantaneous breakdown repairs and either the periodic based or use based inspection policies. The program assumes constant breakdown repair times, d_b, and perfect inspections. When a breakdown occurs, the delay time of each defect that may be present in the system does not expire and defects do not arise. Also, a breakdown being repaired when the system is due for inspection at T is repaired within the inspection time, d_I, along with any other defects. The output, of which, can then be compared to the approximate downtime models (2.18) and (2.36), or the theoretical downtime model using function (A.3), given in the appendix. A sample of output is given in Fig. 5.7 for the canning line case study, Christer and Waller (1984b). The modelling parameters for the case study were, $\alpha = 0.0447$, $\gamma = 1$, $\beta = 1$, $d_b = 0.698$ hrs, $d_I = 0.525$ hrs and $k = 0.101$ hrs^{-1}. The number of inspections carried for the simulation was taken to be $N = 500$. Simulation estimates of downtime per unit time versus the approximate model for expected downtime per unit time, function (2.18). Close agreement is achieved which validates the simulation procedure and, interestingly also the model approximation (2.18) for the selected modelling parameters given.

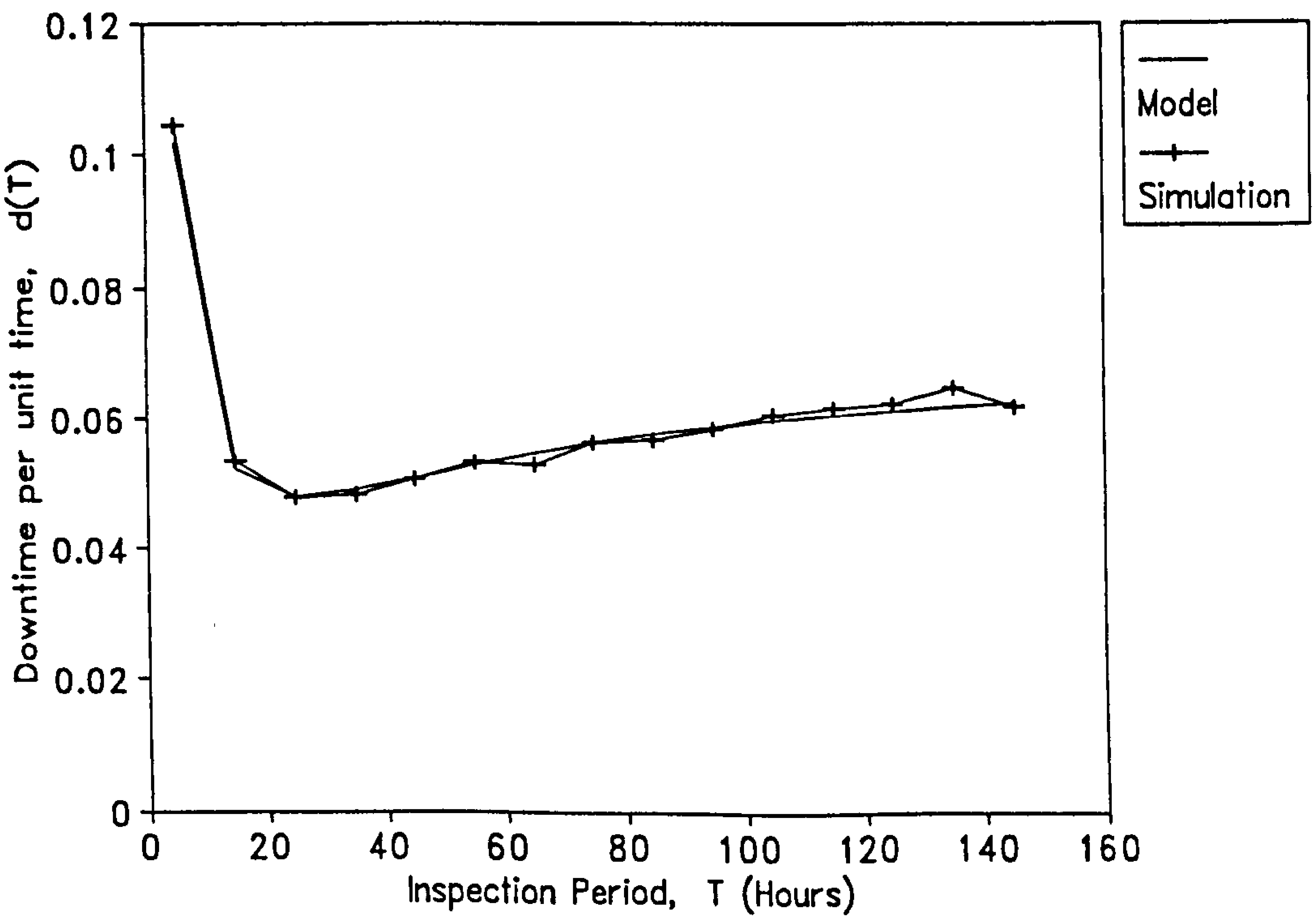

Fig. 5.7. Comparison of simulated downtime versus the model approximation for perfect inspections.

5.4 Estimation Techniques using Simulated Data

Chapters 3 and 4 have discussed various estimation techniques which can be applied to a system given various sets of observational and subjective data. This section verifies some of these methods and demonstrates other options available to the analyst.

5.4.1 Correcting for Bias in Delay Time Parameter Estimates

A program, MLE1, was written in fortran to verify the maximum likelihood procedure given in Chapter 4, for the correction of bias due to the collection of delay times of defects only at breakdowns or to the collection of delay times of defects detected only at inspections. The program uses the likelihood functions L_1 and L_2 given in Section 4.5. A test was conducted using the following input parameters;

$\alpha = 0.3$, $\gamma = 0.8$, $k = 0.5$, $T = 10$, $N = 100$, $\beta = 1$.

The theoretical probability of failure, equation (2.29) is, given the above parameters,

$b(T) = 0.673$.

Over the time period required for 100 simulated inspections, a total number of 485 defects had arisen, resulting in 310 failures and 175 inspected defects. The sample estimate of the proportion of failures, that is the total number of breakdowns (310) divided by the total number of defects (485), is,

$b^{*} = 0.638$.

A 95% confidence interval for the theoretical probability of a breakdown is then given by,

$$b^{*} \pm 1.97\sqrt{0.638(1 - 0.638)/485} \equiv (0.595, 0.681) , \qquad (5.13)$$

which contains the theoretical value. After 100 inspections were performed, the histogram of delay times for the two cases are given in Figures 5.8 and 5.9, which also compares the results with the theoretical expectation.

The bias corrected estimates, for the set of the delay times of inspected defects and breakdowns, $\{h_d\}$ and $\{h_b\}$ respectively, using equations (4.10) and (4.5) in the program MLE1, were,

$\{h_d\}$: $\hat{\alpha} = 0.368$, $\hat{\gamma} = 0.748$, $\hat{b}(T) = 0.683$ and

$\{h_b\}$: $\hat{\alpha} = 0.306$, $\hat{\gamma} = 0.796$, $\hat{b}(T) = 0.677$,

where $\hat{b}(T)$ is the estimated probability that a defect arises as a breakdown, using function (2.29), with the estimated parameters in both cases.

The estimated p.d.fs are also given in Figures 5.8 and 5.9. The chi-square test was carried out on both sets of data assuming the data is distributed with the appropriate estimated conditional p.d.f. In the case of inspected delay times $\chi^2 = 8.63$, and for the case of breakdown delay times, $\chi^2 = 5.3$. Adopting a 5% significant level, we reject the hypothesis if $\chi^2 > 16.9$ (9 degrees of freedom) for the first case, and we reject if $\chi^2 > 18.3$ (10 degrees of freedom) for the second case. Clearly, we can accept the estimated p.d.fs in both cases.

It can be seen that for the case of breakdowns, $\hat{b}(T)$ lies within the confidence limits (5.14) of the probability a defect arises a failure. No updating in this case would be considered necessary. Although, possible updating could be explored for the case $\{h_d\}$ whereby the estimate $\hat{b}(T)$ does not lie within the confidence limits (5.14). However, it can be seen that $\hat{b}(T)$ for this case does lie close to the theoretical value for the simulation, $b(T)$.

The main point, here, is that actual delay time population parameters can be estimated given censored data sets.

5.4.2 Updating Option

An investigation is carried out to estimate the Weibull parameters assuming no bias exists. It will be found that the estimated model $\hat{b}(T)$ will be either greater or less than the observed value b^*, depending on using breakdown or detected delay times. It is then possible to use the scale update method given in Section 3.5.1. The effects of undertaking this approach will be discussed.

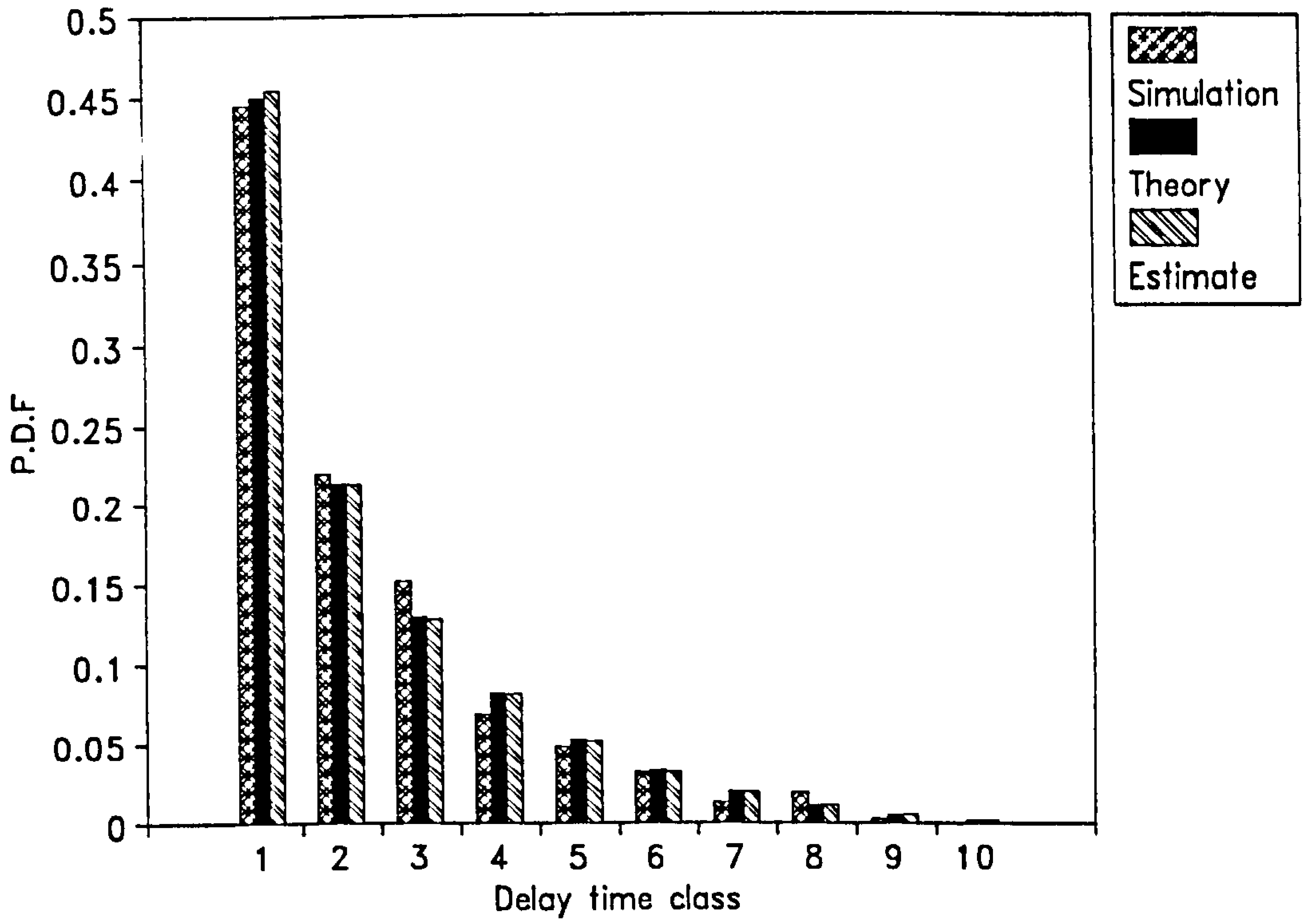

Fig. 5.8. Simulated breakdown delay time sample p.d.f versus theoretical model.

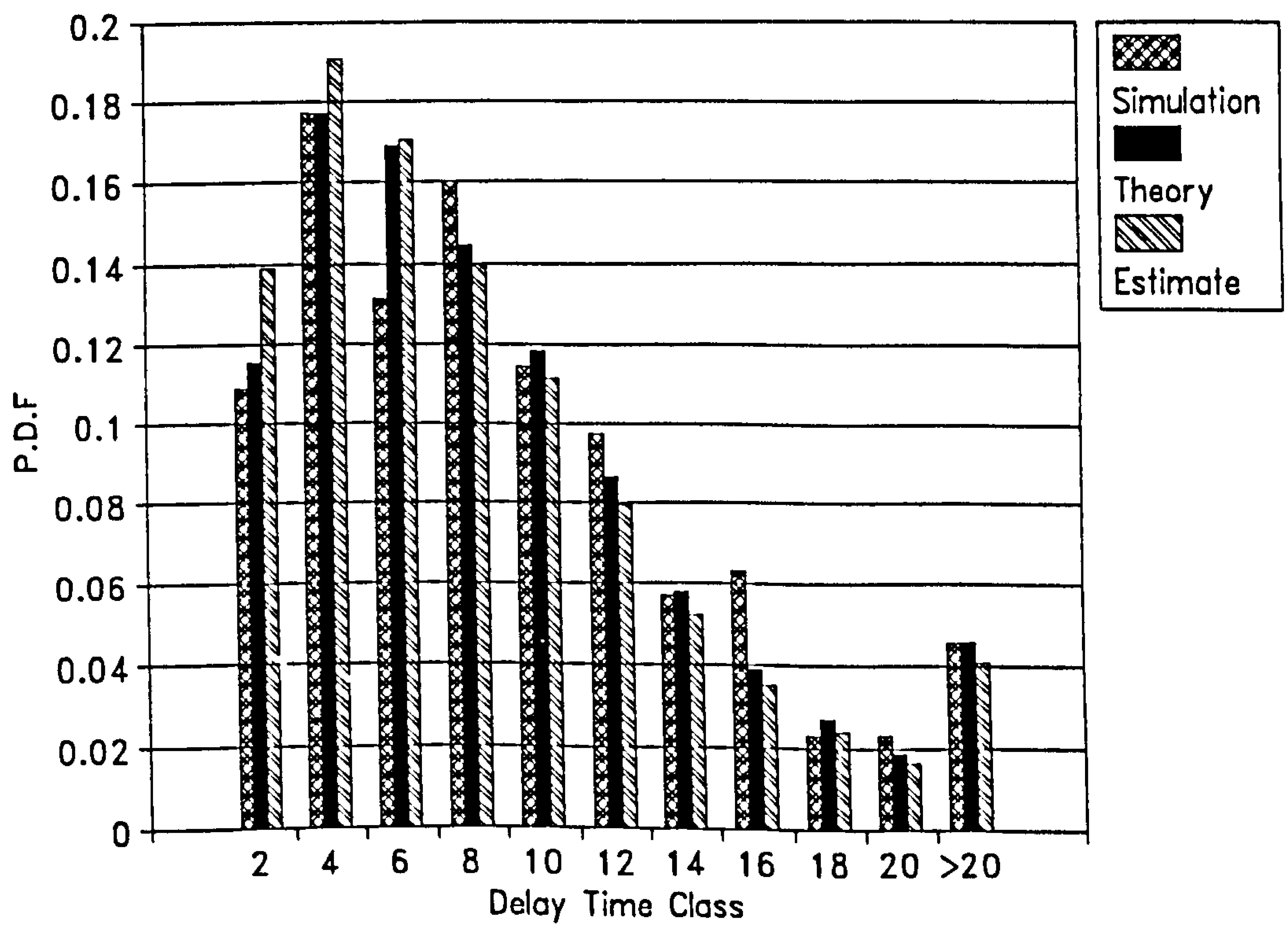

Fig. 5.9. Simulated inspected delay time p.d.f versus theoretical model.

A program, MLE2, was written to estimate the parameters for assumed Weibull distributed data using the Newton-Raphson root finding method. The same set of data was used in the previous section. When not correcting for bias, using MLE2, we obtain,

$$\{h_d\} : \alpha^{\wedge} = 0.112, \ \gamma^{\wedge} = 1.315, \ b^{\wedge}(T) = 0.358$$
$$\{h_b\} : \alpha^{\wedge} = 0.596, \ \gamma^{\wedge} = 0.902, \ b^{\wedge}(T) = 0.825.$$

Clearly, the effects of updating can be investigated in this situation as the sample proportion of defects arising as failures, $b^* = 0.638$, and the theoretical modelling parameters are $\alpha = 0.3$ and $\gamma = 0.8$. The true model for $b(T)$ compared with the above model estimates is given in Figure 5.10. The observation point, b^*, is also labelled.

When respectively updating these two estimate sets by the scale method, using the observation b^*, we obtain,

$$\{h_d\} : \alpha^{\wedge} = 0.249, \ \gamma^{\wedge} = 1.315$$
$$\{h_b\} : \alpha^{\wedge} = 0.256, \ \gamma^{\wedge} = 0.902$$

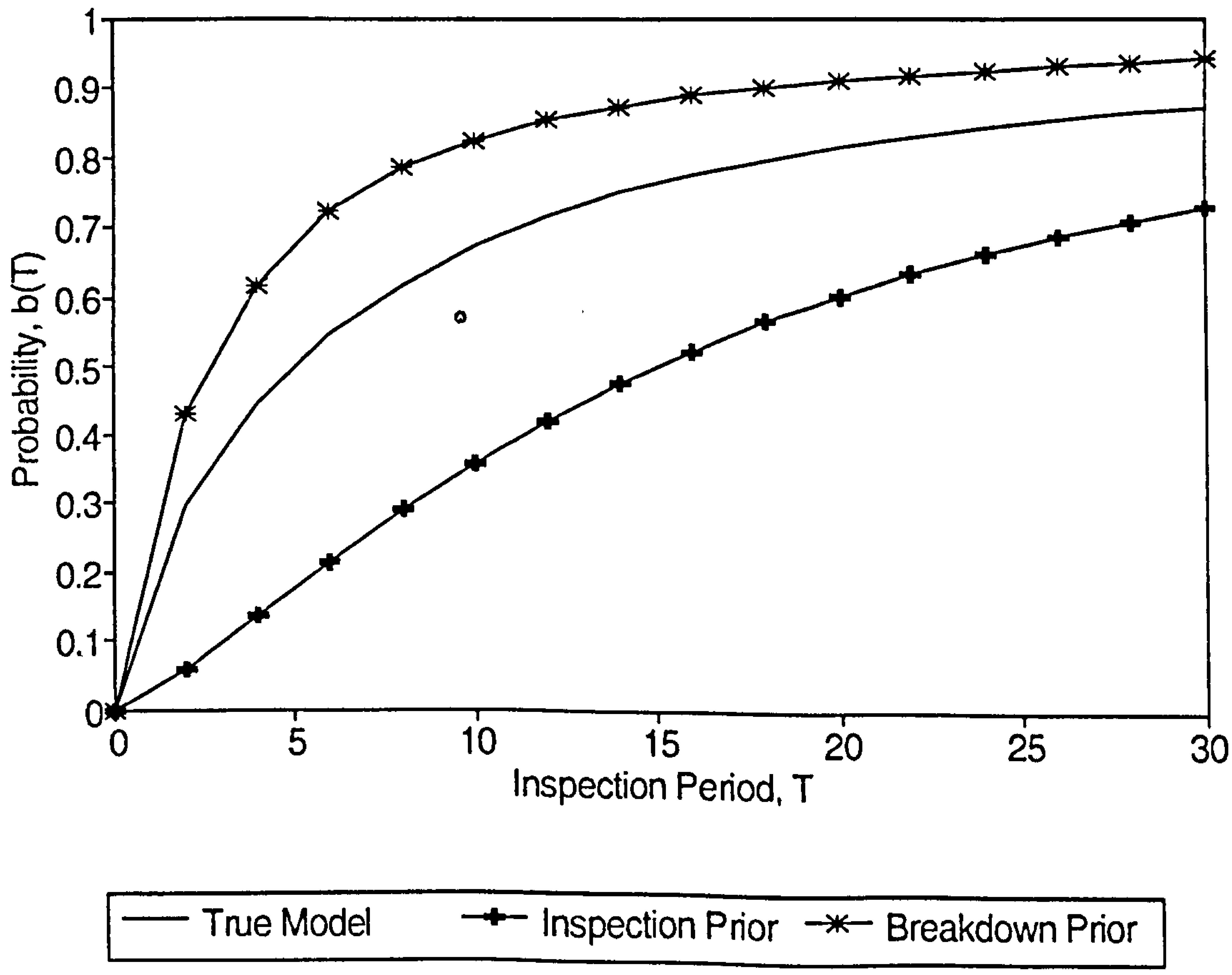

Fig. 5.10. The true model $b(T)$ versus the model estimates.

It can be seen that, in both cases, there is improvement in the estimated scale parameter, that is comparing it to the theoretical value of 0.3.

To investigate decision consequence of these estimating and updating procedures, the model (2.23) of cost is used. The parameters selected were;

c_I = 0.3 (cost of inspection),

c_b = 0.5, (breakdown repair cost),

c_d = 0.2 (defect repair cost) and

d_I = 0.5 (inspection downtime).

The parameters were chosen so that the optimality conditions (2.25) and (2.26) are satisfied. The theoretical optimum inspection period is,

$T^* = 5.16$,

and for the estimated models using the parameter estimates from updating, the optimums are,

Inspection-Update : $T^* = 4.35$

Breakdown-Update : $T^* = 4.95$.

Graphs of the true model versus the model estimates are given in Fig. 5.11. An error in the optimum inspection period is about one time unit for the cost model based on the inspected delay time defects. There is no significant error in the optimum inspection period with the cost model based on breakdown repairs. As can be seen, both the cost models based on the updating procedure, lie beneath the true model. In Section 5.4.4, a test of model fit will be given to decide whether to accept the updated models.

5.4.3 Iteration Method to Capture Scale and Shape Parameters

A program, MLE3, was written to estimate the shape parameter of a Weibull distribution given a set of delay time data observations and a value for the scale parameter. Perfect inspections are assumed. Only minor modifications to MLE1 are necessary to perform this. The re-estimated shape parameter is then used in the updating procedure to produce an updated scale parameter. The procedures can be iterated until a possible convergence.

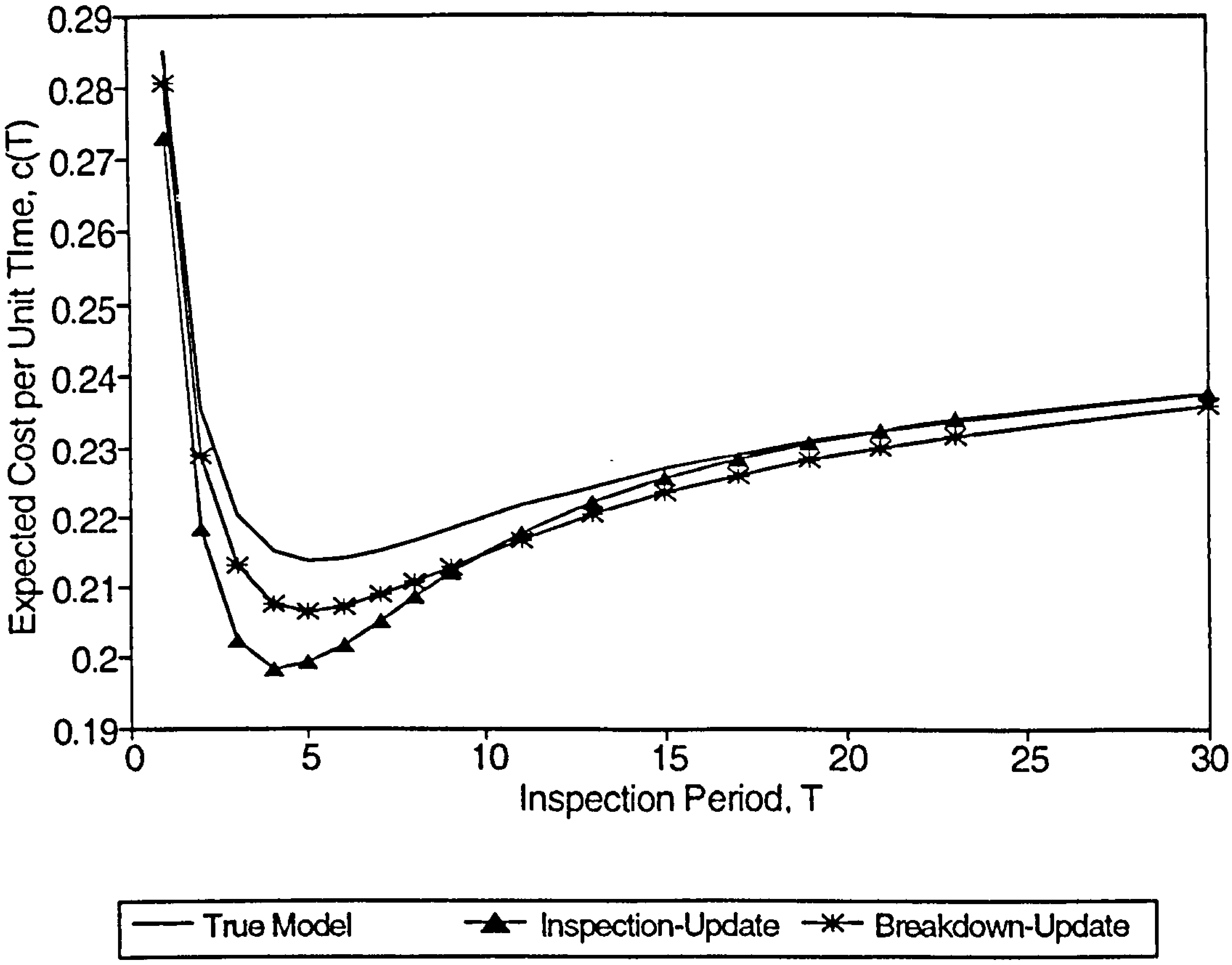

Fig. 5.11. The true cost model versus model estimates.

In both cases of the small data sample, convergence to 3 decimal places, occurs in 4 iterations. The results obtained were,

Detected defects iteration : $\hat{\alpha} = 0.259$, $\hat{\gamma} = 0.805$,

Breakdown repairs iteration : $\hat{\alpha} = 0.260$, $\hat{\gamma} = 0.789$.

Close agreement between the estimated and input parameters ($\alpha = 0.3$, $\gamma = 0.8$) has been achieved through this process. In the case of a large data sample test (5000 defects in all), using the simulation program DTS1 (Section 5.2.1) with a different set of input parameters, even closer agreement was found between the estimated and input parameters. The Weibull parameters selected for the delay time p.d.f of the large sample case, were $\alpha = 0.2$ and $\gamma = 1.2$. The results obtained in 4 iterations were,

Breakdown repairs iteration : $\hat{\alpha} = 0.206$, $\hat{\gamma} = 1.182$

Detected defects iteration : $\hat{\alpha} = 0.206$, $\hat{\gamma} = 1.185$.

5.4.4 Criteria for Deciding on whether to accept the Updated Model

We have seen that a method exists for correcting the recognised bias in delay time parameter estimates obtained from delay time measures captured at only breakdowns or inspection situations. The first method (Section 5.4.2) based upon shearing the scale parameter worked in one numerical case, based upon breakdown delay time estimates, but didn't work in another, based upon inspection delay time estimates. The second method (Section 5.4.3) based upon an iteration updating of both parameters gave satisfactory results to both these numerical cases. What is now required is a means of deciding whether or it necessary to update the estimates, and if so, what techniques to adopt.

A simple method is proposed for deciding on whether to update via the bias correction procedure or through use of the iteration method :

1. Perform a Chi-square test or K-S test on the censored delay times under the hypothesis that the delay times are distributed by the conditional p.d.f $f_b(\xi;T)$, equation (4.5) or $f_d(\xi;T)$, equation (4.10) depending on breakdown-based or inspection-based censoring.

2. If the test fails then either perform the bias correction method or the iteration method. If the tests fail in these cases then the delay times may be assumed not to follow the Weibull distribution.

The Chi-Square values corresponding to each data set obtained by simulation in Section 5.4.1, when not correcting for bias and scale updating has been performed (Section 5.4.2), are as follows;

$\{h_d\} : \hat{\alpha} = 0.249, \hat{\gamma} = 1.315, \chi^2 = 150.2,$

$\{h_b\} : \hat{\alpha} = 0.256, \hat{\gamma} = 0.902, \chi^2 = 9.94.$

Using a 5% significance level, we reject the hypothesis if $\chi^2 > 18.3$ for the first case (10 degrees of freedom), and reject if $\chi^2 > 16.9$ for the second case (9 degrees of freedom).

Therefore, it follows that bias correction or iteration would not, on the basis of the test, be necessary for the case of breakdown delay times. It is clear that bias correction or iteration would be necessary for the detected delay times. Theses two conclusions agree with our previous analysis of the decision consequence for cost.

It is expected that when a decision not to correct for bias is taken, the models for cost and downtime agree closely to the models when a correction is initiated. This is due to the biased distribution and updated distribution being approximately equal.

Chi-square tests have been performed on the bias correction estimation and iteration method. In all cases, these produce a decision not to reject the hypothesis.

An alternative procedure to test whether to correct for bias, would be to use the observed times of breakdowns, if available. A K-S or Chi-Square test could be carried out on the empirical sample of breakdown times measured from the last inspection. The empirical distribution would then be compared to the theoretical distribution, function (3.34), using the updated parameter set. The option is examined below.

5.4.5 Estimation using the observed breakdown times

The section verifies the maximum likelihood and method of moments procedures given in Section 3.7, to estimate delay time parameters when the only data obtainable are observations of breakdown times and the number of defects repaired at inspections. A program MLE4, using equations (3.35), (3.36) and (3.41), was written to estimate the Weibull delay time parameters, assuming perfect inspections ($\beta = 1$) and given this type of observational data.

A test was conducted given the input parameters; $\alpha = 0.1$, $\gamma = 1.3$, $T = 30$, $N = 500$ and $k = 0.1$. The total number of breakdown arrivals from this simulation were 1118 and the total number of defects detected at inspections were 494. The maximum likelihood parameter estimates of α and γ, using the likelihood function (3.41), were $\hat{\alpha} = 0.100$ and $\hat{\gamma} = 1.24$, which are acceptably close to the input parameters. Hence, it follows that delay time modelling parameters can be estimated from only observations of breakdown time and the number of inspection repairs.

We now turn to the method of moments method. The sample mean time of breakdowns was 18.43. In solving the two equations for α and γ, namely,

$$b(T,\alpha,\gamma) = \frac{1}{T}\int_{h=0}^{T}(1 - e^{-(\alpha h)^{\gamma}})dh = 0.694 \quad , \tag{5.14}$$

and

$$m(\alpha,\gamma) = \frac{1}{Tb(\alpha,\gamma)}\int_{h=0}^{T}h(1 - e^{-(\alpha h)^{\gamma}})dh = 18.43, \tag{5.15}$$

the method of moment parameter estimates were $\hat{\alpha} = 0.100$ and $\hat{\gamma} = 1.285$. The estimates are very close to the modelling parameters. Thus, the method provides an alternative estimation procedure.

5.5 Conclusion

It has been seen that the simulation models and various parameter estimation procedures produce results in agreement with theoretical models. This, validates the programming of and the derivation of the delay time models and theory of the previous section. The bias correction and iteration methods estimate the parameters of the delay time distribution to within 10% of the theoretical values for a moderately sized sample, on the tests carried out so far. The method of estimation using only observed data also produces results in accordance with theory. We conclude that these are formal techniques. However, convergence properties of the iteration method needs to be further researched. The answer to the convergence will lie in the form of the log-likelihood functions. E.g, for the case of failure delay times the log-likelihood function, omitting constant terms due to assumed HPP defect arrivals, is given by,

$$L(\{h_b\};\alpha,\gamma) = -F\ln(b(T;\alpha,\gamma)) + \sum_{i=1}^{F}\ln(f(h_i;\alpha,\gamma)) \quad , \tag{5.16}$$

where f is Weibull, F is the number of failures, $b(T)$ is the probability of failure and $\{h_i\}$

is the set of failure delay times, h_b. The likelihood if not correcting for bias is the above function without the $F\ln(b(T))$ term. As the number of observations of h_b increases, the likelihood estimates when correcting for bias will be such that the estimated model $b^{\wedge}(T)$ will tend to $b(T)$ the true model. Hence, the estimates can also be obtained by conditionally selecting values of α and λ such that $b(T) = b^*$, the observed proportion. The maximum likelihood method under this condition is then the procedure to not correct for bias. It is believed, for the work required here, that if convergence occurs through the iteration method, then the estimates will tend to the maximum likelihood estimates when correcting for bias, as the sample size increases.

Investigation into the error of the optimum inspection period when carrying out these method also needs research. A clue could lie in the behaviour of the function $b(T; \alpha, \gamma)$ for values of T, under the non-unique set (α, γ) which satisfy the observation point b^*. The cost and downtime curves under such restrictions, may have optimum inspection periods which lie within a certain calculable interval. However, the measure of error, if the iteration method gives accurate results in most cases, may not now be necessary to consider.

The criteria for deciding whether to perform iteration or bias correction has produced satisfactory results. The suggested decision procedures for deciding whether or not to correct for bias may only work when the biased distribution can be approximated by a Weibull distribution. This occurs in the test of Section 5.4.4, as the failures delay times are almost exponential. A decision to not correct for bias may be made on the status-quo point being satisfied. However, the resulting model for inspection periods other than the current practice may be inaccurate. A further statistical test, such as, for example, comparing the empirical and theoretical distribution of times of breakdowns, would need to be applied.

Overall, simulation programs have been successfully written and tested, and methods have been developed for estimating delay distributions given accurately estimated data in practice. Further work could also lie in the effects of subjective errors when estimating delay time, the possibility of imperfect inspections and convergence properties of the iteration process.

Chapter 6

Application of Delay Time Analysis to Concrete Structures

6.1 Introduction

This chapter gives an account of research supported by the Science and Engineering Research Council (grant: GR/F/61196). The project was a collaborative venture between operational researchers and civil engineers over 3 years. The main objectives were to collect and publish data on the observed rates of deterioration of particular defect types in a large number of concrete bridges and to develop predictive mathematical models that relate inspection frequency to maintenance costs. The motivation was in part associated with the prototype modelling paper for inspection practices of major concrete structures, Christer (1988).

Concrete structures, like other civil engineering structures, deteriorate over their service life. The main cause of this deterioration, in an adequately built structure, can be described as environmental effects. Repair and maintenance represents an ever increasing share of maintenance expenditure on concrete structures and there is, therefore, an economic requirement to quantify and model the deterioration and maintenance process of such structures. Repair options exist at different stages of deterioration with different costs and consequences, and modelling is necessary to aid management decision making to improve the cost effectiveness of maintenance expenditure.

A survey carried out by Queen Mary and Westfield College (QMWC) Concrete Research Group, Rigden et al (1988), indicated that the principal cause of concern amongst engineers with responsibility for structures was their inability to predict rates of change of defective concrete components and to decide on the timing of concrete repairs in order to make cost-effective decisions. Maintenance organisations were found to be spending a great deal of time in analyzing and recording the conditions of components. However, the data recorded was under-utilized, was subject to little by way of analysis, and seldom led to any conclusions. It was not incorporated into any predictive modelling.

There is evidently a need to develop and validate models of the random growth of defects arising in components based upon data that is or could be collected. Christer (1988) highlighted the scope for modelling deterioration and maintenance of concrete structures, based on the concept of delay time, Christer and Waller (1984a), Baker and Wang (1992), Baker and Christer (1994). RILEM (1988) gives an account of engineering factors that mav be necessary to include in a model for concrete deterioration and recognises that a component's life should be modelled at least as two phases, namely defect free (initiation period) and defective (corrosive) period. The main factors which affect the rate of degradation of reinforced concrete are assumed to be environmental conditions, concrete type and concrete cover to the reinforcing steel, see also Currie and Robery (1994).

The chapter reports on the delay time modelling of the growth of defects in concrete bridge components, the analysis of data collected and the development of cost based inspection models. The two phase delay time model is extended to an extra phase in order to model the process of cracking and spalling.

6.2 Deterioration of Concrete Components

Virtually every concrete structure eventually develops detectable cracks at some point in time of a sufficiently long service life. These may be already visible when built or could develop and widen due to environmental factors such as corrosion of steel reinforcement caused by the carbonation of the concrete or the impurities it may contain. If left to deteriorate, spalling will eventually occur, exposing the reinforcement and leading to corrosion, until the structure reaches a point in time when it is perhaps in an unacceptable state for safety or other reasons and a major repair is deemed necessary. Defects deteriorate through a number of definable states where the time scale is measured in years. Data analysis indicates that the time within a state is not deterministic, and that the chance of changing state within a given period is dependent upon the time duration within the current state, that is the deterioration is not Markovian.

In the context of *delay time*, the time interval between the arrival of repairable cracks and severe spalling would normally be the delay time measure, and repairs could be undertaken at non-decreasing costs at any point over this interval of time. However, for the present, the delay time will be split into two phases, namely cracking and spalling. The key phases of the deterioration modelling are now, new to cracking, which occurs at a time, u say, cracking to spalling which occurs over time, h, and spalling to failure when repair is essential, which occurs after a further time period v. Defining a component to be a section of a structure in which only one defect can arise, Fig. 6.1 depicts the deterioration over the cracking and spalling phases.

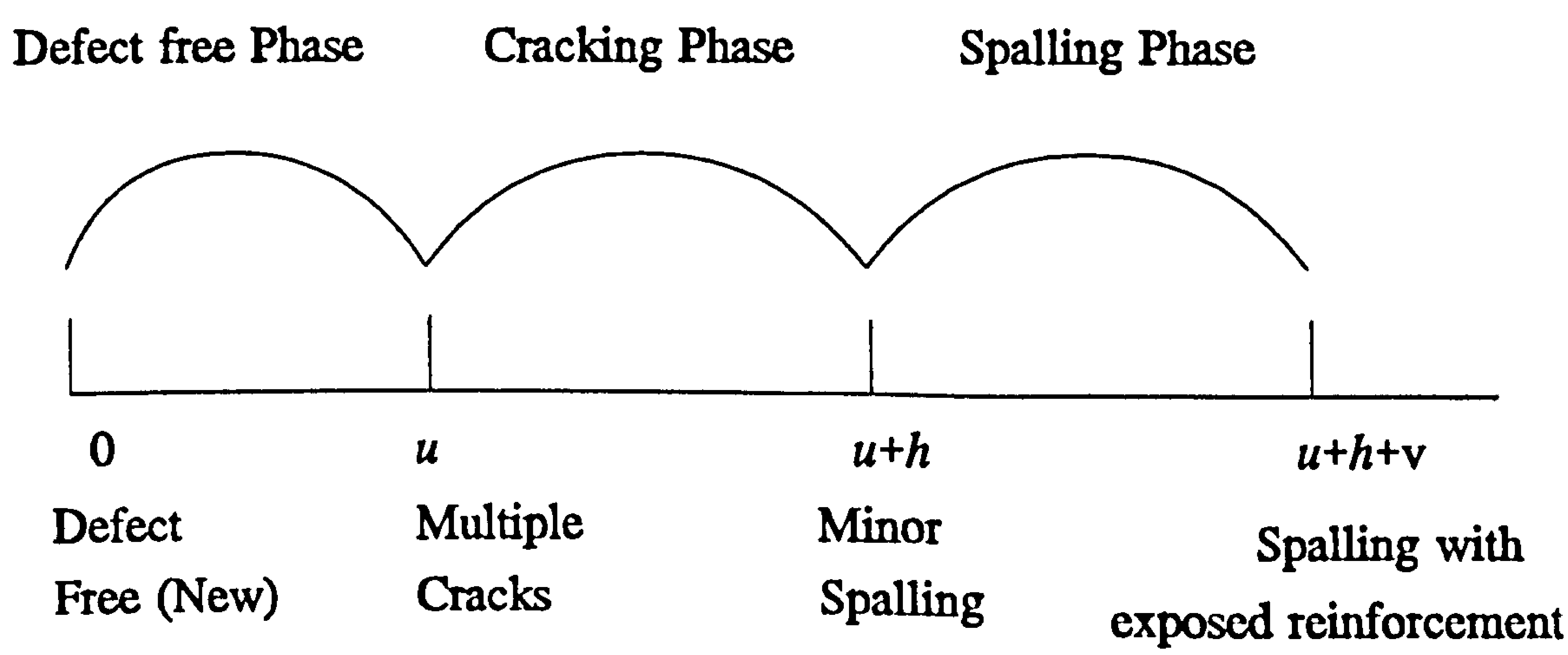

Fig. 6.1. The Deterioration Phases of a Concrete Component.

By dividing the time into respective key phases, the effects of alternative maintenance strategies can be explored. It is noted that the time intervals, u, h and v for this application would be in the order of years, which is in sharp contrast to hours and days observed in the delay time modelling of a mechanical plant, Christer and Waller (1984b). Currie and Robery (1994) model the time from new to cracking and spalling via a chemical process model and estimate the time to cracking at around 25 years and spalling at around 40 years under nominal environmental conditions. However, there was no discussion of the variance of cracking and spalling times. This chapter proposes to model the distribution of time to these states using the inspection records of components that were subjected to varying environmental conditions in order to predict consequences of maintenance and inspection decisions.

6.3. Data Collection and Analysis

Inspection records from London Underground Limited and British Rail of concrete bridges were analyzed by the co-researchers at QMWC. The records contained information of inspection reports of bridge structures which constitutes snapshots of condition spanning more than fifty years in some cases. Data extracted from the records were entered into a linked database organised into three tables concerned with the location, the structure and the defect type. The categories for each table are given in Table 6.1. A unique code is provided for each component which had developed a defect in at least the hairline cracked state. This is then used in linking and querying the database. The structure table contains data on the structure type (e.g an overbridge) and the component type (e.g a flexural beam) which is a member of the structure. The defect table contains data on a set of inspection reports on each defect of a member. Exposure relates to environmental conditions experienced by the component measured on a scale mild, moderate or severe, and could change over the period of inspections undertaken. Urgency is a code relating to the original inspector's judgement on whether to carry out repairs or not. The key fault/action code is a number (1..12) relating to the degree of cracking/spalling of the component or the form of maintenance to be carried out on the component. Current practice is to initiate maintenance on the recommendation of inspectors and engineers following inspection reports. The categories are given in Table 6.2. Fault conditions 1 to 9 are ordered by the research team to represent the perceived stages of degradation. The worst state of the component is recorded.

Complete data on around 700 defect arrivals on 400 bridges are currently held within the database and available for analysis. An example of the development of defects on an overbridge is given in Fig. 6.2. The inspection period adopted was approximately every 2 years before 1950 and every 4 years after 1950.

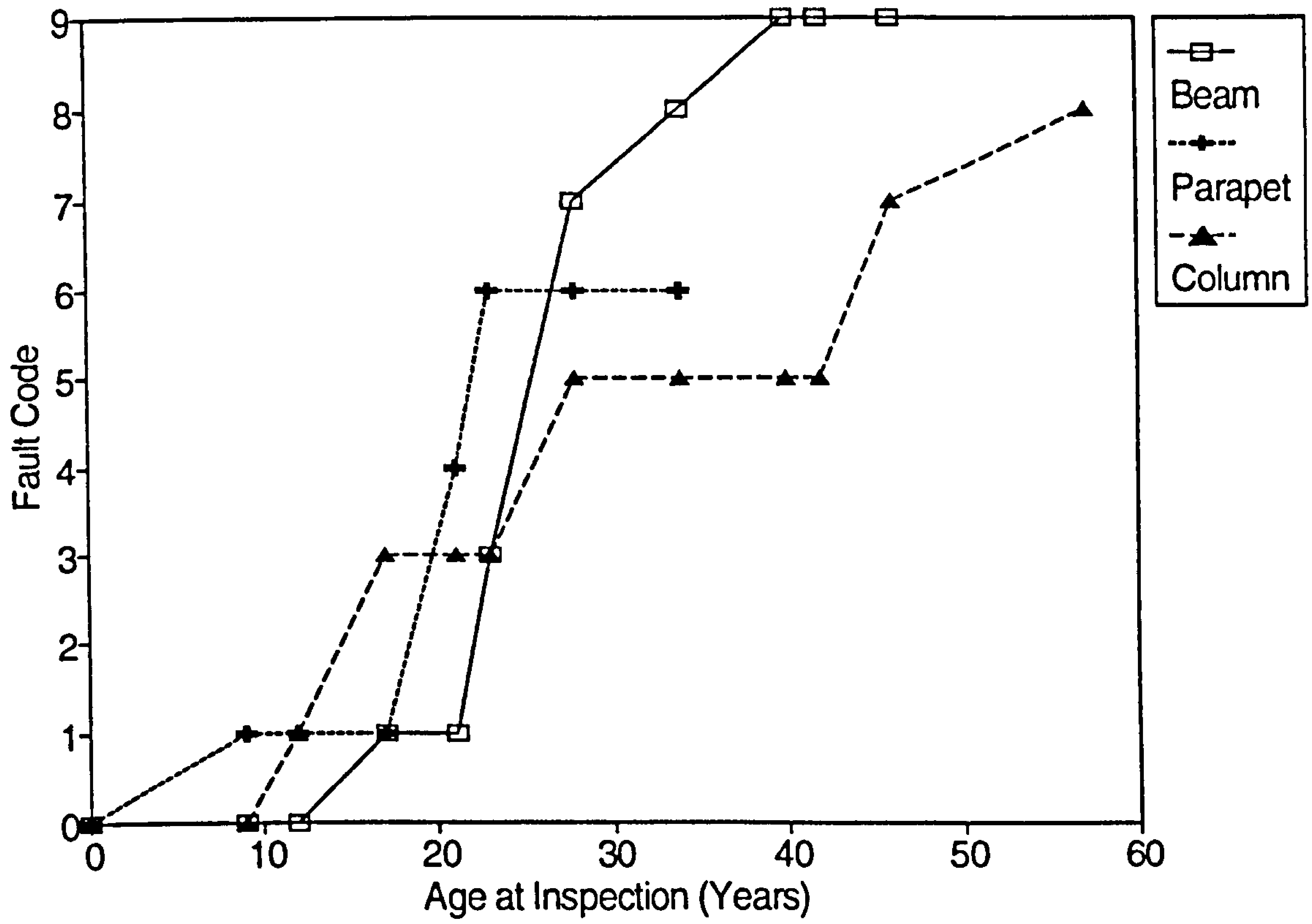

Fig. 6.2. Defect developments on an overbridge built 1928.

Address Table	Structure Table	Defect Table
Code	Code	Code
Address of Structure	Reference	Inspection Year
Comments	Type of Structure	Fault/Action Code
	Year Built	Size of Crack/Spalling
	Component Type	Possible Cause
	Construction	Exposure Condition
	No. of Spans	Urgency
	Maximum Span	Comments

Table 6.1. Categories included in the Database.

Fault Code	Description
1	Hairline Cracks
2	S'ngle Crack (< 1.5mm)
3	Multiple Cracks (< 1.5mm)
4	Single Crack (> 1.5mm, < 3mm)
5	Multiple Cracks (< 3mm)
6	Cracks over 3mm
7	Minor Spalling
8	Spalling with Exposed Reinforcement
9	Severe Spalling
10	Demolition
11	Minor Repairs
12	Major Repairs

Table 6.2. Description of Fault/Action Codes.

In interpreting the model in the context of the data, we require a fault condition corresponding to each of time periods u, h and v. It is recognised that when a component reaches state 3, (multiple cracks), the state is visibly detectable and a repair such as resin injection might be feasible. Hence, we define u to be the age to state 3. Likewise, h is the time interval from state 3 to state 7 (minor spalling), and v is the time interval from state 7 to state 8 (spalling with exposed reinforcement), when a repair is considered necessary. It is to be noted that a component in state 4 would likely have multiple cracks of less than 1.5mm, that is the worst defect defines the state. Hence, a component cannot and will not realistically bypass a state, e.g pass from state 2 to 4. If we refer to the example of defect arrival on the beam in Fig. 6.2, we see that in this case, assuming a 2-6 year inspection period, intervals containing u, h and v can be estimated in years, i.e,

$$\left.\begin{array}{r} u \;\varepsilon\; (21,23) \\ u+h \;\varepsilon\; (23,28) \\ u+h+v \;\varepsilon\; (28,34) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} u \;\varepsilon\; (21,23) \\ h \;\varepsilon\; (0,7) \\ v \;\varepsilon\; (0,11) \end{array}\right. . \qquad (6.1)$$

The data clearly enables bounds upon the estimates of u, h and v to be established.

6.4. Modelling Deterioration of a Component

This section details the methodology used to model the deterioration of a concrete component section subject to one defect arrival over time. Due to defects being recorded only at inspections, a situation of censored data arises. One question which needs to be answered, is whether the times u, h and v are mutually independent. Also, possible dependency on exposure factors may need to be considered. Indications from an earlier data sample show that defect developments within beams and columns share the same p.d.f for u but not for values of h and v. This could be due to differing tensile forces experienced across the element types. Aggressive exposure conditions have indicated an increased rate of defect development which is as expected. As stated earlier, exposure levels can vary over time. Hence, estimation of u, h and v will be undertaken without conditioning on exposure levels, that is we assume exposure is a random factor in defect development.

In the absence of strong indicators to the contrary, we assume the variables are mutually independent and then proceed in deciding on a form or type of distribution for u, h and v for a component type. This choice is aided by histograms of the lower and upper estimates for the variables. Once distribution functions have been decided, maximum likelihood estimation of the parameters of the selected p.d.fs can then be undertaken based upon the interval data. For example, if we let $g(u)$, $f(h)$ and $w(v)$ be the beam component p.d.fs of u, h and v respectively, assuming independence between the phases, then the contribution to the likelihood of the data for the defect arrival on the beam in Fig. 6.2 would be :

$$\int_{u=21}^{23} g(u) \int_{h=23-u}^{28-u} f(h) \int_{v=28-u-h}^{34-u-h} w(v)\, dv\, dh\, du \quad . \tag{6.2}$$

These likelihood functions would be multiplied together across the initial data set for all elements and optimized over the unknown parameters.

Due to defect histories of components having being recorded over an average of 50 years, it is assumed that a component has a probability, p say, that a defect, i.e the deterioration to at least state 3 (multiple cracks), will arise over a 50 year time span. This is to allow for a set of components which are unlikely to develop a fault due to, for example, mild exposure. In effect we are attempting only to model defect arrivals over a 50 year period. In this way, an inspection policy based on reducing costs can be proposed over this finite time horizon. The conditional distribution of u over the 50 year interval will be assumed to be of 'Weibull' form with c.d.f, $G(u)$ say, given by,

$$G(u) \;=\; \frac{[1 \,-\, \exp(-(\alpha_u u)^{\beta_u}))]}{[1 \,-\, \exp(-(50\alpha_u)^{\beta_u})]} \qquad for \quad 0 \le u \le 50 \quad . \tag{6.3}$$

The unconditional p.d.f of u, $g(u)$, is then equal to $pG\ (u)$ in the interval (0, 50), and has not been modelled for $u > 50$. The model is also equivalent to the process whereby a defect on a component will either arrive within 50 years or not at all within its lifetime.

Secondly, it is assumed that a component which develops a defect within 50 years will develop spalling in time, h, where h has c.d.f, $F(h)$ say, given by,

$$F(h) \;=\; 1 \,-\, (1 \,-\, q)(\exp(-(\alpha_h h)^{\beta_h})) \quad . \tag{6.4}$$

The distribution has been selected due to the observation of a substantial number of components which were first detected in the spalling state after a series of defect-free inspections. This suggests that the cracking phase would be small for some components, say in the order of months. The extra parameter q is the finite probability that the delay time, h, to spalling is effectively zero.

The delay time, v, from minor spalling to spalling with exposed reinforcement, will be modelled as exponential with c.d.f, $W(v)$ say, given by,

$$W(v) = 1 - \exp(-(\alpha_v v)) \quad . \tag{6.5}$$

Maximum likelihood estimation will be used to estimate the seven modelling parameters for two component types, namely flexural beams and compressive columns. Results from the two data sets are given and model tests of fit are proposed to test the accuracy of the parameter estimates.

6.4.1 Maximum Likelihood Estimation

A program, ESTu, was written to estimate the Weibull parameters of u and the value of p. The program reads in data on each component of the form $u \in (a, b)$ or $u \in (a, \infty)$, i.e either a defect was detected at age b and the component was defect free at the previous inspection at age a, or the component was detected defect free at the last inspection, age a. The likelihood for the former data type, $u \in (a, b)$, is given by,

$$L_1 = p\frac{\{\exp(-(\alpha_u \min[a,50])^{\beta_u}) - \exp(-(\alpha_u \min[b,50])^{\beta_u})\}}{\{1 - \exp(-(50\alpha_u)^{\beta_u})\}} + I(b)(1 - p) \quad , \tag{6.6}$$

$$\textit{where} \quad a \geq 0, \; I(b) = 0 \; (\textit{for } b \leq 50), \; 1 \; (\textit{for } b > 50) \; .$$

The likelihood for the data type, $u \in (a, \infty)$, is given by,

$$L_2 = p\frac{\{\exp(-(\alpha_u \min[a,50])^{\beta_u}) - \exp(-(50\alpha_u)^{\beta_u})\}}{\{1 - \exp(-(50\alpha_u)^{\beta_u})\}} + (1 - p) \quad , \tag{6.7}$$

which allows for the two possibilities that a defect will develop or not within the 50 year time span, for $a \geq 0$.

A program, ESTh, was written to estimate the Weibull parameters of h, and the parameter, q. To reduce computation time, the program conditions on the estimate of u from ESTu, i.e the parameters (p, α_u, β_u), so that only three parameters need to be estimated, namely (q, α_h, β_h). The program reads in data on each component which has developed a spalling or crack defect within 50 years, i.e $u \in (a, b)$ where $b \leq 50$. The two types of data are of the form, $u + h \in (c, d)$ or $u + h \in (c, \infty)$, i.e either a defect was detected as a spall at age d and the component was cracked (or defect free) at the previous inspection c, or the component was detected cracked at the last inspection, age c. The likelihood for the first data type is given by,

$$L_3 = p \int_{u=a}^{b} G'(u)\{F(d-u) - F(c-u)\}du \quad , \tag{6.8}$$

and the likelihood of the second data type is given by,

$$L_4 = p \int_{u=a}^{b} G'(u)\{1 - F(c-u)\}du \quad . \tag{6.9}$$

A program, ESTuh, was also written to estimate the parameters of u and h without conditioning on u. The parameters obtained using this method produced exactly the same parameter estimates obtained in ESTu and ESTh.

A program ESTv was written to estimate the v phase. Again, and to reduce computation time, the program reads in the parameters (α_u, β_u, q, α_h, β_h) estimated from the programs ESTu, ESTh and ESTuh. Therefore, only the parameter, α_v, needs to be estimated. The program reads in data on each component which has developed a failure or spalling defect and u is known to be less than 50 years. The two types of data are of the form, $u + h + v \in (e, \infty)$ or $u + h + v \in (e, f)$, i.e either a defect was last detected as a spall (not severe) at age e or the component was detected failed at age f. The likelihood for the first data type is given by,

$$L_5 = p \int_{u=a}^{b} G'(u)\left(q(1 - W(e-u))I(b,d) + \int_{h=c-u}^{d} F'(h)(1 - W(e-u-h))dh \right) du \tag{6.10}$$

where $I(b,d) = 0$ (*for* $b \neq d$), 1 (*for* $b = d$) .

The likelihood for the second data type is given by,

$$L_6 = p \int_{u=a}^{b} G'(u)\left(q(W(e-u) - W(f-u))I(b,d) + \int_{h=c-u}^{d} F'(h)(W(e-u-h) - W(f-u-h))dh \right) du \tag{6.11}$$

6.4.2 Estimation on real data

A maximum likelihood calculation was undertaken on data from 195 beam components and 176 column components using the programs EST*u*, EST*h* and EST*v*. A check was also made using the program EST*uh*. The same parameter estimates for *u* and *h* were obtained with this procedure. The results are given in Table 6.3.

Parameter	Beams 195	Simulation (200)	(500)	Columns 176	Simulation (200)	(500)
p	0.665	(0.640)	(0.676)	0.472	(0.445)	(0.482)
α_u	0.035	(0.035)	(0.036)	0.030	(0.024)	(0.029)
β_u	2.08	(1.89)	(2.08)	1.90	(1.63)	(2.10)
q	0.472	(0.446)	(0.434)	0.379	(0.382)	(0.356)
α_h	0.036	(0.033)	(0.036)	0.022	(0.022)	(0.024)
β_h	1.04	(1.02)	(1.01)	2.11	(2.16)	(2.06)
α_v	0.105	(0.135)	(0.110)	0.033	(0.033)	(0.033)
$E(u \mid u < 50)$ (Years)	24.2	(23.3)	(23.7)	25.3	(26.2)	(26.8)
$E(h)$ (Years)	14.3	(16.9)	(15.8)	24.6	(24.9)	(24.3)
$E(v)$ (Years)	9.5	(7.4)	(9.1)	30.3	(30.3)	(30.3)

Table 6.3. Parameter estimate results for beams and columns.

A simulation was undertaken to test the proposed method. Parameter estimates for the beam and column components in Table 6.3 were selected as the theoretical parameters of the simulation. A sample of 200 and 500 components were simulated with an inspection period of 2 years over a 50 year time span. The parameter estimates results are given in brackets in Table 6.3. For the case of beam components, the estimated mean values of *u*, *h* and *v* can be seen to converge to the input parameters. For the case of column components, slight divergence in the mean of *u* for the larger sample can be seen. However, the individual estimated parameters of *u* are closer to the input parameters.

The main point, here, is that it is possible to obtain estimates of the distribution functions of discrete snapshots in the concrete degradation process obtained from available interval and censored data. It then remains to test the fit of the distribution and to model the consequences of maintenance actions.

6.4.3 Test of Model Fit

Due to the censored nature of the data, a test of model fit was developed so as to accommodate the interval type data. This test involves the Kaplan-Meier (K-M) estimate of the reliability function for progressively censored samples, see Kalbfleisch and Prentice (1980). Two K-M estimates are plotted by taking the uncensored observations of u, first at $u = A$, and then at $u = B$, for the interval constrained event, $u \varepsilon (A, B)$. Due to discrete observations of u, the K-M estimator, a nonparameteric representation, $R^*(u)$ say, of the reliability function, $R(u) = 1 - pG(u)$, is given by,

$$R^*(u) = \prod_{j\,:\,u_j < u} \left(1 - \frac{d_j}{n_j}\right) , \tag{6.12}$$

where $\{u_j\}$, is the sample of uncensored u values, d_j is the number of coincidental defective components at inspection time u_j, and n_j is the number of defect free components at time u_j. The distribution of the K-M estimator is approximately normal. The approximate variance of $R^*(u)$, $V^*(u)$ say, is given by,

$$V^*(u) = R^*(u)^2 \sum_{j:\,u_j < u} \frac{d_j}{n_j(n_j - d_j)} . \tag{6.13}$$

The variance can then be used to construct a 95% confidence interval (C-I) estimate for the reliability function of u, $R(u)$, that is $R^*(u) \pm 1.96\sqrt{V^*(u)}$, for $u \leq 50$. The maximum likelihood reliability estimate can then be tested for accuracy by seeing if it lies within the estimated confidence bounds over the 50 year range. Similar procedures apply to time to spalling, $u + h$, and times to spalling with exposed reinforcement, at $u + h + v$.

Plots for beams and columns are given in Figures 6.3 - 6.8. The maximum likelihood reliability fits, formulated using functions (6.3), (6.4) and (6.5), are plotted along with the upper and lower K-M estimates, and the lower (upper) C-I bound for the lower (upper) K-M estimate. The estimated C-I interval plotted would then be expected to be slightly overestimated due to the interval constrained data. It can be seen that close fits between the K-M estimate and the maximum likelihood estimate are achieved for both beams and columns for u and $u + h$ over 50 years, and $u + h + v$ over 35 years for beams and 50 years for columns. The lower and upper K-M estimators can be seen to differ by approximately 4 years. This is due to the inspection period being approximately 4 years. There are also crossovers between the two K-M estimators in the plots for time to failure, Figs. 6.7 and 6.8. This is believed to be due to the small sample size of detected failures. The maximum likelihood fits in all cases lie within the confidence bands of the reliability function. Hence, given the additional confidence, we accept the estimated model for concrete degradation. Interestingly, the hazard function for u for both beams and columns is not strictly increasing over the range (0, 50) years. The hazard has a convex shape with a peak at around the conditional mean value of u. This implies that a beam remaining in the defect free state for around 25 years is unlikely to develop a defect over a 50 year span, say due perhaps to mild environmental conditions or denser concrete.

For a test of independence between phases u and h, the sub-sample of components where spalling was detected was selected. The mid-point of the interval of u, say $u_m = (a+ b)/2$ was selected as a point estimate of u. Given this value of u, the value of h would occur in interval $(c - u_m, d - u_m)$. The mid-point of this interval was then selected as an estimate of h. For a test of independence between phases h and v, the sub-sample of components where spalling with exposed reinforcement was detected was selected. The mid-point of the interval of $u + h$, say $s_m = (c + d)/2$ was selected as a point estimate of $u + h$. Given this value of $u + h$, the value of v would occur in interval $(e - s_m, f - s_m)$. The mid-point of this interval was then selected as an estimate of v. The scatter plots for both cases are given in Figs. 6.9 and 6.10, and clearly indicate independence between states.

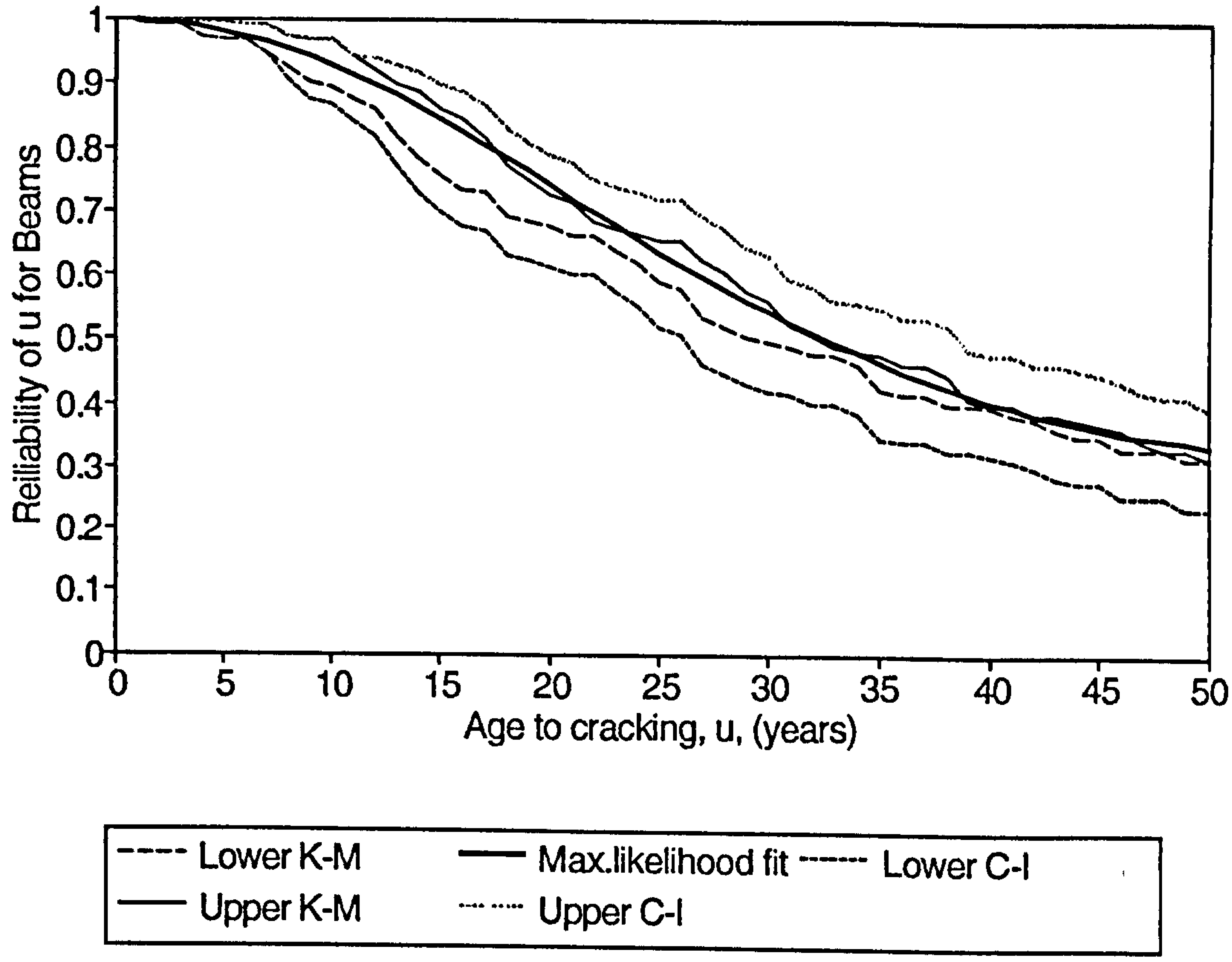

Fig. 6.3. Reliability plots for time to cracking, *u*, for beams.

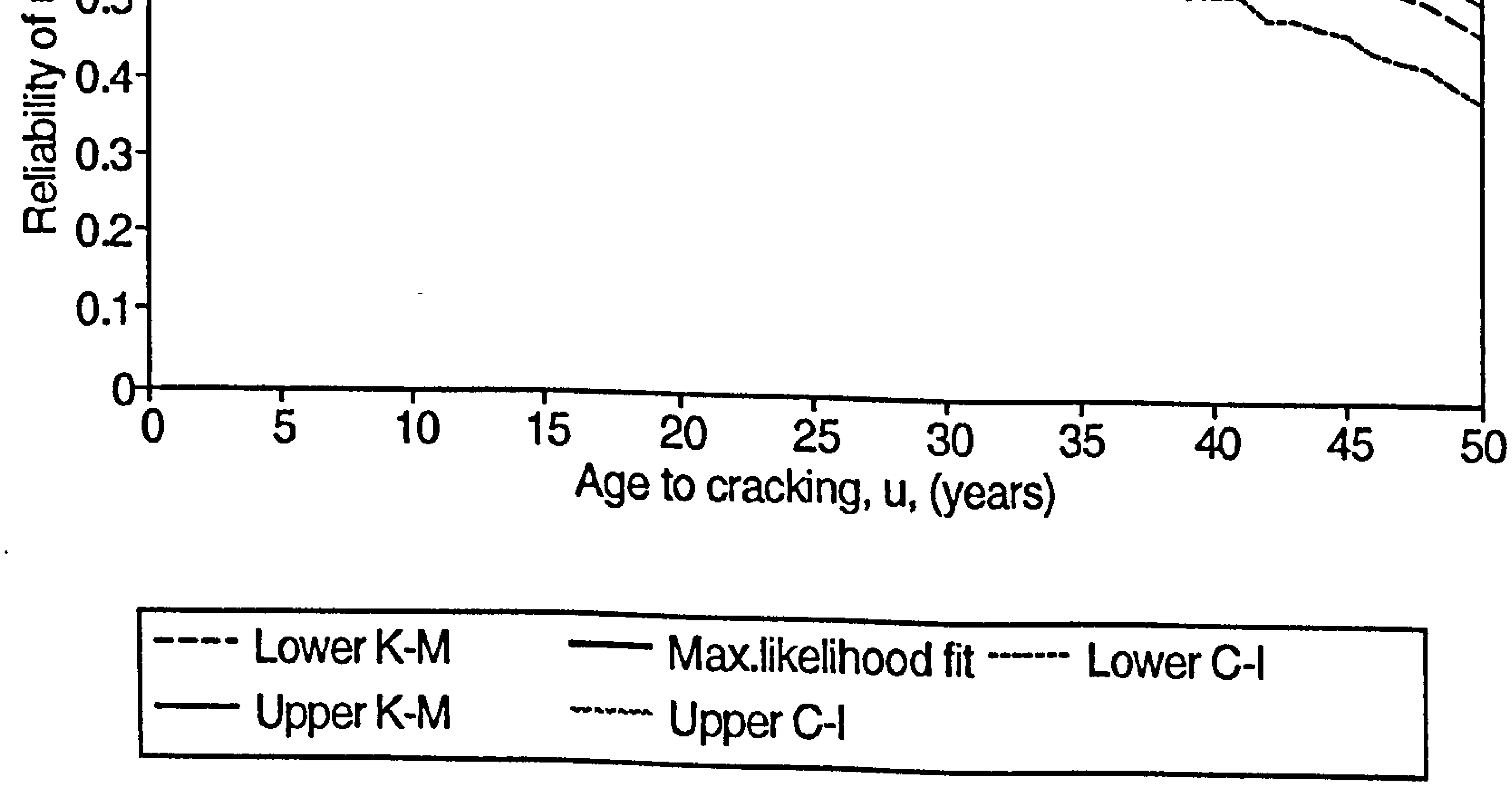

Fig. 6.4. Reliability plots for time to cracking, *u*, for columns.

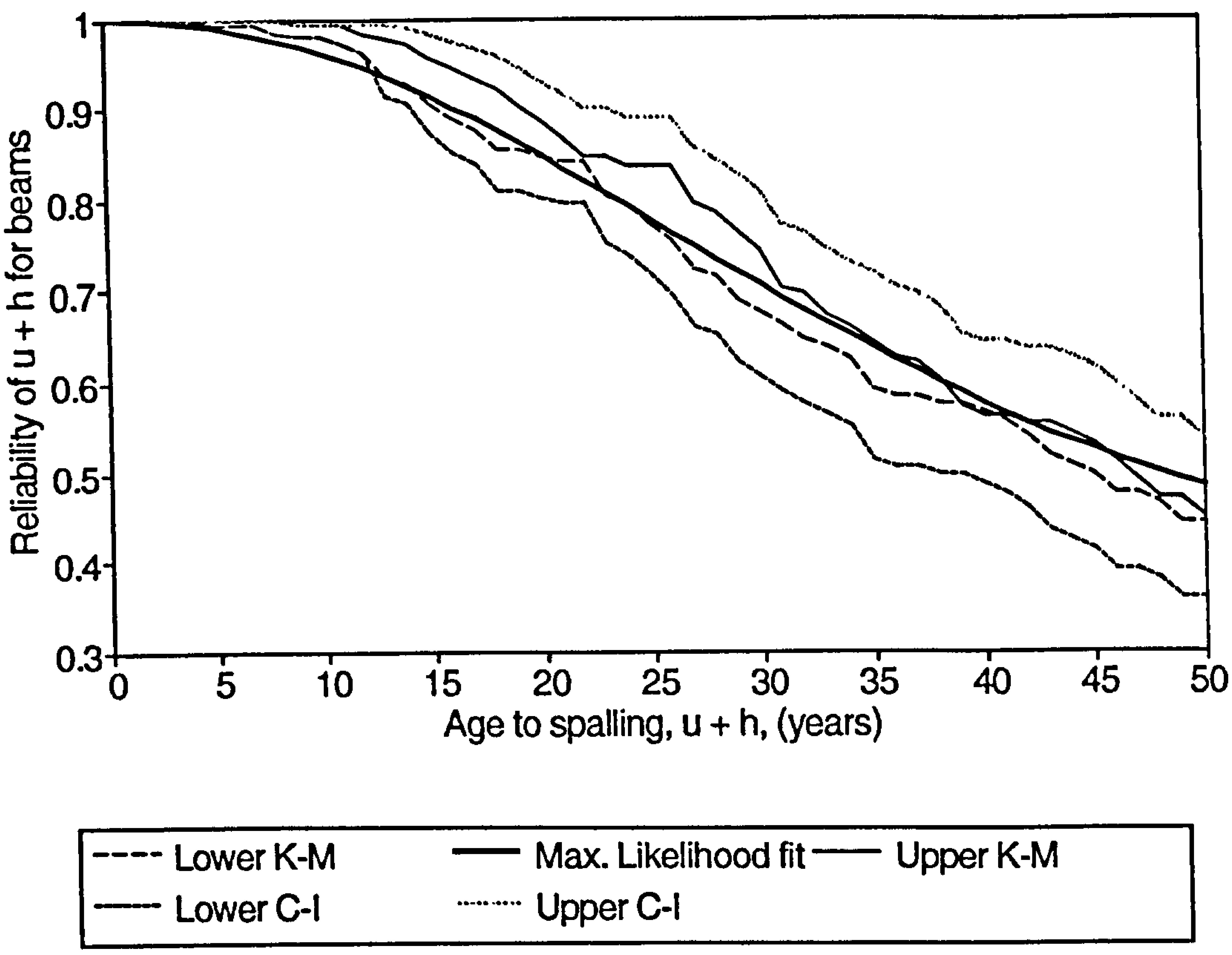

Fig. 6.5. Reliability plots for time to spalling, $u + h$, for beams.

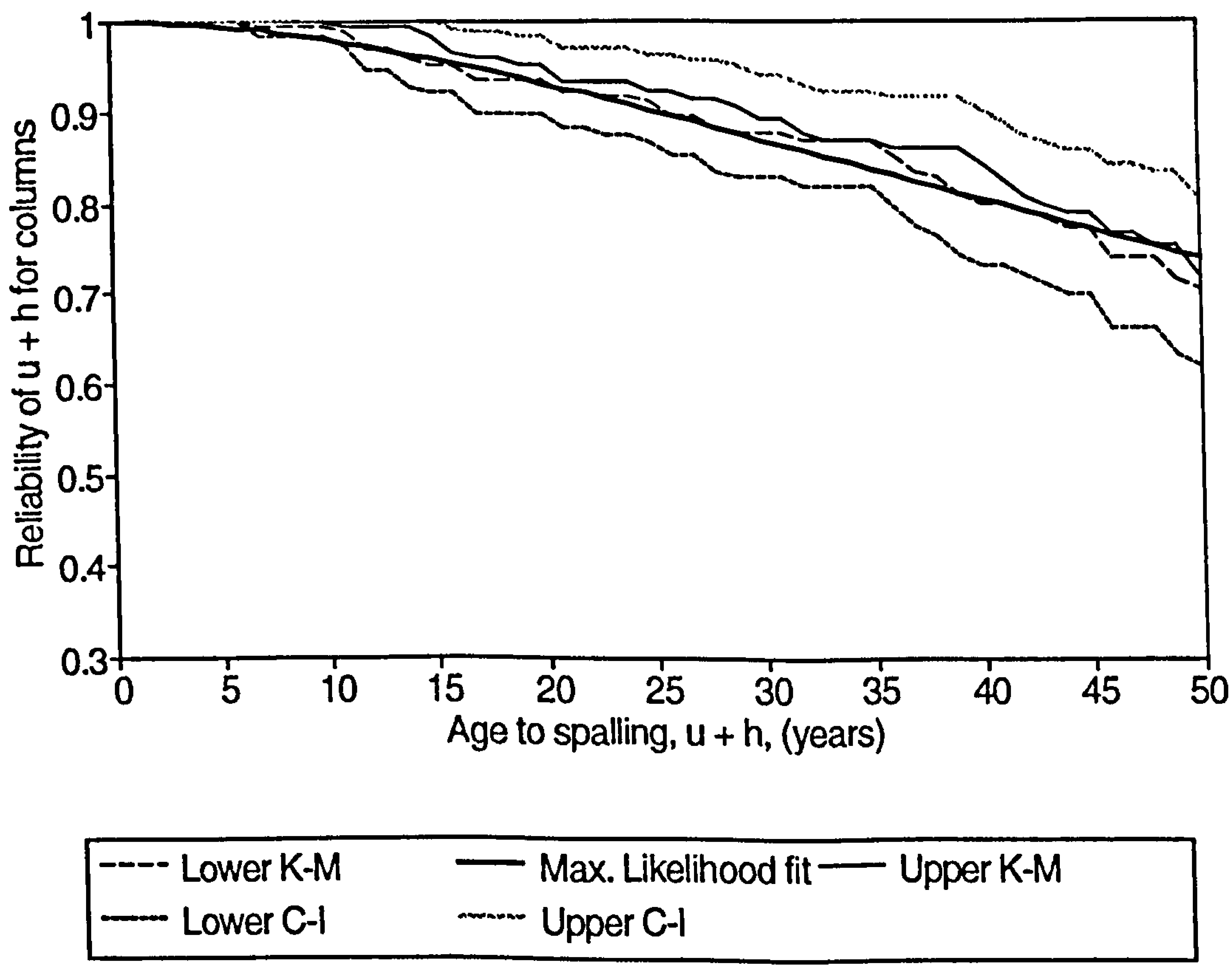

Fig. 6.6. Reliability plots for time to spalling, $u + h$, for columns.

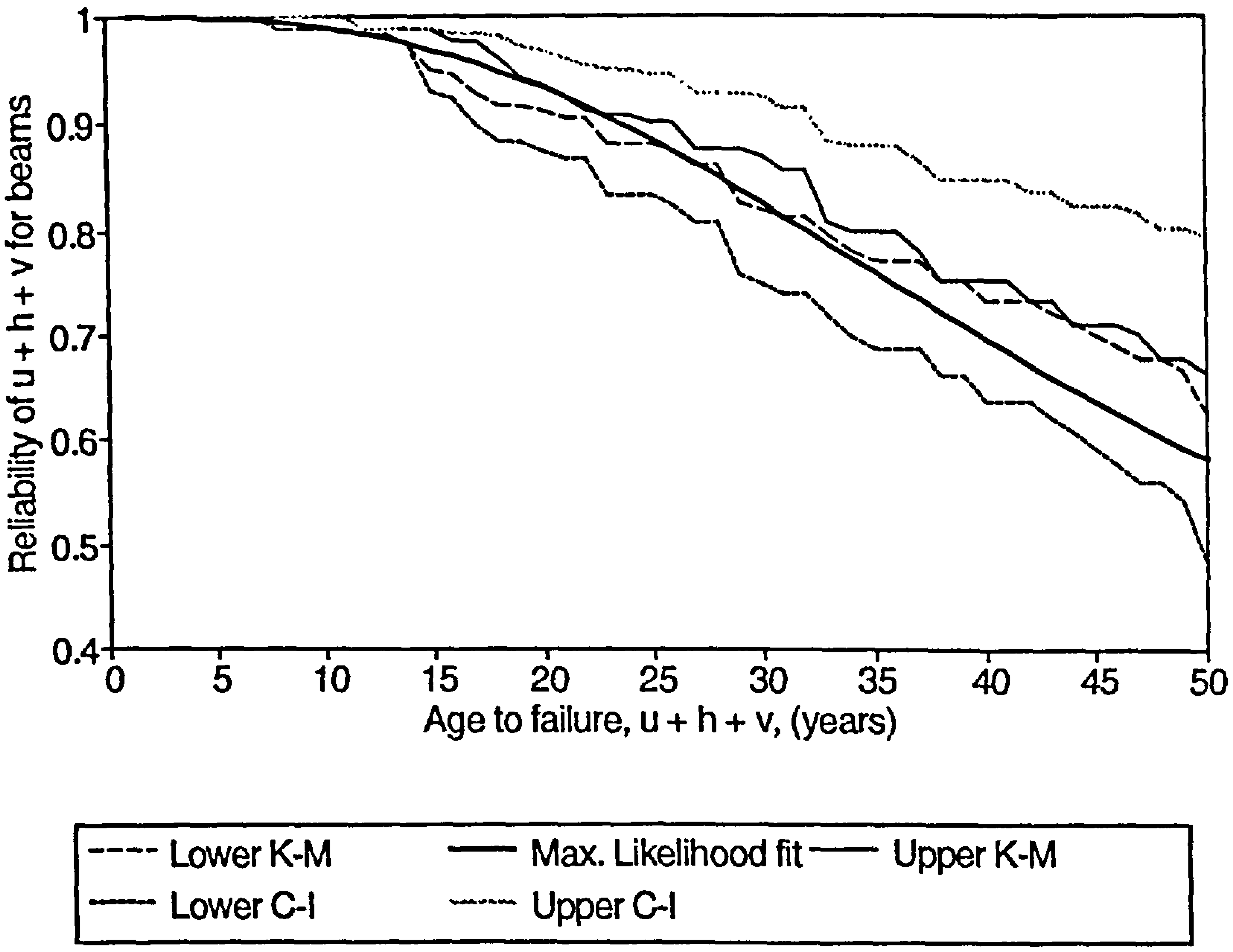

Fig. 6.7. Reliability plots for time to failure, u + h + v, for beams.

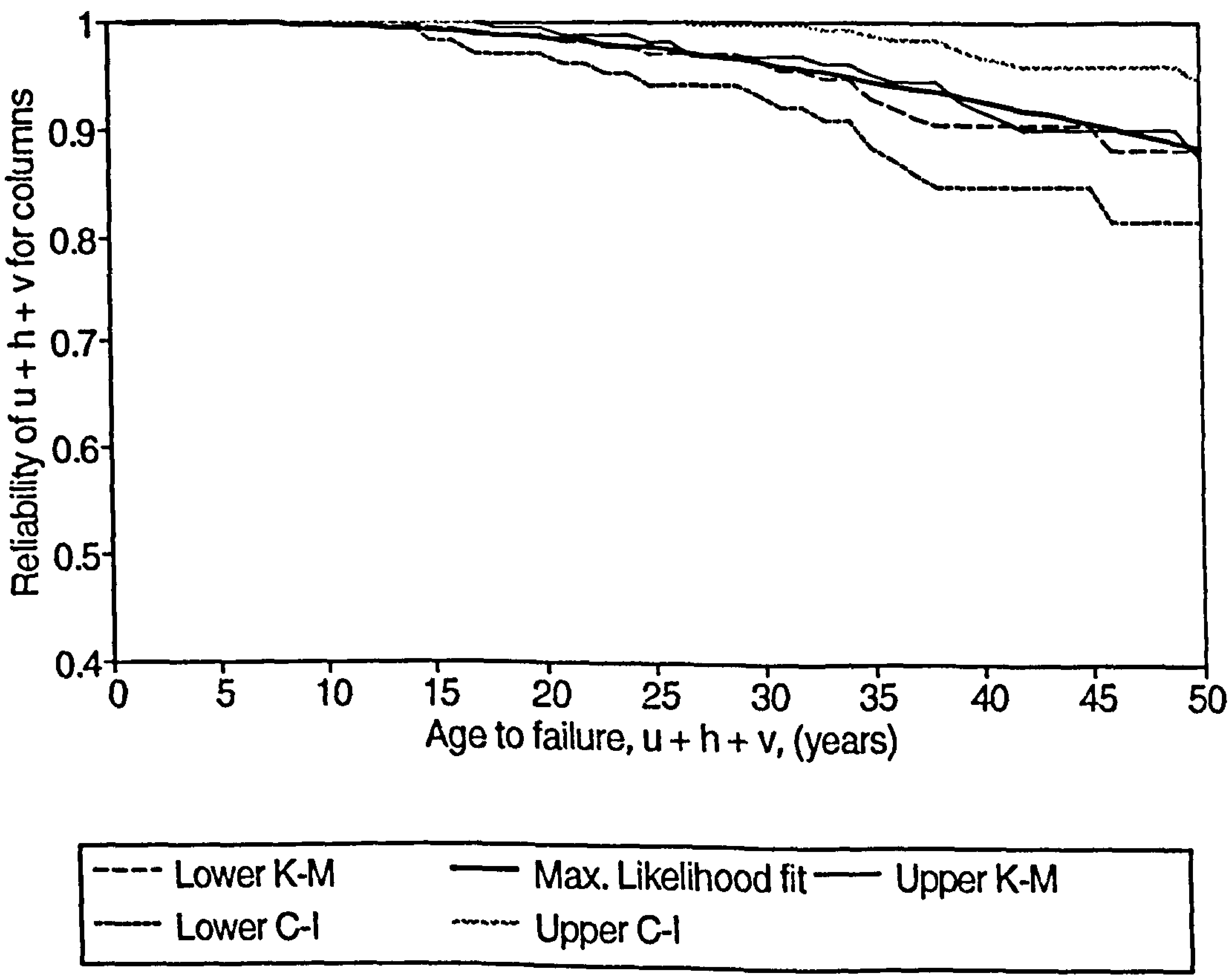

Fig. 6.8. Reliability plots for time to failure, $u + h + v$, for columns.

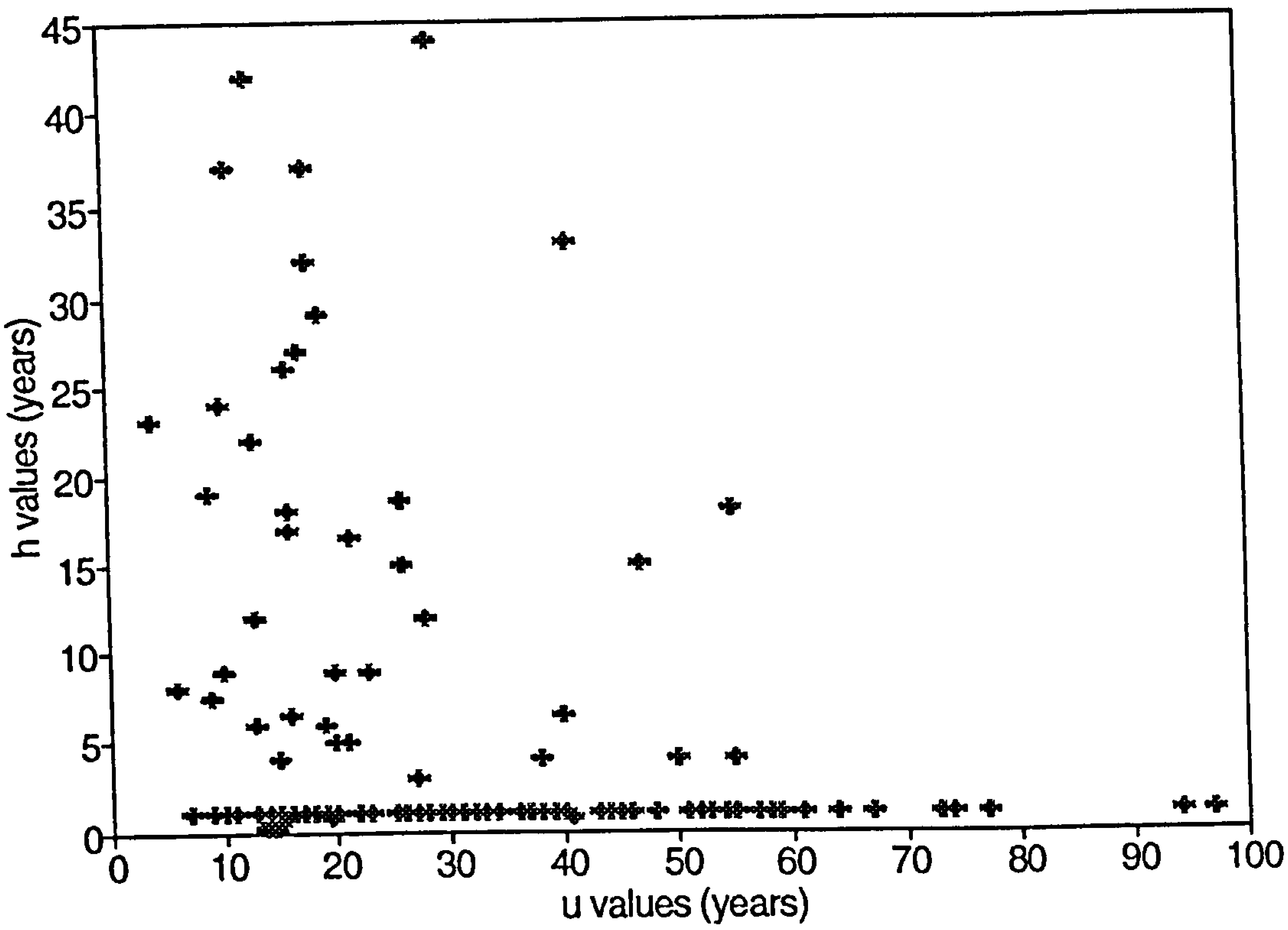

Fig. 6.9. Scatter plot of *u* and *h* estimates for beam and column components.

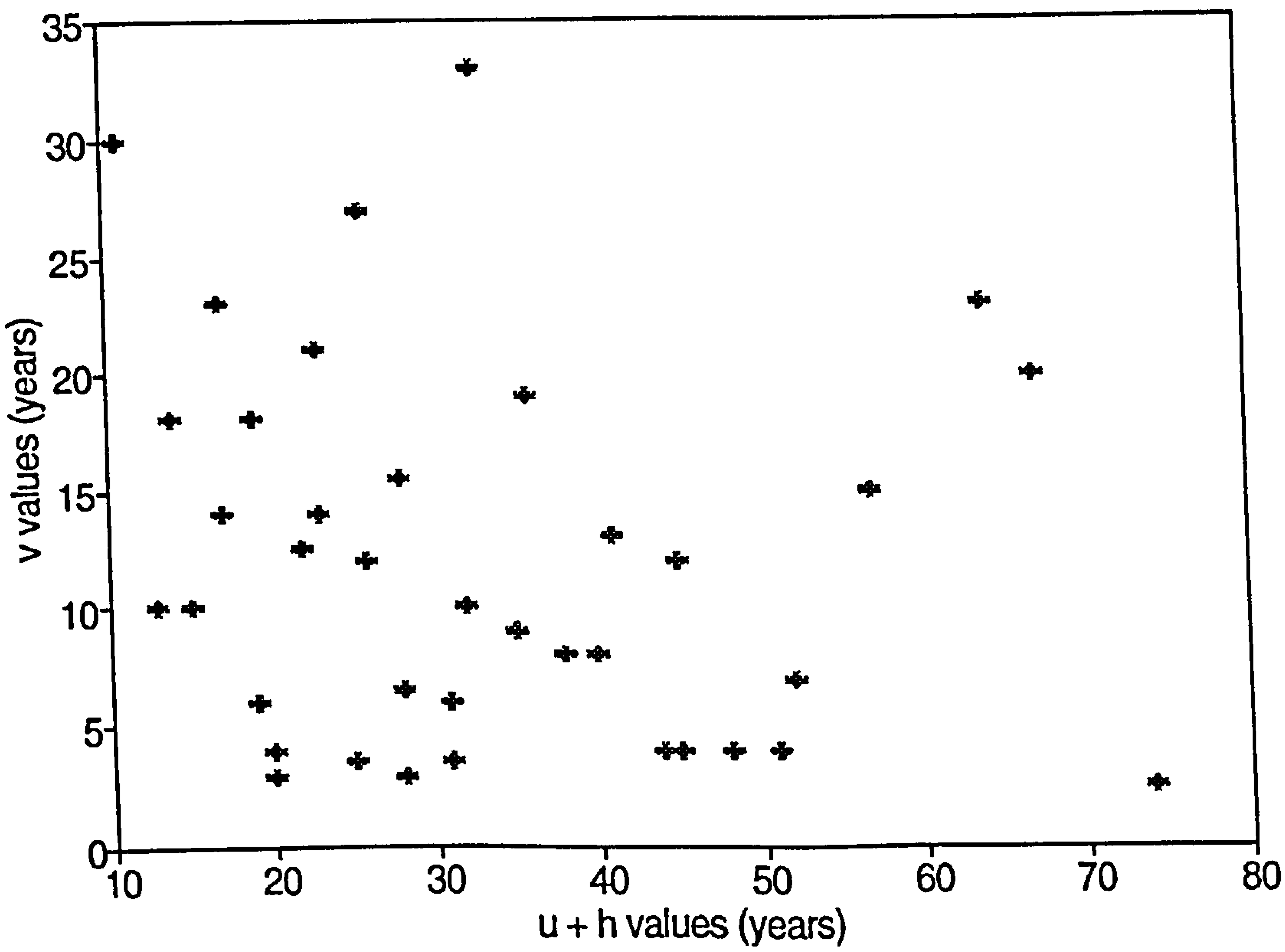

Fig. 6.10. Scatter plot of *h* and *v* estimates for beam and column components.

Having established basic delay time parameters for the various stages of the deterioration process, it is now possible to model the cost consequences of an inspection process.

6.5. Maintenance Models of Cost

When cracks develop in a component, a repair such as resin injection, would vary in cost depending on the location and severity of cracks. Likewise, patch repairs to spalled areas would increase in cost over time when larger areas of the component become affected. When a component reaches the state of necessary major repairs, hazard and safety cost could be incurred until the component is repaired. Frequent effective inspections followed by appropriate action would reduce costs of this type, but a compromise must be sought due to the high cost of inspections. A conceptual example of the cost of a repair over time is given in Fig. 6.11.

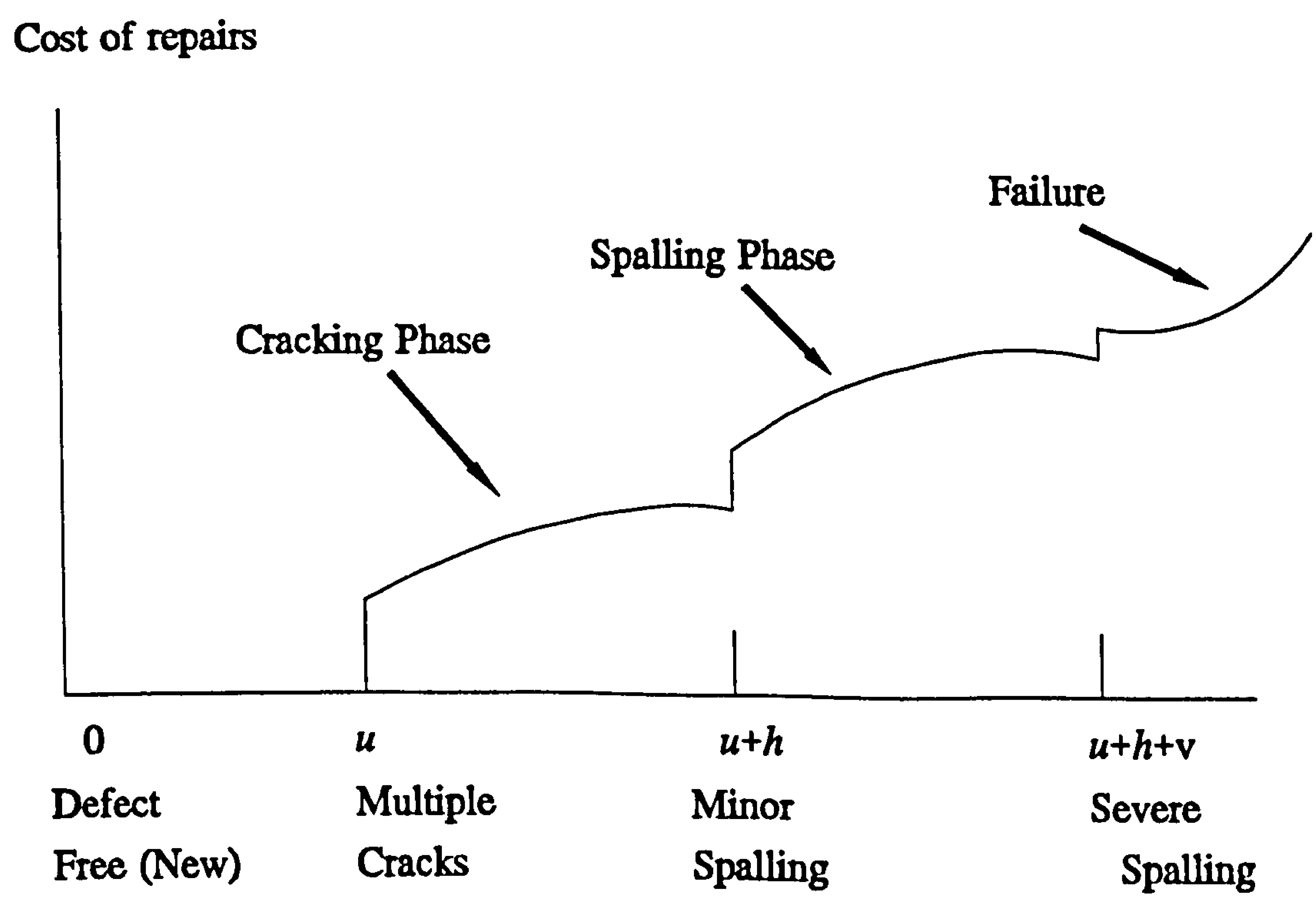

Fig. 6.11. An example of repair costs over the deterioration phase.

Informal and formal meetings were held with the civil engineers at Queen Mary and Westfield College with a view to estimate this cost profile. A questionnaire was designed to be completed by staff at Concrete Repairs Ltd and twelve cases of various defect types at different stages of deterioration were selected for assessments. The questionnaire included background information on the defect, e.g a photograph, description and location of the structure, and the extent of degradation.

Questions were put on the necessary action to establish the cause of the degradation and the costs of inspection, investigation, mobilization and repair of the structure. Questions were also put concerning delay time :

1. Would you consider that this repair should be carried out;
   a. within one year.
   b. in 3-5 years.
   c. after 10 years.
   d. no action would be needed.

2. What would be a reasonable estimate of expected repair cost if:
   a. the fault was left for another ten years.
   b. the fault was repaired when first noticed. How long ago would you estimate this might reasonably have been first noticed.
   c. the fault was left until repair was absolutely essential.

With a sufficient sample size of this subjective type of data, estimates of the repair cost functions for cracking and spalling can be obtained. However, the data will only be of limited value due to the small sample size. The responses to questions, implied the estimator(s) did not fully understand what was being sought. The data was received near to the end of the project completion time, and if received earlier, a revised questionnaire would have been submitted and meetings arranged to clarify further the nature and use of delay time measures.

In an effort to alleviate the problems encountered, it was decided to estimate the expected repair cost of particular states of degradation.
The estimated repair costs per beam or column were:

1. Multiple cracks (£1080).
2. Severe cracks (£1680).
3. Minor spalling (£1370).
4. Severe spalling (£1870).

The costs were based on a typical scenario for a component of a set size with an estimated expected extent and area of degradation when detected in these states. The cost of inspection was estimated as £50. The investigation, access and de-mobilisation cost was estimated at £1650. Anti-carbonation treatment, which can be applied to define defect free components, was estimated at £480. The costs above were accumulated from estimated costs of each of the stages of the repairing process. It can be seen that the estimated cost of minor spalling is lower than that of severe cracks. This is due to the high cost of crack injection fluids. It was also estimated that repair costs after the failed state double in ten years. The current day repair cost was assumed not to be dependent on the age of the component. It is noted that the above costs do not increase uniformly with deteriorating state.

For the modelling phase, the expected repair cost is assumed to increase linearly over each of the phases, cracking and spalling. Penalty factors could also be attached for the time when a defect in the failed state is left unrepaired.

6.1 Single Component Cost Models

A simple model will be given here to show how the consequences of maintenance decisions can be modelled for a component section subject to a single defect. We adopt the following assumptions:

(a) The p.d.fs for u, h and v are $g(u) = pG'(u)$ for $u \in (0, 50)$, $f(h) = F'(h)$ and $w(v) = W'(v)$, respectively.

(b) An inspection is perfect, and of cost I, and is to be first scheduled at time, T.

(c) If at time T a component is found to be in either the cracked or the spalling state repair state at inspection, then it is repaired perfectly to the 'as-new'condition. The repair time is very small compared to T and may be assumed both instantaneous and undertaken at time T.

(d) The expected cost of a crack repair, at time T, is given by $c(y; h)$ where y is the time since the crack first arose, i.e $y = T - u$ at time T, and h is the delay time of the defect, where $h > y$.

(e) The expected cost of spalling repair is given by $s(y; v)$ where y is the time since spalling occurred, i.e. $y = T - u - h$ at time T, where $v > y$.

(f) The expected cost of major repairs is $m(y)$ if inspected in the severe spalling state, where y is time spent in the failed state.

Further to these assumptions, we assume that a defect or failure is only spotted at an inspection. This refelects observed current practice.

Consider the expected repair cost of a defect which is observed as a crack at inspection time T, with associated defect arrival time $u < T$. Let the expected repair cost be denoted by $R_c(T;u)$. For the defect to arise as a crack it must have delay time $h > T - u$. The expected repair cost for a particular $h > 0$ will be $c(T - u;h)$. Hence summing over all possible h,

$$R_c(T; u) = \int_{h=T-u}^{\infty} f(h)\, c(T - u;h)\, dh \quad . \tag{6.14}$$

Next, let the expected repair cost of a defect identified as spalling at inspection time T with, again, associated defect arrival time $u < T$, be denoted by $R_s(T; u)$. For the defect to arise in the spalling state, it must have delay time $h < T - u$. The expected repair cost for a particular v and h will be $s(T - u - h; v)$ for $v > T - u - h$ and $m(T - u - h - v)$ for $v < T - u - h$. Hence summing for all possible h and v, we have,
The expected cost per unit time, $c(T)$ say, over the first interval $(0, T)$, can then be derived,

$$R_s(T;u) = \int_{h=0}^{T-u} S(T,u,h)f(h)dh + qS(T,u,0)$$

$$\text{where} \quad S(T,u,h) = \int_{v=T-u-h}^{\infty} s(T-u-h;v)w(v)dv + \int_{v=0}^{T-u-h} m(T-u-h-v)w(v)dv \ . \qquad (6.15)$$

$$c(T) = \frac{1}{T}\left\{ I + \int_{u=0}^{T} g(u)\{R_c(T;u) + R_s(T;u)\}du \right\} \ . \qquad (6.16)$$

The inspection point T could then be selected so as to minimize this objective function. An adaptive dynamic policy can be undertaken here using information on the state of the defect gathered from each inspection. Once an inspection is completed, the optimum time to next inspection can be determined. Under current assumptions, if a component defect is noticed and repaired to as new at T, then the instant T is a renewal point for the beam. Should no defect be found, the optimum time to the next inspection would be derived from cost function, as before, but with a modified p.d.f for u, namely, $p.g(u + T)/(1 - p.G(T))$, for $u \; \varepsilon \; (T, 50)$. The process is clearly repeatable.

Consider the following numerical example for flexural beams, where the p.d.fs of u, h and v are given by the distributions in the previous section. An example of costs are given by:

$I = 50$, $c(y;h) = 2730 + 600y/h$, $s(y;v)=3020 + 500y/v$ and $m(y) = 3520 + 1870y/10$, which include investigation and access costs for defective components. The costs were estimated by QMWC and based on a typical scenario for a component (beam or column) of an expected area ($40m^2$), with an estimated expected extent and area of degradation when detected in states 3, 6, 7 and 8. As can be seen, the expected repair costs are assumed linear over each delay time phase. The smaller the delay time the faster is the rate of repair cost over the delay time period and vice-versa. The expected costs per unit time over the first inspection point T for beams and columns are given in Fig. 6.12. It can be seen that an optimum occurs at around 4 years for beams and 5 years for columns. The expected cost for columns is lower than beams and more shallow due to the estimated lower rate of deterioration. Interestingly, the current inspection practice is around 4-5 years for the bridges analyzed.

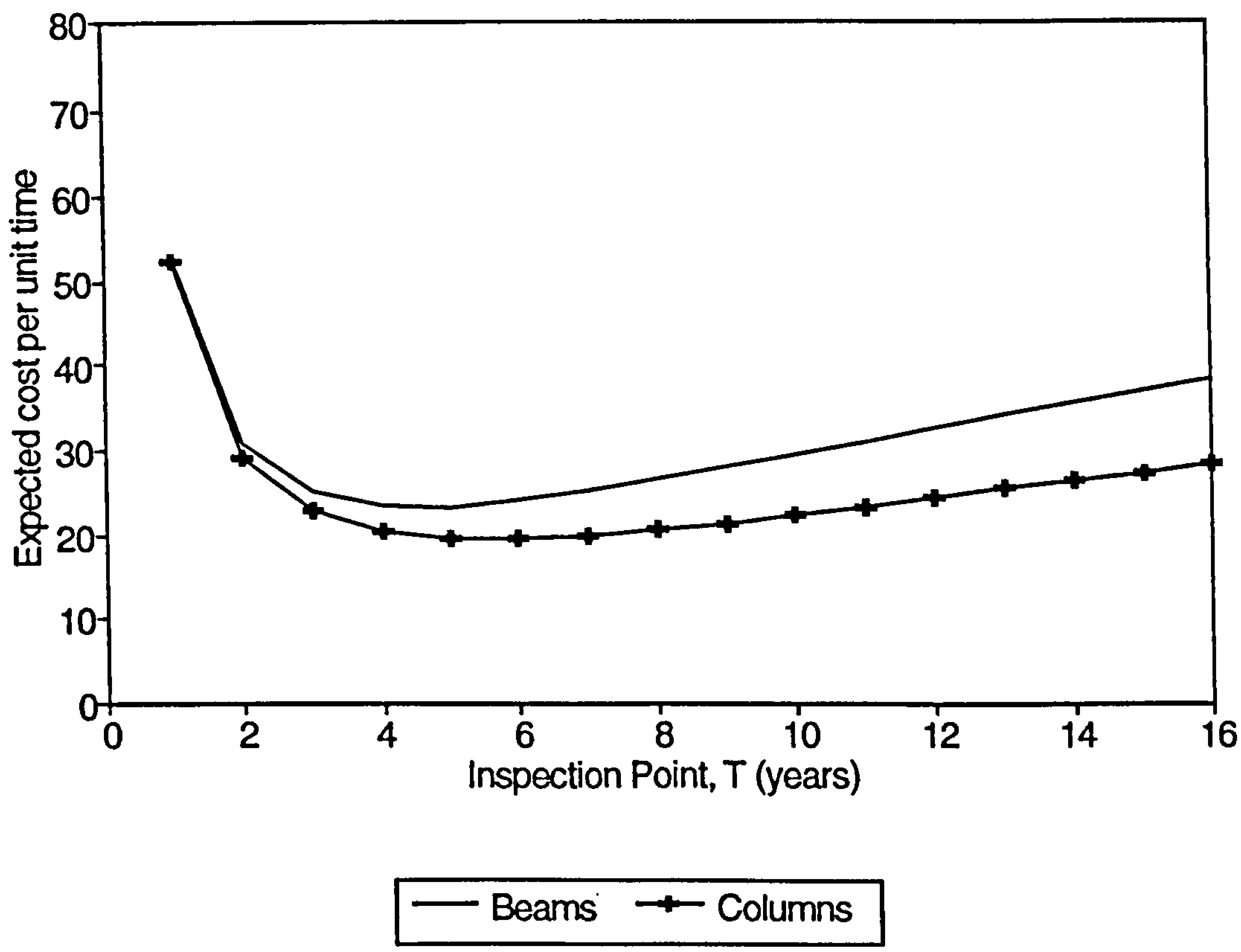

Fig. 6.12. Expected cost per unit time over first inspection point for beams.

6.5.2 Multi-Component Cost Models

In practice, it is usually practical to carry out inspections on all concrete components on a bridge at inspection. Consider a simple bridge consisting of, N_B say, beam components and N_C say, column components. The components are further assumed to be independent. Consider a policy to renovate the bridge periodically every T years with set-up cost I per renovation (involving access and investigation, £1650). Assume that on inspecting a component in at least the multiple cracked state that a repair is initiated, and that the component is restored to the 'as-new' condition. Also, for a component not defective, anti-carbonation treatment is used to restore the component to an assumed 'as-new' state. The expected cost per unit time, $c(T)$ say, is given by,

$$c(T) = \frac{1}{T}(I + N_B(M_B(T) + A) + N_C(M_C(T) + A)) \ , \qquad (6.17)$$

where $M_B(T)$ and $M_C(T)$ are the expected maintenance repair costs, at T, for a beam and column respectively, formulated in function (6.16), using the respective parameterized p.d.fs for u, h and v, functions (6.3), (6.4) and (6.5).

The defect repair costs, $c(y;h) = 1080 + 600y/h$, $s(y;v)=1370 + 500y/v$, $m(y) = 1870(y/10 + 1)$ and $A = 480$, is the anti-carbonation treatment cost.

For a numerical example, assume a bridge consisting of 5 beams and 3 columns. Assume the inspection and repair cost functions are given in Section 6.5.1 and the parameter estimates for u, h and v in Section 6.4. The total expected maintenance cost per unit time, $c(T)$, over the period to the first rennovation T, is given in Fig. 6.13. As can be seen the optimum is around 20-30 years which is considerably in excess of the five year time period for the visual element inspection model of Fig. 6.12. This is due both to the high set-up cost applied at the time of rennovation (£1650 as opposed to £50) and also to the fact that if all elements are defect free or in deterioration states 1 or 2, then an anti-carbonation treatment will be given.

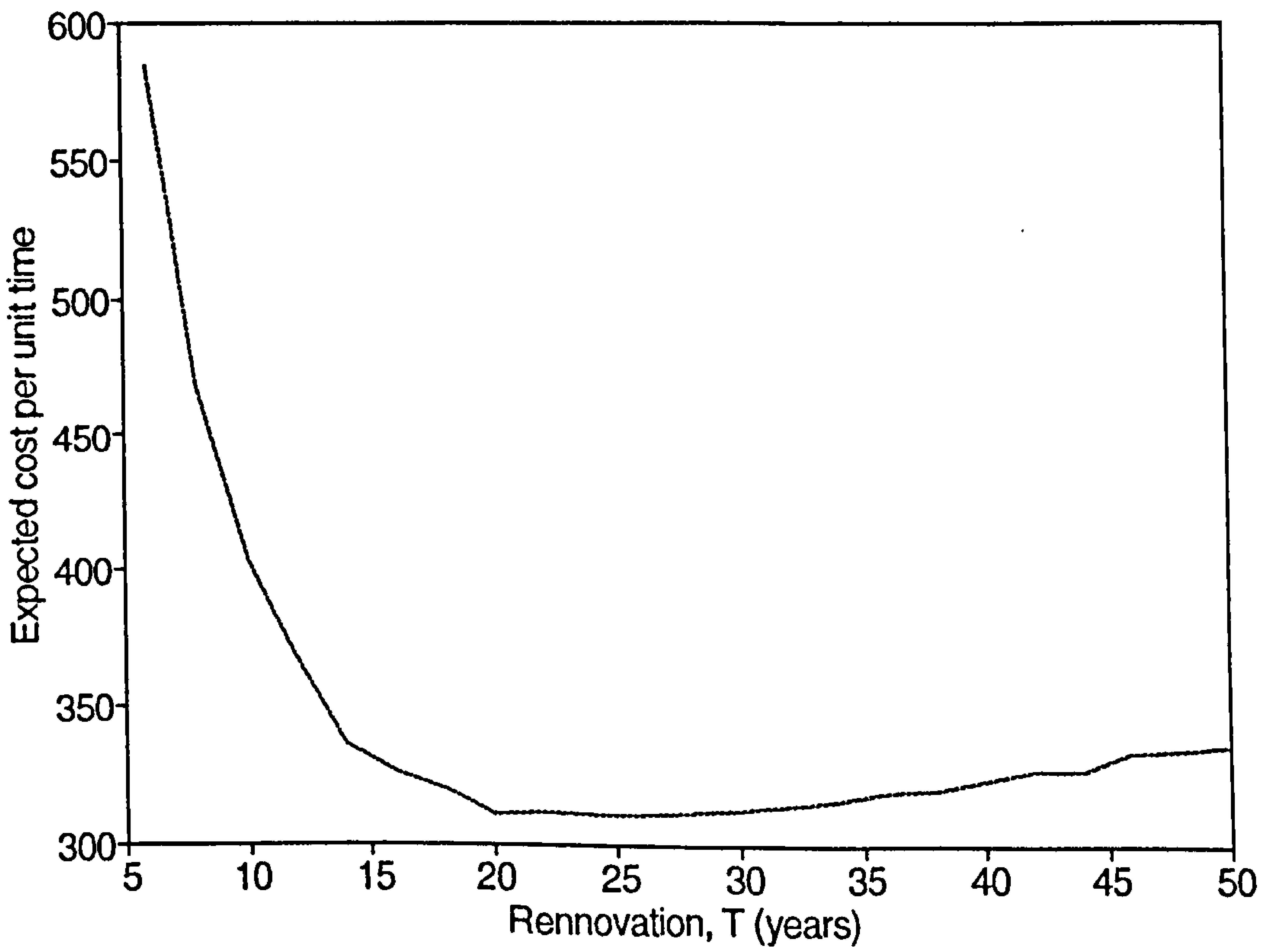

Fig. 6.13. Expected cost per unit time versus rennovation period, *T*, for a bridge.

Next, we consider the effects of a periodic inspection policy, without rennovation, over a 50 year life span for a bridge. Two policies will be considered, that is repairing any defect found at inspection and that of only repairing spalling defects. The current practice of the two companies who provided the data, is to generally repair only the spalling or worse defects. A simulation is used to estimate the expected maintenance cost per unit time, $c(T)$ say, over 50 years. It will be assumed that if more than one component is found defective at inspection then the set-up cost, £1650, will be shared across the components. A sample size of 4000 bridges was selected. The expected cost per unit time for the two policies for the bridge consisting of 5 beams and 3 columns is given in Fig. 6.14. It can be seen that the optimums are around 5 years, that of the current practice. However, the spalling repair policy has a lower cost per unit time. Thus, a recommendation would be to only repair cracks if the need is essential.

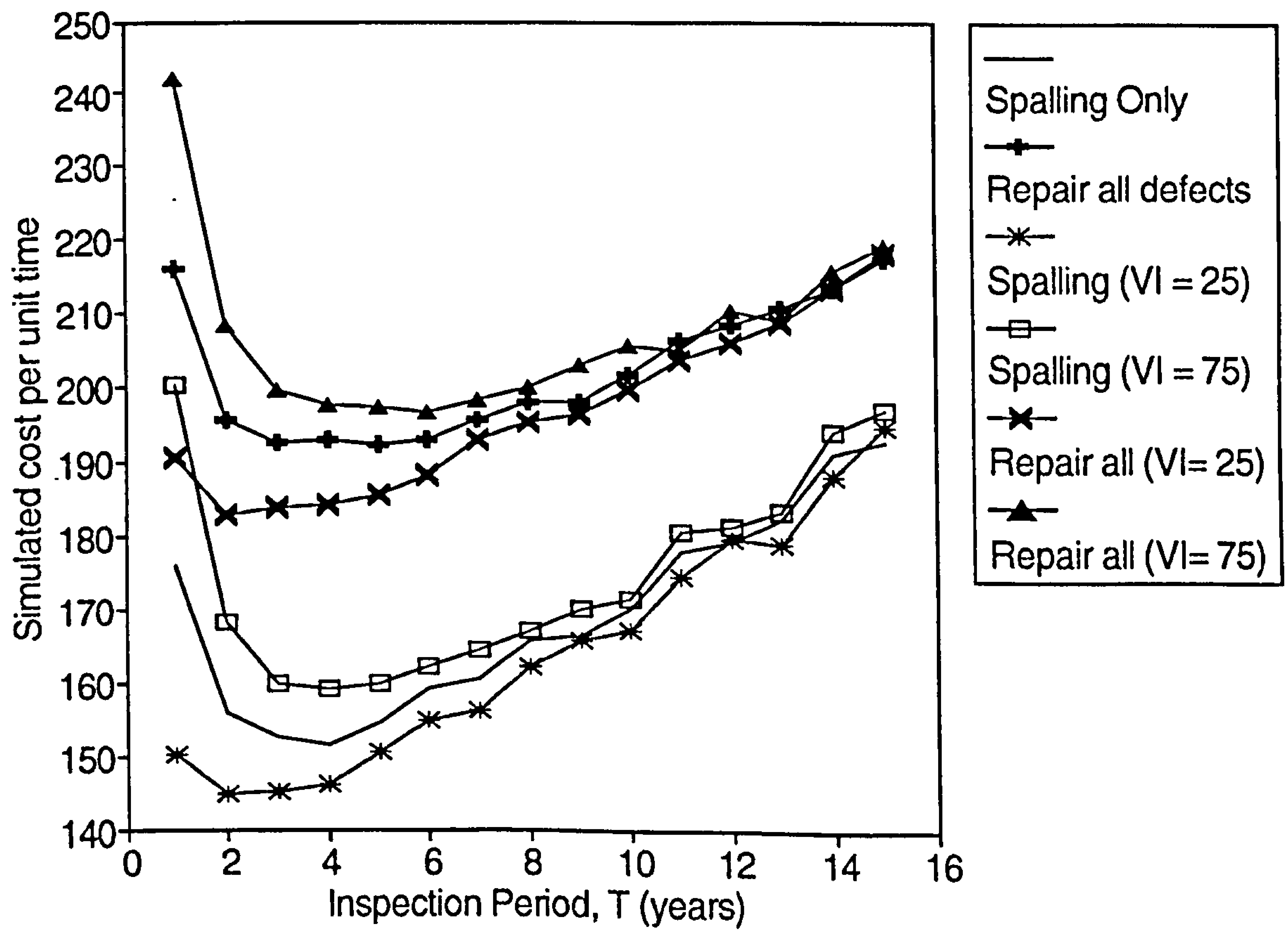

Fig. 6.14. Simulated expected cost per unit time versus inspection period.

For a sensitivity analysis, the subjectively estimated visual inspection cost (£50) was varied to £25 and £75 for both the policies of repairing all defects and of repairing at the spalling only. The simulation outputs for expected cost per unit time over a 50 year time horizon as a function of the inspection period are given in Fig. 6.14. For the case of spalling repair, the optimum inspection is reduced to 2 years for the £25 inspection cost and still at 4 years for the £75 cost. For the case of repairing all defects, the same reduction in the optimum can be seen for the £25 cost. However, the optimum is increased to 6 years for the £75 inspection cost. Clearly, the requirement to a more degradable state before repair does, in this case, lead to notably lower costs. It is also evident that the cost of the inspection is more influential upon the inspection period when all defects are repaired and not postponed to the spalling stage. This is consistent with intuition.

6.6. Conclusion

The chapter has highlighted a method to model the deterioration and maintenance of concrete structures. The concept of delay time is an important ingredient in the modelling process, and this is extended to a four phase classification of a component; namely defect free, cracking, spalling and failure.

Analysis of the database shows the deterioration of components over a series of inspection and initial p.d.fs of the delay time phases have been calculated. A method to estimate parameters with the censored data based using maximum likelihood has been proposed, along with a technique for testing the fit obtained.

Other models for the delay time distribution have been considered. We recall that the probability, p, that a defect will develop within 50 years, was defined due to defect histories being, on average, recorded over this period. If interest is in a maintenance strategy upto a 50 year time span then the time of defect developments beyond 50 years need not be modelled. However, a defect development beyond the 50 year time span is most likely to arise in components subjected to mild and dry environments, see Currie and Robery (1994). Hence, the parameter p introduced in Section 6.4 seems a reasonable parameter to employ for modelling the initial point distribution, $g(u)$, in the assessment of the 50 year condition, equation (6.3). The need to introduce a probability, q, that the cracking phase takes zero time was due to a substantial number of components being detected first in the spalled state. It is known that the cracking phase can be short in some cases when a component is subjected to intermittent wet conditions, see Currie and

Robery (1994). Hence, it is of interest to estimate this parameter.

Parameter estimation programs were successfully written and tested using simulation. The parameter estimates for beams and columns were tested using the Kaplan-Meier estimate of the reliability function, resulting in close fits for u and $u + h$ over 50 years and for $u + h + v$ over 35 years for beams and 50 years for columns, with the maximum likelihood fit lying within 95% confidence bands of the Kaplan-Meier estimate.

Problems that arose when estimating repair costs for concrete components have already been outlined in the text. The main factors in achieving success in the current type of data collection are:

1. The questions need to be fully understood by the organization involved in estimating.
2. Regular meetings with the repair organization and operational researchers are important to clarify the nature and use of the information sought.
3. A large data sample is required, especially to estimate bi-variate cost functions.
4. Timely return of costing information so that, if necessary, revised questionnaires can be prepared within the project time.

The recent paper by Wang (1995) would have assisted us considerably here, and the concepts will be utilised in developments when distributions of costs are required. Close collaboration and continuous contact with civil engineers is also necessary to fully understand the process being modelled, so that accurate predictions of concrete deterioration and cost consequences of inspection and repair practices can be formulated.

Single component and multi-component cost models were formulated incorporating the estimated model for deterioration and the estimated cost functions. The single component model recommends to inspect beams at around 5 years and columns at 6 years. The cost model for a bridge suggests to overhaul at around 20-30 years. The recommended periodic inspection policy for a bridge is to inspect every 5 years, repairing only spalling defects, which interestingly is approximately the current practice of the companies involved. Further research would lie in investigating the effect of inspections prior to an overhaul or in some form of age based policy. Other extensions relate to the stochastic distribution nature of costs and the non-perfect nature of inspections. The sensitivity analysis, on varying the visual inspection cost, has shown that the cost of inspection has greater influence upon the optimum inspection period for the case of repairing all defects, compared to the practice of repairing only at the spalling stage.

The main conclusion of the research program is that if the appropriate input data is collected in a coherent and methodical fashion, it is possible to utilise it to model deterioration rate, and further, the potential exists to model cost consequences of inspection policies and thereby optimize inspection practices.

Chapter 7

Conclusion

7.1 Literature Review

Chapter 1 has presented an overview of past and current developments in maintenance modelling. It is evident that delay time models and other new models are now increasingly being applied and tested through case studies. However, there is evidence of a deficiency of models for maintenance that takes into account the physical process, be it chemical, electrical or mechanical, that leads to a component failure. Geraerds (1972) regards the selection of statistical models for component and system failure behaviour as a subsection of a complex maintenance model of an organisation, that also takes into account such factors as maintenance planning and control, designs of systems, inventory problems and the feedback of results. Dekker et al (1995) consider also the planning of the maintenance activities for a group of components with different estimated optimal policies. It is shown that combining the maintenance activities, by delaying or bringing forward planned maintenance for some components with increased cost penalty, can reduce overall maintenance costs. This is due to the setup cost being shared. Hence, the necessary fusion between mathematical models and organizational planning and constraints are evolving in the maintenance field.

Over the past ten years, since the first paper, Christer and Waller (1984a), delay time modelling has undergone considerable development and is increasingly being accepted as an important concept for the real world modelling of maintenance of components and systems. There have been models which have touched on the concept, for example, Cox (1957, p.121) introduced a wear model such that a component can be defect free or enter a defective state prior to failure. This is equivalent to having a finite probability of zero delay time. An inspection model is presented to take into account this effect. Cozzoloni (1968) formulates a model whereby a system is assumed to have an unknown number of defects after a planned maintenance activity with each defect having a delay to cause failure. It is shown that the process of breakdowns will be a non-homogeneous Poisson process, as with the delay time system model. Butler (1979) classifies a component as functioning, functioning but defective, and failed. An inspection is also assumed to possibly increase the chance of failure due to the chance of observing a component in the defective state. However, a Markov model is formulated, thereby restricting the distribution of u and h to exponential. Lewis (1972) suggests an accumulation model whereby defective components which are detectable and do not cause failures, i.e having infinite delay time, are repaired at a failure. The expected repair time is then correlated with operating time.

Statistical methods, testing and policy formulations are evidently being developed and formalised for the delay time model with the growing experience through applications. It is important that statistical tests are carried out in confirming all postulated assumptions, e.g the renewal assumption of a perfect inspection. A method to identify the optimal and feasible policy type (e.g periodic or age-based inspections) for a component or system also needs to be addressed.

7.2 Delay Time Models for Repairable Systems

It has been seen that the concept of delay time can be used in modelling maintenance of a complex system. The NHPP model for the arrival process of failures of a repairable system, endorsed by Ascher and Feingold (1984), incorporates the concept of delay time by allowing the ROCOF be a convolution of the defect rate and delay time p.d.f under the assumption of independence. In this way, the expected number of defects detected at inspections can also be modelled. However, the assumption of an NHPP for a system will need to be tested for a specific case.

The case of imperfect inspection with NHPP defect arrivals has not been considered here and is an area for further research which is underway elsewhere. The inspection point for this case is not a system renewal implying that the defect arrival rate $g(u)$ cannot be considered identical in each inspection interval. This increases the modelling complexity.

Clearly, the downtime and cost of other inspection policies can also be investigated. For example, a policy could be to inspect after the n'th failure occurrence or when a particular length of operating time t has elapsed. The decision variables would be n and t. The use-based inspection policy is when $n = \infty$. The case, $n = 1$, implies an age based replacement policy, for the case of perfect inspections.

Under restricted circumstances, it has been shown that the system can be modelled by a Markov process in continuous time, see also Duyn Schouten and Wartenhorst (1994). This model could also be expanded to the more realistic case when there are a finite number of defect prone components within a system.

A criticism of these models is that ageing of the system after each inspection has not been modelled. Ageing can be modelled by assuming non-identical defect rates, $g(u)$, over each inspection interval. It is also possible to allow the delay time of a defect to be dependent on u and the inspection interval in which the defect occurred, see Christer and Wang (1992). The process of breakdowns then would not necessarily be an NHPP.

The type of model selected is directly dependent on assumptions on how the system is used, type and quality of maintenance and the deterioration processes over time. The purpose of this chapter has been to introduce the basic nature of the delay time concept and the variety of models that may be constructed.

7.3 Updating and Estimating Parameters for Delay Time Models.

Several formal techniques of updating delay time models have been presented. These have been based on the existence of subjective data to decide a prior or type of delay time distribution. The prior is then parameterized under a linear transform and the uniqueness and existence of a solution to modelling the "status quo" is discussed. It has been found that a unique solution exists under a simple scale transform and a set of solutions under the more general linear case. The effects of changing the model via the parameter, β (the probability a defect is detected at inspection), that is for $\beta = 1$ to $\beta \neq 1$ and vice-versa, or simply varying β, as another updating option has also been investigated which highlights the variety of updating options.

The effects of the change in the downtime model and consequently the optimum inspection period has been demonstrated for various updating techniques. Further research could lie in predicting the behaviour of the optimum for updating options, modelling parameters and delay time p.d.f types.

Another method of parameter estimation has been proposed based on observed times of breakdown and the defects detected at inspections. For this method, the prior distribution type can be assumed and the parameters are then determined using the method of moment or maximum likelihood technique. It has been seen that the observed failure times can be also be used in a test for fitting a model and in deciding upon a delay time prior when no delay time data is available.

Delay times may not only be biased subjectively, but also from a censored data set. Delay times, for example, may only be estimated from the failures which occur over a data collection survey. Hence, an observational bias enters the problem.

7.4 Bias in the Initiation and Delay Time Parameter Estimates

It has been seen that a statistical bias can exist in the data leading to parameter estimates of delay time and initiation time for censored data. Breakdown based observations give rise to an underestimate of delay time and initiation time whilst observations based upon defect repairs at inspection give rise to overestimates. The bias has been shown to be dependent on inspection frequency and the perfectness of inspections.

Methods based on maximum likelihood have been proposed to cope with the bias and lead to the estimation of the actual initiation time and delay time distribution.

In the case of censored data, it may be possible that parameters for the delay time distribution can be estimated to an acceptable degree of accuracy by updating procedures instead of performing the bias correction. Some form of iteration method could be proposed.

7.5 Simulation Study of the Delay Time Process

It has been seen that the simulation models and various parameter estimation procedures produce results in agreement with theoretical models. This, validates the programming of and the derivation of the delay time models and theory of the previous section. The bias correction and iteration methods estimate the parameters of the delay time distribution to within 10% of the theoretical values for a moderately sized sample, on the tests carried out so far. The method of estimation using only observed data also produces results in accordance with theory. We conclude that these are formal techniques. However, convergence properties of the iteration method needs to be further researched. The answer to the convergence will lie in the form of the log-likelihood functions. It is believed, for the work required here, that if convergence occurs through the iteration method, then the estimates will tend to the maximum likelihood estimates when correcting for bias, as the sample size increases.

Investigation into the error of the optimum inspection period when carrying out these method also needs research. A clue could lie in the behaviour of the function $b(T; \alpha, \gamma)$ for values of T, under the non-unique set (α, γ) which satisfy the observation point b^*, where α and ω are the scale and shift parameters respectively. The cost and downtime curves under such restrictions, may have optimum inspection periods which lie within a certain calculable interval. However, the measure of error, if the iteration method gives accurate results in most cases this may not now be necessary to consider.

The criteria for deciding whether to perform iteration or bias correction produces satisfactory results. The suggested decision procedures for deciding whether or not to correct for bias may only work when the biased distribution can be approximated by a Weibull distribution. This occurs in the test of section 5.4.4, as the failures delay times are almost exponential. A decision to not correct for bias may be made on the status-quo point being satisfied. However, the resulting model for other inspection periods than the current practice may be inaccurate. A further statistical test, such as, for example, comparing the empirical and theoretical distribution of times of breakdowns, would need to be applied.

Overall, simulation programs have been successfully written and tested, and methods have been developed for estimating delay distributions given accurately estimated data in practice. Further work could also lie in the effects of subjective errors when estimating delay time, the possibility of imperfect inspections and convergence properties of the iteration process.

7.6 Application of Delay Time Analysis to Concrete Structures

The chapter has highlighted a method to model the deterioration and maintenance of concrete structures. The concept of delay time is an important ingredient in the modelling process, and this is extended to a four phase classification of a component; namely defect free, cracking, spalling and failure.

Analysis of the database shows the deterioration of components over a series of inspection and initial p.d.fs of the delay time phases have been calculated. A method to estimate parameters with the censored data based using maximum likelihood has been proposed, along with a technique for testing the fit obtained.

Other models for the delay time distribution have been considered. We recall that the probability, p, that a defect will develop within 50 years, was defined due to defect histories being, on average, recorded over this period. If interest is in a maintenance strategy upto a 50 year time span then the time of defect developments beyond 50 years need not be modelled. However, a defect development beyond the 50 year time span is most likely to arise in components subjected to mild and dry environments, see Currie and Robery (1994). Hence, the parameter p introduced in Section 4 seems a reasonable parameter to employ for modelling the initial point distribution, $g(u)$, in the assessment of the 50 year condition, equation (6.3). The need to introduce a probability, q, that the cracking phase takes zero time was due to a substantial number of components being detected first in the spalled state. It is knwown that the cracking phase can be short in some cases when a component is subjected to intermittent wet conditions, see Currie and Robery (1994). Hence, it is of interest to estimate this parameter.

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The main conclusion of the research program is that if the appropriate input data is collected in a coherent and methodical fashion, it is possible to utilise it to model deterioration rate, and further, the potential exists to model cost consequences of inspection policies and thereby optimize inspection practices.

Appendix

Stochastic Case Model for Downtime

An outline is given here on obtaining a more general model for downtime, that is relaxing assumption (b), $d_b \ll T$, from Section 2.5. What is required is the expected cumulative downtime over a real time interval, say (0, T). It is a more complicated option because at time T the system could either be operating or down for repair. Hence, the model would be dependent on knowledge of the p.d.f of the breakdown repair time. Chilcott and Christer (1991) develop an iterative equation method for a similar problem. Dagpunar and Jack (1993) derive a minimal repair model for a system with the assumption of constant breakdown repair times. Hence, a need could arise for this refinement.

Let D_T, denote the cumulative breakdown repair time over interval (0, T). If no failures have arisen in interval (0, T) then this implies $P\{D_T = 0\} = e^{-B(T)}$, due to NHPP breakdown arrivals. Consider the joint event; {the system is up at time T, D_T is in the small interval (x, $x + dx$), n failures have arisen over (0, T) and have been completely repaired}. This is equivalent to having only n breakdowns in operating time $t = T - x$ and the cumulative breakdown time of n complete repairs in interval (x, $x + dx$). The joint probability of this event is, $P\{B_{T-x} = n\}z_n(x)dx$, where $z_n(x)$ is the n-fold convolution of the p.d.f for each breakdown repair time and $P\{B_t = n\}$ is given by function (2.4).

Secondly, consider the above joint event above when the system is now down at time T, and the n+1'th breakdown is being repaired at time T. D_T will now have a contribution from the last and incomplete repair. This event is equivalent to n failures in operating time $t = T - x$, an additional failure in operating time interval ($T - x$, $T - x + dx$) and x spanning the interval of the n'th and n+1'th cumulative breakdown repair time. The probability of the joint event under consideration, is $P\{B_{T-x} = n\}\{Z_n(x) - Z_{n+1}(x)\}r(x)dx$, denoting $Z_n(x)$ as the c.d.f form of $z_n(x)$. Summing the above probabilities for all n, it follows that the p.d.f of D_T, say $\pi(x;\, T)$, for the interval, $0 < x \leq T$, is given by,

$$\pi(x;T) = \sum_{n=0}^{\infty} B(T-x)^n \frac{e^{-B(T-x)}}{n!}\{z_n(x) + r(T-x)\{Z_n(x) - Z_{n+1}(x)\}\} \ , \qquad \text{(A.1)}$$

letting $z_0(x) = 0$ and $Z_0(x) = 1$. Note that the integral of $\pi(x;\, T)$ over the interval $(0, T)$ will equal $P\{D_T > 0\} = 1 - e^{-B(T)}$, i.e the probability of non-zero downtime in interval $(0, T)$.

Hence, the expected cumulative downtime, say $D_R(T)$, is given by,

$$D_R(T) = \int_{x=0}^{T} x\pi(x;T)\,dx \qquad \text{(A.2)}$$

Due to the complexity of the p.d.f, it is possible that Laplace transforms could be used or some other numerical method could be employed to evaluate the function. The revised expected downtime per unit time model, $d_R(T)$ say, that is given by,

$$d_R(T) = \frac{D_R(T) + d_I}{T + d_I} \ , \qquad \text{(A.3)}$$

can then be used to select the optimum inspection period. Also, validation of the approximate model for selected modelling parameters could be undertaken.

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